Using the generalised invariant formalism: a class of conformally flat pure radiation metrics with a negative cosmological constant

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Abstract. We demonstrate an integration procedure for the generalised invariant formalism by obtaining a subclass of conformally flat pure radiation spacetimes with a negative cosmological constant. The method used is a development of the methods used earlier for pure radiation spacetimes of Petrov types O and N respectively. This subclass of spacetimes turns out to have one degree of isotropy freedom, so in this paper we have extended the integration procedure for the generalised invariant formalism to spacetimes with isotropy freedom,

1. GIF integration procedure

The GIF integration procedure [10], [7] is a generalisation to the generalised invariant formalism (GIF) [17], [18], [19] of an integration method originally proposed by Held [14], [15] and developed by Edgar and Ludwig [3], [5], [6] in the GHP formalism [11]. It consists of manipulating all the equations of the formalism in an attempt to construct a complete and involutive set of tables involving first derivative GIF spinor operators. The aim is for the set to include

- a table for each of four real zero-weighted scalars ('coordinate candidates'),
- a table for one complex (non-trivially) weighted scalar
- a table for a second spinor $\mathbf{I}_A$ (which is not parallel to the first dyad spinor $\mathbf{o}_A$).

An important element in this method is to recognise that much information resides in the GIF commutator equations (as well as in the GIF Ricci and Bianchi equations) and in order that all this information is extracted it is essential that the commutators should be applied explicitly to these five scalars, as well as to the new spinor $\mathbf{I}$ [10].

Once these tables have been found, then the new spinor $\mathbf{I}$ can be identified with the second dyad spinor $\boldsymbol{\iota}$ in the GHP formalism, and so the spinor equations in the GIF translate to simpler scalar equations in the GHP formalism.

In the 'optimal situation', all of these five scalars and the spinor $\mathbf{I}$ will be *intrinsic* to GIF, and will be generated *directly* by manipulations and rearrangements of intrinsic elements of the GIF formalism; the generic class of the conformally flat pure radiation spacetimes provided an example of this [10]. In less than optimal situations, there is a failure to generate the full quota of five scalars and/or the spinor $\mathbf{I}$ *directly* within the GIF formalism; in such cases, it is essential that we find these 'missing' quantities, and so they have to be introduced *indirectly, via their tables*, and since they are not intrinsic, we will refer to them as *complementary.*

This technique of introducing a 'missing' coordinate candidate indirectly by its table had earlier been developed in the closely related integration procedure within the GHP formalism [5], [6], [16] where it was understood that the absence of such a coordinate candidate is associated with the presence of a Killing vector. In situations where the 'missing' coordinate candidate has a counterpart in the generic case, we can 'copy' the table structure for this generic case, but ensure that our new complementary coordinate candidate is 'freed' from direct links with any other elements of the formalism; the special non-generic subclass of the conformally flat pure radiation spacetimes provided an example of this [10]. On other occasions, we may not have the advantage of a generic case from which we can get hints, and in such cases we will need to carefully study the structure of the other equations, especially the commutators, to guess, and then check to confirm the validity of, an appropriate table; this was the approach in [7].

In GIF we can also encounter the situation where the second spinor $\mathbf{I}$ fails to be generated directly and uniquely in terms of intrinsic elements of GIF; this will happen in spacetimes which have one or two degrees of null isotropy freedom.

We wish to obtain more experience by investigating spacetimes with 'missing' coordinate candidates, as well as with isotropy freedom; the spaces we will now investigate provide us with these opportunities.

So, in this paper we investigate a special class of conformally flat spaces whose Ricci tensor has a Ricci scalar as well as a pure radiation component; equivalently these can be considered as conformally flat pure radiation spacetimes with a cosmological constant. Specifically we concentrate on a particular subclass which has the desirable properties; furthermore, in this paper we will consider only the generic case for this subclass. The derivation will require additional techniques and provide additional insights compared to the analysis of conformally flat pure radiation spacetimes without the cosmological constant in [10]; in particular, we will find that the spacetimes identified in this paper have the isotropy freedom of a one-parameter group of null rotations. The details of the integration procedure for the complete subclass is given in [8], where we consider also the special cases excluded from this paper; and preliminary results for the other subclass of conformally flat spaces whose Ricci tensor has a Ricci scalar (with unrestricted sign) as well as a pure radiation component were reported in [9].

