Several types of one-sided partial orders in rings

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Abstract

In this paper, we investigate one-sided sharp partial orders and one-sided core and dual core partial orders in rings. Moreover, their characterizations and relations with other partial orders are given.

Keywords: group inverses, Moore-Penrose inverses, core inverses, one-sided sharp partial orders, one-sided core partial orders

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1 Introduction

There exist specific partial orders in the mathematical literature, such as star partial orders [3], minus partial orders [4], sharp partial orders [7], core and dual core partial orders [2], diamond partial orders [5] and so on. Recently, Marovt [6] and Mosić [8] introduced one-sided sharp partial orders and one-sided core partial orders in rings, respectively. Moreover, some properties of them were investigated.

In this article, we further study properties and characterizations of one-sided sharp partial orders and one-sided core and dual core partial orders in rings. The paper is organized as follows. In Section 2, notations and definitions are presented and several known results of generalized inverses and partial orders are given. In Section 3, one condition of Marovt's one-sided sharp partial orders is weakened. Also, we give the existence criterion such that b is group invertible if a is below b with respect to left sharp partial order. In Section 4, we present several characterizations of one-sided core partial orders and their relations with other types of partial orders.

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2 Preliminaries

Assume that R is a ring with unity 1. An element $a \in R$ is (von Neumann) regular if there exists an element $x \in R$ such that $a = axa$. Such an x is called an inner inverse of a , and is denoted by a^- . The symbol $a\{1\}$ denotes the set of all inner inverses of a .

We call $a \in R$ to be group invertible if there exists an element $b \in R$ such that

- (i) $ab = ba$,
- (ii) $bab = b$,
- (iii) $aba = a$.

A b satisfying the above conditions (i)-(iii) is called a group inverse of a . It is unique if it exists, and is denoted by $a^\#$. We denote by $R^\#$ the set of all group invertible elements in R .

In 1987, Mitra [7] introduced the sharp partial order of square matrices over a field. Let A and B be matrices of index 1. A is below B with respect to the sharp partial order (write $A \leq^\# B$) if $A^\#A = A^\#B$ and $AA^\# = BA^\#$. Recently, Marovt [6] introduced left and right sharp partial orders in a ring R , as extensions of the sharp partial order. Given $a \in R$, we define the sets ${}^0a = \{x \in R : xa = 0\}$ and $a^0 = \{x \in R : ax = 0\}$. Following [6], let $a, b \in R^\#$. Then (i) $a \nless b$ if and only if $a^\#a = a^\#b$ and ${}^0b \subseteq {}^0a$. (ii) $a \leq^\# b$ if and only if $aa^\# = ba^\#$ and $b^0 \subseteq a^0$.

A ring R is called a unital $*$ -ring if it is a ring with unity 1 and an involution $*$ satisfying $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. Let R be a unital $*$ -ring. We call $a \in R$ to be $\{1, 3\}$ -invertible if there exists some $x \in R$ satisfying equations $axa = a$ and $(ax)^* = ax$. Such an x is called a $\{1, 3\}$ -inverse of a , and is denoted by $a^{(1,3)}$. An element $a \in R$ is $\{1, 4\}$ -invertible if there exists some $y \in R$ such that $aya = a$ and $(ya)^* = ya$. Such a y is called a $\{1, 4\}$ -inverse of a , and is denoted by $a^{(1,4)}$. By $a\{1, 3\}$ and $a\{1, 4\}$ we denote the sets of all $\{1, 3\}$ -inverses and $\{1, 4\}$ -inverses of a , respectively. It is well known that $a \in R$ is Moore-Penrose invertible if and only if it is both $\{1, 3\}$ -invertible and $\{1, 4\}$ -invertible. The standard notion of the Moore-Penrose inverse can be found in [9]. The Moore-Penrose inverse of $a \in R$ is unique if it exists, and is denoted by $a^\dagger$. More results on Moore-Penrose inverses can be found in [13, 14, 15].

