Curvature Collineations for Type-*N* Robinson-Trautman Space-Times

E. G. L. R. VAZ¹ and C. D. COLLINSON

Department of Applied Mathematics, University of Hull, Hull, England

Received September 15, 1982

Abstract

The metric of type-*N* Robinson-Trautman space-times is generated by a real function P satisfying certain field equations. Canonical forms for P are obtained under the assumption that at least one curvature collineation exists. In order to give an example of the improper subgroup structure of a group of curvature collineations all the curvature collineations are determined for the space-times corresponding to one of the two canonical forms.

§(1): *Introduction*

A vector field ξ is said to generate a curvature collineation [1] if along the integral curves of ξ the Riemann tensor remains invariant, i.e., if

$$\mathcal{L} R^\alpha_{\beta\gamma\delta} = 0 \quad (1)$$

where $\mathcal{L}$ denotes the Lie derivative along ξ . The set of linearly independent solutions $\xi_1, \dots, \xi_r$ to the above equation form an r -parameter invariance group. For empty space-times the Ricci tensor vanishes and equation (1) becomes

$$\mathcal{L} C^\alpha_{\beta\gamma\delta} = 0 \quad (2)$$

This is an integrability condition for the equation

$$\mathcal{L} g_{\alpha\beta} = \phi g_{\alpha\beta} \quad (3)$$

which defines a conformal motion (ϕ is a real function). Hence any field ξ satisfying (3) also satisfies (2): a conformal motion of an empty space-time is a spe-

¹Departamento de Matemática, Universidade do Minho, Braga, Portugal.

cial curvature collineation. Other special curvature collineations are homothetic and Killing motions, corresponding to $\phi = \text{const}$ and $\phi = 0$, respectively.

Following Ref. 2 a necessary condition for ξ to generate a curvature collineation of an empty space-time is given by

$$\begin{aligned} h_{\alpha\beta} &= \frac{1}{2}\phi g_{\alpha\beta}, & \text{if the space-time is not of Petrov type } N \\ h_{\alpha\beta} &= \frac{1}{2}\phi g_{\alpha\beta} + \alpha l_\alpha l_\beta, & \text{if the space-time is of Petrov type } N \\ (h_{\alpha\beta} &= \mathcal{L} g_{\alpha\beta}) \end{aligned} \quad (4)$$

Here ϕ and α are real functions and l is the unique (in direction) principal vector of the Weyl tensor. Therefore the only empty space-times that can admit a *proper* curvature collineation, i.e., a collineation which does not satisfy (3), are of Petrov type N . The curvature collineations for the plane-fronted gravitational waves have been found by Collinson [2] and McIntosh and Halford [3].² The present authors have shown, by explicit calculation, that type- N empty space-times with twisting geodesic rays do *not* admit a proper curvature collineation [4]. Hall [5] has also derived the result in an elegant manner, his method depending on the fact that the metric tensor of such space-times is uniquely determined (up to a constant scaling factor) by the curvature tensor [6]. Here the only other type- N empty space-times, namely, those having diverging hypersurface orthogonal geodesic rays (Robinson-Trautman space-times [7]) are considered.

With the coordinate functions $(x^1, x^2, x^3, x^4) \equiv (u, r, \xi, \bar{\xi})$ and the tetrad given in [8] and [9] the metric of the type- N Robinson-Trautman space-times can be written in terms of a real function P independent of r and satisfying the field equation

$$U^0 = -4P^2 \frac{\partial^2(\log P)}{\partial \xi \partial \bar{\xi}} \quad (5)$$

where U^0 is a real function of u alone. The metric is invariant under the transformation

$$\begin{aligned} u' &= s(u), & l'^\alpha &= \frac{ds}{du} l^\alpha \\ r' &= r \left[\frac{ds}{du}(u) \right]^{-1}, & n'^\alpha &= \left(\frac{ds}{du} \right)^{-1} n^\alpha \\ \xi' &= h(\xi), & m'^\alpha &= \frac{dh}{d\xi} \left| \frac{dh}{d\xi} \right|^{-1} m^\alpha \end{aligned} \quad (6)$$

Under this transformation

$$P' = P \left(\frac{ds}{du} \right)^{-1} \left| \frac{dh}{d\xi} \right| \quad \text{and} \quad U'^0 = U^0 \left(\frac{ds}{du} \right)^2$$

² Additional work has been done by Katzin and Levine [11] and Aichelburg [12].

In Section 2 the following is proved:

Theorem. The type- N Robinson–Trautman space-times which admit at least one curvature collineation are characterized by the existence of appropriate co-ordinate systems in which the function P takes either the canonical form

$$\log P(u, \zeta, \bar{\zeta}) = \zeta h_1(u) + \bar{\zeta} \bar{h}_1(u) \quad (7)$$

or the canonical form

$$P(u, \zeta, \bar{\zeta}) = e^{G(u)} \frac{ah(u + \zeta)\bar{h}(u + \bar{\zeta}) + \bar{b}h(u + \zeta) + b\bar{h}(u + \bar{\zeta}) + k}{[h'(u + \zeta)\bar{h}'(u + \bar{\zeta})]^{1/2}} \quad (8)$$

where a, b, k are constants, a and b being real, and $h' = dh/d(u + \zeta)$, etc. In both cases the empty space field equations are satisfied.

It is also shown in passing that

Corollary. A conformal motion of a Robinson–Trautman type- N space-time is necessarily homothetic.

This last result generalizes a result given in Ref. 10. In Section 3 the class of space times generated by P given by (7) is considered. *All* the curvature collineations are determined for such space-times together with the (improper) subgroups of Killing and homothetic motions. An example of the nonexistence of improper subgroups is given.