2. GIF

In this section we will give summaries from [18] of the relevant parts of the GIF which are needed in this paper. In the GIF the role of the spin coefficients κ, σ, ρ and τ is taken up by spinor quantities $\mathbf{K}$, $\mathbf{S}$, $\mathbf{R}$ and $\mathbf{T}$

$$\begin{aligned}
\mathbf{K} &= \kappa \\
\mathbf{S}_{A'} &= \sigma\bar{\mathbf{o}}_{A'} - \kappa\bar{\boldsymbol{\iota}}_{A'} \\
\mathbf{R}_A &= \rho\mathbf{o}_A - \kappa\boldsymbol{\iota}_A \\
\mathbf{T}_{AA'} &= \tau\mathbf{o}_A\bar{\mathbf{o}}_{A'} - \rho\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'} - \sigma\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'} + \kappa\boldsymbol{\iota}_A\bar{\boldsymbol{\iota}}_{A'}
\end{aligned} \tag{1}$$

These spinors are invariant under null rotations, and under the usual spin and boost transformations have respective weights $\{\mathbf{3},\mathbf{1}\}, \{\mathbf{3},\mathbf{0}\}, \{\mathbf{2},\mathbf{1}\}, \{\mathbf{2},\mathbf{0}\}$.

The GIF differential operators $\text{\th}$, $\eth$, $\text{\th}'$ and $\eth'$, which act on properly weighted symmetric spinors to produce symmetric spinors of different valence and weight, may all be defined in terms of an auxiliary differential operator $\mathcal{D}_{ABA'B'}$ acting on a spinor $\boldsymbol{\eta}$ of weight $\{\mathbf{p},\mathbf{q}\}$,

$$\begin{aligned}
\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} &= \mathbf{o}_A\bar{\mathbf{o}}_{A'}\nabla_{BB'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \\
&-(\mathbf{p}\bar{\mathbf{o}}_{A'}\nabla_{BB'}\mathbf{o}_A \quad + \mathbf{q}\mathbf{o}_A\nabla_{BB'}\bar{\mathbf{o}}_{A'})\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}}
\end{aligned} \tag{2}$$

The GIF operators are obtained by contraction with $\mathbf{o}$ and $\bar{\mathbf{o}}$, and symmetrizing.

$$(\text{\th}\boldsymbol{\eta})_{AC_1\ldots C_NA'C_1'\ldots C_{N'}'} = \sum_{sym}\mathbf{o}^B\bar{\mathbf{o}}^{B'}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'} \quad (3)$$

$$(\boldsymbol{\eth}\boldsymbol{\eta})_{AC_1\ldots C_NA'B'C_1'\ldots C_{N'}'} = \sum_{sym}\mathbf{o}^B\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'} \quad (4)$$

$$(\boldsymbol{\eth}'\boldsymbol{\eta})_{ABC_1\ldots C_NA'C_1'\ldots C_{N'}'} = \sum_{sym}\bar{\mathbf{o}}^{B'}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'} \quad (5)$$

$$(\text{\th}'\boldsymbol{\eta})_{ABC_1\ldots C_NA'B'C_1'\ldots C_{N'}'} = \sum_{sym}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'} \quad (6)$$

where $\sum_{sym}$ indicates symmetrization over all free primed and unprimed indices.

In our calculations, we will need to know the result of contracting $\text{\th}'\boldsymbol{\eta}$ with $\mathbf{o}$ and $\bar{\mathbf{o}}$ respectively, as well as analogous contractions on the other operators. In the case of a scalar field η, from (6)

$$(\text{\th}'\eta)_{ABA'B'}\bar{\mathbf{o}}^{B'} = \tfrac{1}{2}\{(\boldsymbol{\eth}'\eta)_{ABA'} - q\overline{\mathbf{T}}_{A'(A}\mathbf{o}_{B)}\eta\} \quad (7)$$