An element a in a unital $*$ -ring R is called core invertible [11] if there exists an $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$. Such an x is called the core inverse of a . Dually, a is called dual core invertible if there exists $y \in R$ such that $aya = a$, $Ry = Ra$ and $yR = a^*R$. Such a y is called the dual core inverse of a . The core inverse and the dual core inverse of a are unique, if they exist, and are denoted by $a^\oplus$ and $a_{\oplus}$, respectively. As

usual, by $R^\dagger$, $R^{(1,3)}$, $R^{(1,4)}$, $R^\oplus$ and $R_\oplus$ we denote the sets of all Moore-Penrose invertible, $\{1,3\}$ -invertible, $\{1,4\}$ -invertible, core invertible and dual core invertible elements in R , respectively. It is known ([12]) that $a \in R^\oplus$ if and only if $a \in R^\# \cap R^{(1,3)}$, and that $a \in R_\oplus$ if and only if $a \in R^\# \cap R^{(1,4)}$. In this case, $a^\oplus = a^\#aa^{(1,3)}$ and $a_\oplus = a^{(1,4)}aa^\#$. More properties about core inverses and dual core inverses can be seen [16].

Using core and dual core inverses, the core and dual core partial orders were given in [2]. Given core invertible complex matrices A, B , A is below B with respect to the core partial order (denoted by $A \leq^\oplus B$) if $A^\oplus A = A^\oplus B$ and $AA^\oplus = BA^\oplus$. If A and B are dual core invertible, A is below B with respect to the dual core partial order (denoted by $A \leq_\oplus B$) if $A_\oplus A = A_\oplus B$ and $AA_\oplus = BA_\oplus$. Later, Mosić [8] extended core and dual core partial orders to one-sided core and dual core partial orders in rings. Indeed, given $a \in R^\oplus$ then

- (i) a is below b under the left core partial order, denoted by $a \leq^{l,\oplus} b$, if $a^\oplus a = a^\oplus b$ and $aR \subseteq bR$,
- (ii) a is below b under the right core partial order, denoted by $a \leq^{r,\oplus} b$, if $aa^\oplus = ba^\oplus$ and $Ra \subseteq Rb$.

Given $a \in R_\oplus$ then

- (iii) a is below b under the left dual core partial order, denoted by $a_{l,\oplus} \leq b$, if $a_\oplus a = a_\oplus b$ and $aR \subseteq bR$,
- (iv) a is below b under the right dual core partial order, denoted by $a \leq_{r,\oplus} b$, if $aa_\oplus = ba_\oplus$ and $Ra \subseteq Rb$.

3 One-sided sharp partial orders

In [6] it is assumed $a, b \in R^\#$ in when considering one-sided sharp partial orders, and their properties and characterizations are presented. Inspired by [6], we further study one-sided sharp partial orders only by assuming $a \in R^\#$, and give an existence criterion of $b^\#$ provided that $a \sharp \leq b$.

Definition 3.1. Let $a, b \in R$ with $a \in R^\#$. We say that a is below b under left sharp partial order and write $a \sharp \leq b$ if $a^\#a = a^\#b$ and $aR \subseteq bR$.

The relation defined in Definition 3.1 above is a partial order. Clearly, $\sharp \leq$ is reflexive. Let $a, b \in R$ with $a \sharp \leq b$ and $b \sharp \leq a$. Then $bR \subseteq aR$ and hence $b = aa^\#b$. From $a^\#a = a^\#b$, we have $a = aa^\#a = aa^\#b = b$.

For the proof of transitivity, let $a \sharp \leq b$ and $b \sharp \leq c$. Then we have $a^\#a = a^\#b$ and $b^\#b = b^\#c$. Hence, $a^\#a = a^\#b = (a^\#)^2abb^\#b = (a^\#)^2(abb^\#)c = (a^\#)^2ac = a^\#c$ by the

condition (iii) $a = ab^\#b$ in Proposition 3.3 below.