Although the definition of the differential operators is quite complicated, the fact that they take symmetric spinors to symmetric spinors means that we can write down the equations in a more compact and index free notation. In the compacted notation (7) and its analogues become

$$(\text{\th}'\eta)\cdot\bar{\mathbf{o}} = \tfrac{1}{2}\{(\boldsymbol{\eth}'\eta) - q\overline{\mathbf{T}}\eta\} \quad (8)$$

$$(\text{\th}'\eta)\cdot\mathbf{o} = \tfrac{1}{2}\{(\boldsymbol{\eth}\eta) - p\mathbf{T}\eta\} \quad (9)$$

$$(\boldsymbol{\eth}'\eta)\cdot\mathbf{o} = \tfrac{1}{2}\{(\text{\th}\eta) - p\mathbf{R}\eta\} \quad (10)$$

$$(\boldsymbol{\eth}\eta)\cdot\bar{\mathbf{o}} = \tfrac{1}{2}\{(\text{\th}\eta) - q\overline{\mathbf{R}}\eta\} \quad (11)$$

$$(\text{\th}'\eta)\cdot\mathbf{o}\cdot\bar{\mathbf{o}} = \tfrac{1}{4}\{(\text{\th}\eta) - p\mathbf{R}\eta - q\overline{\mathbf{R}}\eta\} \quad (12)$$

For a valence (1,0)-spinor $\boldsymbol{\eta}_A$ of weight $\{\mathbf{p},\mathbf{q}\}$ we get

$$(\text{\th}'\boldsymbol{\eta})\cdot\mathbf{o} = \tfrac{1}{3}\{\text{\th}'(\boldsymbol{\eta}\cdot\mathbf{o}) + (\boldsymbol{\eth}\boldsymbol{\eta}) - (\mathbf{p}-1)\mathbf{T}\boldsymbol{\eta}\} \quad (13)$$

$$(\text{\th}'\boldsymbol{\eta})\cdot\bar{\mathbf{o}} = \tfrac{1}{3}\{\text{\th}'(\boldsymbol{\eta}\cdot\bar{\mathbf{o}}) + (\boldsymbol{\eth}'\boldsymbol{\eta}) - \mathbf{q}\overline{\mathbf{T}}\boldsymbol{\eta}\} \quad (14)$$

An alternative way to define the GIF operators is via the GHP operators $\text{\th}, \eth, \eth', \text{\th}'$,

$$\begin{aligned}
&\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'} \\
&= (\text{\th}'\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'})\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} \\
&-(\eth'\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'})\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'} - (\eth\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'})\mathbf{o}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} \\
&-(\text{\th}\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'})\mathbf{o}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'} \\
&+(\mathbf{p}\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'}\mathbf{T}_{BB'} + \mathbf{q}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{B'}\overline{\mathbf{T}}_{B'B})\boldsymbol{\eta}_{C_1\ldots C_NC_1'\ldots C_{N'}'}
\end{aligned} \quad (15)$$

where $\text{\th}'$, $\eth'$, $\eth$ and $\text{\th}$ are the ordinary GHP operators applied to spinors.

In the case of a scalar field we can therefore transfer from the GIF to GHP formalism via

$$\begin{aligned}
(\text{\th}'\eta)_{ABA'B'} = {} & (\text{\th}'\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\eth'\eta - q\bar{\tau}\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} \\
& -(\eth\eta - p\tau\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} + (\text{\th}\eta - p\rho\eta - q\bar{\rho}\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} \\
& -p\sigma\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - q\bar{\sigma}\mathbf{o}_A\mathbf{o}_B\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'} + p\kappa\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} + q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'}
\end{aligned} \quad (16)$$

$$
\begin{aligned}
(\boldsymbol{\eth}'\eta)_{ABA'} &= (\eth'\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'} - (\text{Þ}\eta - p\rho\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'} \\
&\quad + q\bar{\sigma}\mathbf{o}_A\mathbf{o}_B\boldsymbol{\iota}_{A'} - p\kappa\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'} - q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'} \qquad (17)\\
(\boldsymbol{\eth}\eta)_{AA'B'} &= (\eth\eta)\mathbf{o}_A\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\text{Þ}\eta - q\bar{\rho}\eta)\mathbf{o}_A\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} \\
&\quad + p\sigma\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - p\kappa\boldsymbol{\iota}_A\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}B') - q\bar{\kappa}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'} \qquad (18)\\
(\boldsymbol{\text{Þ}}\eta)_{AA'} &= (\text{Þ}\eta)\mathbf{o}_A\mathbf{o}_B + p\kappa\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'} - q\bar{\kappa}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'} \ . \qquad (19)
\end{aligned}
$$

3. The equations

In the usual way, we choose

$$\boldsymbol{\Phi}_{ABA'B'} = \Phi\, \mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} \qquad (20)$$

where $\Phi(=\Phi_{22})$ is a real scalar field of weight $\{2,2\}$; all the other curvature components, except the Ricci scalar Λ, vanish.

For this class of spaces $\mathbf{K}=\mathbf{0}, \mathbf{S}=\mathbf{0}, \mathbf{R}=\mathbf{0}$, but

$$\mathbf{T}_{AA'} = \tau\mathbf{o}_A\bar{\mathbf{o}}_{A'} \qquad (21)$$

Since τ and Φ_{22} are both invariant under the group of null rotations, they can be used instead of their GIF spinor equivalents; this gives a considerable simplification in the GIF notation.

The GIF Ricci and Bianchi equations are:

$$
\begin{aligned}
\boldsymbol{\text{Þ}}\tau &= 0\\
\boldsymbol{\eth}\tau &= \tau^2\\
\boldsymbol{\eth}'\tau &= \tau\bar{\tau} + 2\Lambda \qquad (22)\\
\boldsymbol{\text{Þ}}\Phi &= 0\\
\boldsymbol{\eth}\Phi &= \tau\Phi\\
\boldsymbol{\eth}'\Phi &= \bar{\tau}\Phi \qquad (23)\\
\boldsymbol{\text{Þ}}\Lambda &= 0\\
\boldsymbol{\eth}\Lambda &= 0\\
\boldsymbol{\eth}'\Lambda &= 0\\
\boldsymbol{\text{Þ}}'\Lambda &= 0 \qquad (24)
\end{aligned}
$$

In addition we have the GIF commutators from [18] which, for this paper, specialise to:

$$
\begin{aligned}
(\boldsymbol{\text{Þ}}\boldsymbol{\text{Þ}}' - \boldsymbol{\text{Þ}}'\boldsymbol{\text{Þ}})\boldsymbol{\eta} &= (\bar{\tau}\boldsymbol{\eth} + \tau\boldsymbol{\eth}')\boldsymbol{\eta} + (\mathbf{p}-N)\Lambda\boldsymbol{\eta} + (\mathbf{q}-N')\Lambda\boldsymbol{\eta} \qquad (25)\\
(\boldsymbol{\text{Þ}}\boldsymbol{\eth}' - \boldsymbol{\eth}'\boldsymbol{\text{Þ}})\boldsymbol{\eta} &= 2\Lambda(\boldsymbol{\eta}\cdot\bar{\mathbf{o}}) \qquad (26)\\
(\boldsymbol{\eth}\boldsymbol{\eth}' - \boldsymbol{\eth}'\boldsymbol{\eth})\boldsymbol{\eta} &= -(\mathbf{p}-N)\Lambda\boldsymbol{\eta} + (\mathbf{q}-N')\Lambda\boldsymbol{\eta} \qquad (27)\\
(\boldsymbol{\text{Þ}}'\boldsymbol{\eth} - \boldsymbol{\eth}\boldsymbol{\text{Þ}}')\boldsymbol{\eta} &= -\tau\boldsymbol{\text{Þ}}'\boldsymbol{\eta} - \Phi(\boldsymbol{\eta}\cdot\mathbf{o}) \qquad (28)
\end{aligned}
$$

where $\boldsymbol{\eta}$ is a general symmetric spinor of weight $\{\mathbf{p},\mathbf{q}\}$ and with N unprimed and N' primed indices; $(\boldsymbol{\eta}\cdot\mathbf{o})$ is the $(N-1,N')$-spinor $\boldsymbol{\eta}_{A_1....A_NA_1....A_{N'}}\mathbf{o}^{A_N}$, and $(\boldsymbol{\eta}\cdot\bar{\mathbf{o}})$ is the $(N,N'-1)$-spinor $\boldsymbol{\eta}_{A_1....A_NA_1....A_{N'}}\bar{\mathbf{o}}^{A_{N'}}$, and if the contraction is not possible then these terms are set to zero.

We emphasize that these GIF equations (22)-(28) contain all the information for the type O pure radiation metrics with non-zero Ricci scalar.