Remark 3.2. Definition 3.1 above differs from Marovt's left sharp partial order in R . However, the difference vanishes when $b \in R^\#$. Indeed, the implication $Ra \subseteq Rb \Rightarrow b^0 \subseteq {}^0a$ is trivial. If $b \in R^\#$, then $b^0 \subseteq {}^0a$ implies $Ra \subseteq Rb$ by $b(1 - bb^\#) = 0$.

Proposition 3.3. *Let $a, b \in R$. Then*

- (i) *If $a \in R^\#$ with $a^\#a = a^\#b$ and $bR \subseteq aR$, then $a = b$.*
- (ii) *If $b \in R^\#$ with $aR \subseteq bR$, then $a = bb^\#a$.*
- (iii) *If $a \nless b$ and $b \in R^\#$, then $a = abb^\#$ and $aa^\# = ab^\#$.*

Proof. (i) Note that $bR \subseteq aR$ ensures that $b = ax$ for some $x \in R$. Then $b = aa^\#ax = aa^\#b$, which combines with $a = aa^\#a = aa^\#b$ giving $a = b$.

(ii) As $aR \subseteq bR$, there exists some $y \in R$ such that $a = by = bb^\#by = bb^\#a$ since $b \in R^\#$.

(iii) Since $a \nless b$, i.e., $a^\#a = a^\#b$ and $aR \subseteq bR$, it follows $a = aa^\#b$, and consequently $Ra \subseteq Rb$. This implies $a = abb^\#$ since $b \in R^\#$. Hence, $ab^\# = aa^\#ab^\# = aa^\#bb^\# = a^\#abb^\# = a^\#a = aa^\#$. $\square$

It follows from Proposition 3.3 (iii) above that if $a, b \in R^\#$ and $a \nless b$, then $a^\# = aa^\#b^\# = ab^\#a^\#$.

We next present an existence criterion of $b \in R^\#$, provided that $a \in R^\#$ and $a \nless b$. First, an auxiliary lemma is given.

Lemma 3.4. *Let $a, b \in R^\#$ and $a \nless b$. Then $a^\# = aa^\#b^\# = bb^\#a^\#$.*

Proof. The proof is essentially given in [6, Lemma 2.6 and Theorem 2]. $\square$

Theorem 3.5. *Let $a, b \in R$ with $a \in R^\#$ such that $a \nless b$. Then $b \in R^\#$ if and only if $c = b - baa^\# \in R^\#$. In this case, $b^\# = a^\# + c^\# - c^\#ba^\#$ and $c^\# = b^\# - b^\#aa^\#$.*

Proof. Suppose $b \in R^\#$. To show that $c = b - baa^\# \in R^\#$, it is sufficient to find an element $x \in R$ satisfying (i) $cx = xc$, (ii) $cxc = c$ and (iii) $xcx = x$. Indeed, $x = b^\# - b^\#aa^\#$ is the group inverse of c .

(i) Notice from Lemma 3.4 that $bb^\#aa^\# = aa^\#$ and $a^\# = aa^\#b^\#$. Then, we have $cx = (b - baa^\#)(b^\# - b^\#aa^\#) = bb^\# - aa^\#$. Again, it follows from $a^\#a = a^\#b$ that $xc = (b^\# - b^\#aa^\#)(b - baa^\#) = bb^\# - aa^\#$. Thus, $cx = xc$.

(ii) $cxc = (bb^\# - aa^\#)(b - baa^\#) = b - baa^\# - aa^\#b + aa^\#baa^\# = c - aa^\#a + aa^\#aaa^\# = c$.

(iii) $xcx = (b^\# - b^\#aa^\#)(bb^\# - aa^\#) = b^\# - b^\#aa^\# - b^\#aa^\#bb^\# + b^\#aa^\# = b^\# - b^\#aa^\#bb^\# = b^\# - b^\#ab^\# = b^\# - b^\#aa^\# = x$.