In this present paper we shall only be concerned with the special subclass where

$$\tau\bar{\tau} + \Lambda = 0 \qquad (29)$$

which of course means that we will only be considering *a negative cosmological constant*, and it will be convenient to write $\lambda \equiv \pm\sqrt{-\Lambda}$.

4. The integration procedure: the generic case.

4.1. Constructing a table for **I** *and applying commutators.*

The Riemann tensor and the spin coefficients supply three real scalars which can easily be rearranged to give one real zero-weighted $(\tau\overline{\tau})$ and two real weighted scalars, Φ and $\arg(\tau/\overline{\tau})$. In this special case, the real zero-weighted scalar $(\tau\overline{\tau}) = \lambda^2$ and is constant; and in order for convenient presentation we use the weighted scalars

$$\mathcal{P} = \sqrt{\frac{\tau}{\overline{\tau}}}, \tag{30}$$

$$\mathcal{Q} = \sqrt{\Phi} \tag{31}$$

where $\mathcal{P}$ is a complex scalar of weight $\{1,-1\}$ and $\mathcal{P}\overline{\mathcal{P}} = 1$; $\mathcal{Q}$ is a real scalar of weight $\{-1,-1\}$. (As well as $\Phi = \mathcal{Q}^2 \neq 0 \neq \Lambda$, we are assuming $\tau = \lambda\mathcal{P} \neq 0$, and so each of $\mathcal{P},\ \mathcal{Q}$, will always be defined and different from zero.)

So we can replace the Ricci and Bianchi equations with the table for $\overline{\mathcal{P}}\mathcal{Q}$

$$\begin{aligned}
\text{þ}(\overline{\mathcal{P}}\mathcal{Q}) &= 0\\
\eth(\overline{\mathcal{P}}\mathcal{Q}) &= -\lambda\mathcal{Q}/2\\
\eth'(\overline{\mathcal{P}}\mathcal{Q}) &= 3\lambda\mathcal{Q}\overline{\mathcal{P}}^2/2\\
\text{þ}'(\overline{\mathcal{P}}\mathcal{Q}) &= \mathbf{J}
\end{aligned} \tag{32}$$

where we have completed the table with some spinor $\mathbf{J}$, which is as yet undetermined.

From (8) and (9) we can deduce that

$$\text{þ}'(\overline{\mathcal{P}}\mathcal{Q})\cdot\bar{\mathbf{o}} = \eth'(\overline{\mathcal{P}}\mathcal{Q}) \tag{33}$$

$$\text{þ}'(\overline{\mathcal{P}}\mathcal{Q})\cdot\mathbf{o} = \eth(\overline{\mathcal{P}}\mathcal{Q}) + 2\tau\overline{\mathcal{P}}\mathcal{Q} = \eth(\overline{\mathcal{P}}\mathcal{Q}) + 2\lambda\mathcal{Q} \tag{34}$$

Substituting (32) we can then write

$$\mathbf{J} = -3\lambda\overline{\mathcal{P}}\mathcal{Q}\,(\mathcal{P}\mathbf{I} + \overline{\mathcal{P}\mathbf{I}})/2 \tag{35}$$

where we have introduced the additional factors in order that the new spinor $\mathbf{I}$ has the following required simple properties

$$\mathbf{I}\cdot\bar{\mathbf{o}} = 0, \qquad \mathbf{I}\cdot\mathbf{o} = -1 \tag{36}$$

Hence $\mathbf{I}$ is a $(1,0)$ valence spinor, and we conclude that its weight is $\{-\mathbf{1},\mathbf{0}\}$, from

$$\mathbf{J}_{ABA'B'} = -\overline{\mathcal{P}}\mathcal{Q}\left(3\lambda\mathcal{P}/2\right)\mathbf{I}_{(A}\mathbf{o}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - \overline{\mathcal{P}}\mathcal{Q}\left(3\lambda\overline{\mathcal{P}}/2\right)\bar{\mathbf{I}}_{(A'}\bar{\mathbf{o}}_{B')}\mathbf{o}_A\mathbf{o}_B \tag{37}$$

Note that the spinor $\mathbf{I}$, as defined above, is *not* given uniquely in terms of the elements of the GIF formalism and so is not an intrinsic spinor; $\mathbf{I}$ is only defined up to the freedom of a one dimensional null rotation

$$\mathbf{I} \to \mathbf{I} + i\epsilon\bar{\mathcal{P}}\,\mathbf{o} \tag{38}$$

where ϵ is an arbitrary real zero-weighted scalar.

When we apply the commutators $\overline{\mathcal{P}}\mathcal{Q}$ via its table, the only notrivial information is the partial table

$$\begin{aligned}
\text{þ}(\mathcal{P}\,\mathbf{I} + \overline{\mathcal{P}}\,\bar{\mathbf{I}}) &= -2\lambda\\
\eth(\mathcal{P}\,\mathbf{I} + \overline{\mathcal{P}}\,\bar{\mathbf{I}}) &= \lambda\mathcal{P}(\mathcal{P}\mathbf{I} + \overline{\mathcal{P}\mathbf{I}})\\
\eth'(\mathcal{P}\,\mathbf{I} + \overline{\mathcal{P}}\,\bar{\mathbf{I}}) &= \lambda\overline{\mathcal{P}}(\mathcal{P}\mathbf{I} + \overline{\mathcal{P}\mathbf{I}})
\end{aligned} \tag{39}$$

We can complete this table with

$$\text{þ}'(\mathcal{P}\,\mathbf{I}+\overline{\mathcal{P}}\,\overline{\mathbf{I}}) \;=\; \mathbf{K} \tag{40}$$

and following the same procedure as for the completion of the table for $\overline{\mathcal{P}}\mathcal{Q}$, we find

$$\mathbf{K}=\mathcal{Q}^2K-\lambda(\mathcal{P}\mathbf{I}+\overline{\mathcal{P}}\,\overline{\mathbf{I}})^2 \tag{41}$$

where K is a zero-weighted real scalar, as yet undetermined; the factor $\mathcal{Q}^2$ was introduced so that the unknown scalar would be zero-weighted.

Therefore we do not obtain *directly* a table for $\mathbf{I}$. However, it is essential that we do obtain a table for a second spinor; therefore we introduce one particular spinor Ɨ from the class of spinors $\mathbf{I}$ (which we noted were defined up to the freedom (38)), by its table

$$\begin{aligned}
\text{þ}\text{Ɨ} &= -\lambda\overline{\mathcal{P}}\\
\eth\text{Ɨ} &= \lambda\overline{\mathcal{P}}\,\overline{\text{Ɨ}}\\
\eth'\text{Ɨ} &= \lambda\overline{\mathcal{P}}\,\text{Ɨ}\\
\text{þ}'\text{Ɨ} &= \overline{\mathcal{P}}\mathcal{Q}^2W-\lambda\mathcal{P}\text{Ɨ}^2-\lambda\overline{\mathcal{P}}\text{Ɨ}\overline{\text{Ɨ}}
\end{aligned} \tag{42}$$

which we have completed in the usual way with W, a zero-weighted *complex* scalar, as yet undetermined. This table is clearly consistent with (39), (40) and (41) with $K=W+\overline{W}$.

The theory requires that we next apply the commutators to Ɨ, yielding a partial table for W,

$$\begin{aligned}
\text{þ}W &= 0\\
\eth W &= -\mathcal{P}\Big(1-\lambda(W+\bar{W})\Big)\\
\eth' W &= 0
\end{aligned} \tag{43}$$

This partial table satisfies the relevant commutators, so therefore the table (42) for Ɨ is completely compatible with the remainder of the equations, and so we can adopt Ɨ as the second spinor.

4.2. Completing all the tables.

We also need tables for four zero-weighted real scalars. Putting

$$W=M-iB+1/2\lambda \tag{44}$$

the partial table (43) for (complex) W yields partial tables for (real) M and B respectively; we complete in the usual way to get

$$\begin{aligned}
\text{þ}M &= 0\\
\eth M &= \lambda\mathcal{P}M\\
\eth' M &= \lambda\overline{\mathcal{P}}M\\
\text{þ}' M &= \mathcal{Q}M^{3/2}R-\lambda\mathcal{P}M\text{Ɨ}-\lambda\overline{\mathcal{P}}M\overline{\text{Ɨ}}
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
\text{þ}B &= 0\\
\eth B &= i\lambda\mathcal{P}M\\
\eth' B &= -i\lambda\overline{\mathcal{P}}M\\
\text{þ}' B &= \mathcal{Q}M^{1/2}(G+RB)-i\lambda\mathcal{P}M\text{Ɨ}+i\lambda\overline{\mathcal{P}}M\overline{\text{Ɨ}}
\end{aligned} \tag{46}$$

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where R and G as usual are real zero-weighted scalars, as yet undetermined. (The additional factors introduced alongside R, G respectively are to ensure that these new scalars are zero-weighted, and also themselves satisfy very simple tables.)