Therefore, $c = b - baa^\# \in R^\#$ and $c^\# = b^\# - b^\#aa^\#$.

Conversely, as $c = b - baa^\# \in R^\#$, then $ca^\# = a^\#c = 0$ by $a^\#a = a^\#b$. Also, it follows $c^\#a = ac^\# = 0$ since $c^\#a = (c^\#)^2ca^\#a^2$ and $ac^\# = a^2a^\#c(c^\#)^2$. Hence, $cc^\# = (b - baa^\#)c^\# = bc^\# - ba^\#ac^\# = bc^\#$. Let us now show that $y = a^\# + c^\# - c^\#ba^\#$ is the group inverse of b .

(i) $byb = b(a^\# + c^\# - c^\#ba^\#)b = ba^\#a + bc^\#(b - ba^\#b) = ba^\#a + bc^\#c = ba^\#a + cc^\#c = ba^\#a + c = b$ since $c = b - ba^\#a$.

(ii) $yb = (a^\# + c^\# - c^\#ba^\#)b = a^\#b + c^\#b - c^\#ba^\#b = a^\#a + c^\#b - c^\#ba^\#a = a^\#a + c^\#(b - ba^\#a) = a^\#a + c^\#c$. Again, from $aR \subseteq bR$ and $byb = b$, we conclude that $a = bya = ba^\#a - bc^\#ba^\#a$, and consequently $aa^\# = ba^\# - bc^\#ba^\#$. We get $by = ba^\# + bc^\# - bc^\#ba^\# = aa^\# + bc^\# = aa^\# + cc^\#$.

(iii) $yby = (a^\# + c^\# - c^\#ba^\#)(aa^\# + cc^\#) = a^\# + c^\# - c^\#ba^\# = y$ since $a^\#c = c^\#a = 0$.

Therefore, $b \in R^\#$ and $b^\# = a^\# + c^\# - c^\#ba^\#$. $\square$

Let us recall the fact [10, Proposition 7] that $b \in R^\#$ if and only if there exist some $s, t \in R$ such that $b = b^2s = tb^2$. Moreover, $b^\# = tbs$. Hence, there is an alternative proof for $b^\# = a^\# + c^\# - c^\#ba^\#$ in the “if” part of Theorem 3.5 above. Notice that the conditions $byb = b$ and $by = yb$ imply $b = b^2y = yb^2$, and consequently $b^\# = yby$. A direct calculation gives $b^\# = yby = (a^\#a + c^\#b - c^\#ba^\#a)(a^\# + c^\# - c^\#ba^\#) = a^\# + c^\# - c^\#ba^\#$.

Dually, we give the definition of the right sharp partial order.

Definition 3.6. Let $a, b \in R$ with $a \in R^\#$. We say that a is below b under the right sharp partial order and write $a \leqslant^\# b$ if $aa^\# = ba^\#$ and $Ra \subseteq Rb$.

The left sharp partial order may not be the right sharp partial order in general. A counterexample can be found in [6, page 8].

Let $a \in R^\#$. Then a is below b under the sharp partial order if $a^\#a = a^\#b$ and $aa^\# = ba^\#$. We remark that a is below b under the sharp partial order if and only if it is below b under both left and right sharp partial orders.

Marovt [6] has considered relations among the sharp partial order, the left and the right sharp partial orders, under the conditions $a, b \in R^\#$. Inspired by [6, Theorem 7], we present another characterization of $a \leqslant^\# b$, provided that $a \in R^\#$.

Theorem 3.7. Let $a \in R^\#$. Then the following conditions are equivalent:

- (i) $a \leqslant^\# b$.
- (ii) $a \leqslant b$ and $ab = ba$.
- (iii) $a \leqslant^\# b$ and $ab = ba$.