When we apply the commutators to the above tables for M and B we obtain the following partial tables for R and G respectively

$$\begin{aligned} \text{þ}R &= 0 = \text{þ}G \\ \eth R &= 0 = \eth G \\ \eth' R &= 0 = \eth' G \end{aligned} \tag{47}$$

It is clear that these four scalars, M, B, R, G are not functionally independent; hence, tentatively adopting M, B, R as our three coordinate candidates, we complete the table for R in the usual way with the zero-weighted scalar Y, as yet undetermined,

$$\begin{aligned} \text{þ}R &= 0 \\ \eth R &= 0 \\ \eth' R &= 0 \\ \text{þ}' R &= \mathcal{Q} Y M^{1/2} \end{aligned} \tag{48}$$

Application of the commutators to R gives

$$\begin{aligned} \text{þ}Y &= 0 \\ \eth Y &= 0 \\ \eth' Y &= 0 \end{aligned} \tag{49}$$

It is clear that we have extracted all the information which is available *directly* from the tables for $\mathcal{P}, \mathcal{Q}$ and $\mathbb{I}$; we have identical partial tables for R, G, Y which means that they are functionally dependent. Clearly we now need a fourth table with a non-zero þ component.

So we introduce a real zero-weighted scalar N indirectly via a table which is consistent with the commutators. Beginning with the simplest possibilities and the commutators (25), (26), (27), we are led to

$$\begin{aligned} \text{þ}N &= \frac{M^{3/2}}{\mathcal{Q}} \\ \eth N &= -\frac{M^{3/2}}{\mathcal{Q}} \bar{\mathbb{I}} \\ \eth' N &= -\frac{M^{3/2}}{\mathcal{Q}} \mathbb{I} \\ \text{þ}' N &= \frac{\mathcal{Q} M^{1/2}}{2} U + \frac{M^{3/2}}{\mathcal{Q}} \mathbb{I} \bar{\mathbb{I}} \end{aligned} \tag{50}$$

where we have completed the table in the usual way with U, a real zero-weighted scalar, as yet undetermined.

Applying the remaining commutators gives the partial table for U,

$$\begin{aligned} \text{þ}U &= \frac{3M^{3/2}}{\mathcal{Q}} R \\ \eth U &= -2\mathcal{P} M(M + iB + 1/2\lambda) - \frac{3M^{3/2}}{\mathcal{Q}} R \bar{\mathbb{I}} \\ \eth' U &= -2\overline{\mathcal{P}} M(M - iB + 1/2\lambda) - \frac{3M^{3/2}}{\mathcal{Q}} R \mathbb{I} \end{aligned} \tag{51}$$

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When the relevant commutators are applied to U it is found that they are identically satisfied, and so N can be taken as our fourth coordinate candidate.

We have now obtained all the information about these spaces in an explicit form.

4.3. The tables in GHP scalar operators

If we identify the spinor $\mathbf{I}$ with the second dyad spinor $\boldsymbol{\iota}$ of the GHP formalism, then the above tables can all be translated into GHP formalism using (16)-(19):

$$\begin{aligned} þM &= 0 \\ \eth M &= \lambda \mathcal{P} M \\ \eth' M &= \lambda \overline{\mathcal{P}} M \\ þ' M &= \mathcal{Q} R M^{3/2} \end{aligned} \tag{52}$$

$$\begin{aligned} þB &= 0 \\ \eth B &= i\lambda \mathcal{P} M \\ \eth' B &= -i\lambda \overline{\mathcal{P}} M \\ þ' B &= \mathcal{Q}(G + RB) M^{1/2} \end{aligned} \tag{53}$$

$$\begin{aligned} þR &= 0 \\ \eth R &= 0 \\ \eth' R &= 0 \\ þ' R &= \mathcal{Q} Y M^{1/2} \end{aligned} \tag{54}$$