Proof. (i) $\Rightarrow$ (ii) As $a \leq^{\#} b$, then $a^{\#}b = a^{\#}a = aa^{\#} = ba^{\#}$. By the double commutativity of the group inverse, we get $(a^{\#})^{\#}b = b(a^{\#})^{\#}$, i.e., $ab = ba$. Notice that $a = aa^{\#}a = ba^{\#}a$. Then $aR = ba^{\#}aR \subseteq bR$, which together with $a^{\#}a = a^{\#}b$ implies $a \leq^{\#} b$.

(ii) $\Rightarrow$ (iii) From $ab = ba$, it follows $a^{\#}b = ba^{\#}$ and hence $aa^{\#} = a^{\#}a = a^{\#}b = ba^{\#}$. Since $a^{\#}a = a^{\#}b$, we obtain $a = aa^{\#}b$ and $Ra \subseteq Rb$.

(iii) $\Rightarrow$ (i) Note that $ab = ba$ ensures that $a^{\#}$ and b commute. Then the implication is straightforward. $\square$

4 One-sided core and dual core partial orders

Let us recall the conceptions of left and right star partial orders [1]. Given $a, b \in R$, we say that a is below b under the left star partial order ($a * \leq b$) if $a^*a = a^*b$ and $aR \subseteq bR$, and that a is below b under the right star partial order ($a \leq * b$) if $aa^* = ba^*$ and $Ra \subseteq Rb$. We next consider the relations among one-sided core partial orders, sharp partial orders and star partial orders.

Theorem 4.1. *Let $a \in R^{\oplus}$. Then*

- (i) $a^{l,\oplus} \leq b$ if and only if $a * \leq b$.
- (ii) $a \leq^{r,\oplus} b$ if and only if $a \leq^{\#} b$.

Proof. “ $\Rightarrow$ ” As $a^{l,\oplus} \leq b$, then $a^{\oplus}a = a^{\oplus}b$ and $aR \subseteq bR$. Multiplying $a^{\oplus}a = a^{\oplus}b$ by a on the left side gives $a = aa^{\oplus}b = (aa^{\oplus})^*b$, and consequently $a^*a = a^*(aa^{\oplus})^*b = (aa^{\oplus}a)^*b = a^*b$. So $a * \leq b$.

“ $\Leftarrow$ ” Since $a * \leq b$, we have $a^*a = a^*b$ and hence $a = (a^{\oplus})^*a^*a = (a^{\oplus})^*a^*b = aa^{\oplus}b$. Pre-multiplying $a = aa^{\oplus}b$ by $a^{\oplus}$ concludes $a^{\oplus}a = a^{\oplus}b$. It follows from $aR \subseteq bR$ that $a^{l,\oplus} \leq b$.

(ii) Suppose $a \leq^{r,\oplus} b$. Then $aa^{\oplus} = ba^{\oplus}$ and $Ra \subseteq Rb$. Multiplying $aa^{\oplus} = ba^{\oplus}$ by $aa^{\#}$ on the right side gives $aa^{\#} = ba^{\#}$, which together with $Ra \subseteq Rb$ guarantee $a \leq^{\#} b$.

Conversely, as $a \leq^{\#} b$, i.e., $aa^{\#} = ba^{\#}$ and $Ra \subseteq Rb$, then we get $aa^{\#}aa^{(1,3)} = ba^{\#}aa^{(1,3)}$, that is $aa^{\oplus} = ba^{\oplus}$ since $a^{\oplus} = a^{\#}aa^{(1,3)}$, as required. $\square$

Dually, we have the following result.

Theorem 4.2. *Let $a \in R_{\oplus}$. Then*

- (i) $a_{l,\oplus} \leq b$ if and only if $a \leq^{\#} b$.
- (ii) $a \leq^{r,\oplus} b$ if and only if $a \leq * b$.

In a $*$ -ring R , the condition $a^\oplus a = a^\oplus b$ implies $a = aa^\oplus b$, and consequently $Ra \subseteq Rb$. Also, $aa^\oplus = ba^\oplus$ gives $aR \subseteq bR$. Hence, $a \leq^\oplus b$ if and only if $a^{l,\oplus} \leq b$ and $a \leq^{r,\oplus} b$. We combine Theorem 4.1 and Theorem 4.2 to get the following corollary.

Corollary 4.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \leq^\oplus b$ if and only if $a \leq^\# b$ and $a * \leq b$.*

Recall that the minus partial order is defined in the following way. For $a, b \in R$, where a is regular, we write $a \leq^- b$ if $a^- a = a^- b$ and $aa^- = ba^-$ for $a^- \in a\{1\}$. A characterization of the minus partial order is that $a \leq^- b$ if and only if there exists $b^- \in b\{1\}$ such that $a = bb^- a = ab^- b = ab^- a$.

We next consider under what conditions one-sided core and dual core partial orders are equivalent to the minus partial order. First, we give the following lemma.

Lemma 4.4. *Let $a, b \in R$. Then*

- (i) *If $a, b \in R^\oplus$ and $a^{l,\oplus} \leq b$, then $a \leq^- b$.*
- (ii) *If $a, b \in R_\oplus$ and $a_{l,\oplus} \leq b$, then $a \leq^- b$.*
- (iii) *If $a, b \in R^\oplus$ and $a \leq^{r,\oplus} b$, then $a \leq^- b$.*
- (iv) *If $a, b \in R_\oplus$ and $a \leq_{r,\oplus} b$, then $a \leq^- b$.*

Proof. (i) As $a^{l,\oplus} \leq b$, i.e., $a^\oplus a = a^\oplus b$ and $aR \subseteq bR$, then we have $a = aa^\oplus a = aa^\oplus b$ and hence $Ra \subseteq Rb$, which gives $a = xb = xbb^\oplus b = ab^\oplus b$ for some $x \in R$. Note also that $aR \subseteq bR$ ensures $a = bb^\oplus a$. By a direct calculation, we have $ab^\oplus a = a(a^\oplus a)b^\oplus a = a(a^\oplus b)b^\oplus a = aa^\oplus bb^\oplus a = aa^\oplus a = a$. Therefore, $a = bb^\oplus a = ab^\oplus b = ab^\oplus a$ for $b^\oplus \in b\{1\}$. We hence conclude $a \leq^- b$.

(ii) Given $a_{l,\oplus} \leq b$, we get $a = aa_\oplus b$, which implies $Ra \subseteq Rb$. Also, we obtain $a = bb_\oplus a = ab_\oplus b$ from $aR \subseteq bR$ and $Ra \subseteq Rb$. As $b_\oplus$ is an inner inverse of a , indeed, $ab_\oplus a = aa_\oplus ab_\oplus a = aa_\oplus bb_\oplus a = aa_\oplus a = a$, then there exists $b_\oplus \in b\{1\}$ such that $a = bb_\oplus a = ab_\oplus b = ab_\oplus a$. Thus, $a \leq^- b$.

(iii) and (iv) can be obtained similarly. □

Theorem 4.5. *Let $a, b \in R$. Then we have*

- (i) *If $a, b \in R^\oplus$, then $a^{l,\oplus} \leq b$ if and only if $a \leq^- b$ and $Ra = Raa^\oplus b$.*
- (ii) *If $a, b \in R_\oplus$, then $a_{l,\oplus} \leq b$ if and only if $a \leq^- b$ and $Ra = Rab$.*
- (iii) *If $a, b \in R^\oplus$, then $a \leq^{r,\oplus} b$ if and only if $a \leq^- b$ and $aR = baR$.*
- (iv) *If $a, b \in R_\oplus$, then $a \leq_{r,\oplus} b$ if and only if $a \leq^- b$ and $aR = ba_\oplus aR$.*

Proof. Suppose $a^{l,\oplus} \leq b$. Then, by Lemma 4.4, $a \leq^- b$. Also, $a^\oplus a = a^\oplus b$ gives $a = aa^\oplus b$. Thus, $Ra = Raa^\oplus b$.