$$\begin{aligned} þN &= \frac{M^{3/2}}{\mathcal{Q}} \\ \eth N &= 0 \\ \eth' N &= 0 \\ þ' N &= \frac{\mathcal{Q} M^{1/2}}{2} U \end{aligned} \tag{55}$$

where

$$\begin{aligned} þG &= 0 = þY \\ \eth G &= 0 = \eth Y \\ \eth' G &= 0 = \eth' Y \end{aligned} \tag{56}$$

and

$$\begin{aligned} þU &= \frac{3M^{3/2}}{\mathcal{Q}} R \\ \eth U &= -2\mathcal{P} M (M + iB + 1/2\lambda) \\ \eth' U &= -2\overline{\mathcal{P}} M (M - iB + 1/2\lambda) \ . \end{aligned} \tag{57}$$

Before we can adopt the coordinate candidates as coordinates, we must confirm that they are functionally independent. Assuming that $M \neq 0$, we can easily confirm that B, R, N cannot be constant; an examination of the determinant formed from the four tables (52), (53), (54) and (55) shows that the four coordinate candidates are indeed functionally independent — providing $M \neq 0 \neq Y$. We are restricting ourselves to this generic case in this paper; the excluded special cases with $Y = 0$ and $M = 0$ can be found in [8].

4.4. Constructing the metric
We make the obvious choice of the coordinate candidates as the coordinates

$$r = R, \qquad n = N, \qquad m = M, \qquad b = B$$

We now write down the tetrad vector in these coordinates from the respective tables,

$$\begin{aligned} l^i &= \frac{1}{\mathcal{Q}}(0,\ m^{3/2},\ 0,\ 0) \\ m^i &= \mathcal{P}\Big(0,\ 0,\ \lambda m,\ i\lambda m\Big) \\ \overline{m}^i &= \overline{\mathcal{P}}\Big(0,\ 0,\ \lambda m,\ -i\lambda m\Big) \\ n^i &= \mathcal{Q}\Big(Ym^{1/2},\ Um^{1/2}/2,\ rm^{3/2},\ (rb+G)m^{1/2}\Big) \end{aligned} \tag{58}$$

where G, Y and U can be found from their respective tables, in the chosen coordinate system.

Trivially $Y = \nu_1(r)$ and $G = \nu_2(r)$ where $\nu_2(r)$ is a completely arbitrary function of r, whereas $\nu_1(r)$ is an arbitrary function of r excluding the zero function, since we are assuming $Y \neq 0$.

From (57), in this coordinate system, we can integrate to get

$$U = 3rn - \frac{b^2}{\lambda} - \frac{m^2}{\lambda} - \frac{m}{\lambda^2} + \nu_3(r) \tag{59}$$

where $\nu_3(r)$ is a completely arbitrary function of r.

It follows immediately from the equation

$$g^{ij} = 2l^{(i}n^{j)} - 2m^{(i}\bar{m}^{j)}$$

that the metric g^{ij}, in r, n, m, b coordinates, is given by

$$g^{ij} = m^2 \begin{pmatrix} 0 & \nu_1(r) & 0 & 0 \\ \nu_1(r) & U & mr & (rb+\nu_2(r)) \\ 0 & mr & -2\lambda^2 & 0 \\ 0 & (rb+\nu_2(r)) & 0 & -2\lambda^2 \end{pmatrix} \tag{60}$$

where U is given by (59).

5. Summary and Discussion
We have shown here how the method in [10] can be developed to investigate the more complicated situation where there is a non-zero cosmological constant; the additional special cases omitted here can be found in [8].

We have also used the GIF to construct the other subclass of spacetimes of this type — those with the condition $\Lambda \neq -\tau\bar{\tau}$; in this case there was no isotropy, but a richer Killing vector structure. The calculations were longer and some preliminary results were given in [9].

Although there have recently been a number of investigations of pure radiation spacetimes with non-zero cosmological constant for different Petrov types of Weyl tensors [1],[2],[21],[12],[13], these investigations generally seem to be built around a non-zero Weyl tensor, and it is not clear whether the whole class of conformally flat spaces are included as special cases. On the other hand, in this paper and in [8] we have investigated the spacetimes with a formalism which is directly suited to the class, and the explicit metrics found here are in simple form. The next task is to compare the whole class of these spacetimes found via GIF, with the conformally flat limits of these various other investigations.

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