Conversely, as $a \leq^- b$, then $a = aa^-b = ba^-a$ for some $a^- \in a\{1\}$, and consequently $aR = ba^-aR \subseteq bR$. Next, we will prove that $a^\oplus a = a^\oplus b$. From $a(1 - a^-a) = 0$ and $Ra = Raa^\oplus b$, it follows $aa^\oplus b(1 - a^-a) = 0$. That is $aa^\oplus b = aa^\oplus ba^-a = aa^\oplus a = a$. Thus, $a^\oplus a = a^\oplus aa^\oplus b = a^\oplus b$, as required.

(ii) Suppose $a_{l,\oplus} \leq b$. Then $a \leq^- b$ by Lemma 4.4. We know that $a_\oplus a = a_\oplus b$ concludes $a = aa_\oplus b$, and consequently $a = aa^\# b$ since $aa_\oplus = aa^\#$. Then $Ra = Raa^\# b = Rab$ by noting $Ra = Raa^\#$.

For the converse, as $a \leq^- b$, then $a = aa^-b = ba^-a$ and $aR \subseteq bR$. Since $Ra = Rab$, we get $ab(1 - a^-a) = 0$. So, $ab = aba^-a = a^2$. Pre-multiplying $ab = a^2$ by $a^{(1,4)}a^\#$ implies $a_\oplus a = a_\oplus b$.

(iii) and (iv) can be proved by a similar way of (i) and (ii). $\square$

It follows from [6, Lemma 3.2] that if $a, b, a - b \in R^\#$ then $a \# \leq b$ if and only if $(b - a) \# \leq b$. It is natural to ask whether one-sided core partial orders have a similar property, i.e., whether $a^{l,\oplus} \leq b$ is equivalent to $(b - a)^{l,\oplus} \leq b$. Theorem 4.7 below illustrates the possibility, under certain conditions.

Lemma 4.6. *Let $a, b \in R$. Then*

- (i) $a * \leq b$ if and only if $(b - a) * \leq b$.
- (ii) $a \leq * b$ if and only if $(b - a) \leq * b$.

Proof. (i) Suppose $a * \leq b$. Then $a^*a = a^*b$ and $aR \subseteq bR$. It follows that $(b - a)^*a = 0$, and hence $(b - a)^*(b - a) = (b - a)^*b$. Also, $aR \subseteq bR$ implies $b - a \in bR$, and consequently $(b - a)R \subseteq bR$, as required.

For the converse, note that $(b - a) * \leq b$. Then $b - (b - a) = a * \leq b$

(ii) can be proved similarly. $\square$

Theorem 4.7. *Let $a, b - a \in R^\oplus$. Then*

- (i) $a^{l,\oplus} \leq b$ if and only if $(b - a)^{l,\oplus} \leq b$.
- (ii) $a \leq^{r,\oplus} b$ if and only if $(b - a) \leq^{r,\oplus} b$.

Proof. (i) It follows from Theorem 4.1 and Lemma 4.6 that $a^{l,\oplus} \leq b$ if and only if $a * \leq b$ if and only if $(b - a) * \leq b$. Again, by Theorem 4.1, we know that $(b - a)^{l,\oplus} \leq b$ if and only if $(b - a) * \leq b$. Hence, $a^{l,\oplus} \leq b$ if and only if $(b - a)^{l,\oplus} \leq b$.

(ii) can be proved similarly. $\square$

Analogously, we obtain similar properties of other one-sided core partial orders. The details are left to the reader.

It is known that $a \leq^{\oplus} b$ if and only if $a^{l,\oplus} \leq b$ and $a \leq^{r,\oplus} b$. Applying Theorem 4.7, we get the following result.

Corollary 4.8. *Let $a, b - a \in R^{\oplus}$. Then $a \leq^{\oplus} b$ if and only if $(b - a) \leq^{\oplus} b$.*

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