§(2): Canonical Forms for the Metric

For the null tetrad referred to in the previous section, equations (2) and (5) give, after some simplification,

$$D\xi_1 = 0 \quad (9a)$$

$$\delta\xi_3 = -2\alpha\xi_3 \quad (9b)$$

$$\delta\xi_1 = 0 \quad (9c)$$

$$\delta\xi_2 + \Delta\xi_3 = \bar{\nu}\xi_1 + \mu\xi_3 \quad (9d)$$

$$D\xi_2 + \Delta\xi_1 + \delta\bar{\xi}_3 + \bar{\delta}\xi_3 = 2(\mu + \gamma)\xi_1 - 2\rho\xi_2 + 2\alpha\xi_3 + 2\bar{\alpha}\bar{\xi}_3 \quad (9e)$$

$$D\xi_3 = -\rho\xi_3 \quad (9f)$$

$$\xi_1\Delta\psi_4 - \xi_3\bar{\delta}\psi_4 + [\Delta\xi_1 - D\xi_2 - 2\bar{\delta}\xi_3 + 2\xi_1(\mu + \gamma) - \rho\xi_2]\psi_4 = 0 \quad (9g)$$

together with equations determining ϕ and α

$$\alpha = 2(\Delta\xi_2 - \nu\xi_3 - \bar{\nu}\bar{\xi}_3) \quad (9h)$$

$$\phi = 2(D\xi_2 + \Delta\xi_1) \quad (9i)$$

Equations (9a-g) are now integrated to determine P under the assumption that at least one curvature collineation exists. The null tetrad components of the vector field ξ generating the collineation are also found. The last two equations then give the functions ϕ and ξ for such a collineation.

Equations (9a) and (9c) imply that ξ_1 is a function of r alone. From (9f) and (9b) the r dependence of ξ_3 is obtained:

$$\xi_3 = \frac{rF^0}{P} \quad (10)$$

where f^0 is an arbitrary function independent of r and $\bar{\zeta}$. The r dependence of ξ_2 is obtained from (9e) and used in (9g) to give

$$\xi_2 = -U^0 \xi_1 + g^0 r \quad (11)$$

where

$$g^0 = \dot{\xi}_1 + 2 \left(\frac{\partial F^0}{\partial \zeta} + \frac{\partial \bar{F}^0}{\partial \bar{\zeta}} \right) + \xi_1 \frac{\partial(\log P)}{\partial u} - 4F^0 \frac{\partial(\log P)}{\partial \zeta} - 4\bar{F}^0 \frac{\partial(\log P)}{\partial \bar{\zeta}} \quad (12)$$

and

$$\frac{\partial}{\partial u} (\xi_1 \psi_4^0) + 2\psi_4^0 \left[-\xi_1 \frac{\partial(\log P)}{\partial u} - 2 \frac{\partial F^0}{\partial \zeta} + 2F^0 \frac{\partial(\log P)}{\partial \zeta} \right] - 2F^0 \frac{\partial \psi_4^0}{\partial \zeta} = 0 \quad (13)$$

(the dot denotes the derivative with respect to u). Equation (9d) then gives that F^0 is a function of ζ alone and also the equation

$$\frac{\partial g^0}{\partial \bar{\zeta}} + \xi_1 \frac{\partial^2(\log P)}{\partial u \partial \bar{\zeta}} = 0 \quad (14)$$

To complete the integration we assume that the space-time admits at least one curvature collineation, ξ being the corresponding vector field, and then use the transformation leaving the form of the metric invariant to adapt the null tetrad to the assumed collineation. As ξ_2 and ξ_3 depend on F^0 and ξ_1 , these two functions have to be analyzed and several cases arise.

Case 1. If ξ_1 and F^0 vanish for all values of u and ζ , respectively, then from (10), (11), and (12) it follows that $\xi_1 = \xi_2 = \xi_3 = 0$, i.e., no curvature collineation exists.

Case 2. Suppose that F^0 vanishes for all values of ζ and $\exists u_1: \xi_1(u_1) \neq 0$. As ξ_1 has been assumed to be well behaved ξ_1 does not vanish in some neighborhood of u_1 . In such a neighborhood equations (12) and (14) yield $\partial^2(\log P)/\partial u \partial \bar{\zeta} = 0$ and the space-time is flat.

Case 3. Suppose that ξ_1 vanishes for all values of u and $\exists \zeta_1: F^0(\zeta_1) \neq 0$. Again F^0 is assumed to be well behaved so that it does not vanish in some neighborhood of ζ_1 . Using (6) in that neighborhood it is possible to set $F^0(\zeta) = i$, the

freedom left in the transformation being

$$\begin{aligned} u' &= s(u), \quad r' = r \left[\frac{ds}{du}(u) \right]^{-1}, \quad \zeta' = \zeta + \text{const} \\ l'^\alpha &= \frac{ds}{du} l^\alpha, \quad n'^\alpha = \left(\frac{ds}{du} \right)^{-1} n^\alpha, \quad m'^\alpha = m^\alpha \end{aligned} \quad (15)$$

with $P' = P(ds/du)^{-1}$ and $U^{0'} = U^0(ds/du)^2$. As g^0 is real, (14) gives that g^0 is a function of u alone. Integrating (12) with $\xi_1 = 0$ and $F^0 = i$ gives

$$\log P(u, \zeta, \bar{\zeta}) = (\zeta - \bar{\zeta}) G(u) + \log f(u, \zeta + \bar{\zeta}) \quad (16)$$

where $G(u) = -g^0(u)/4i$. The field equation (6) becomes

$$e^{-2(\zeta - \bar{\zeta})G} U^0 = -4f^2 \frac{\partial^2(\log f)}{\partial(\zeta + \bar{\zeta})^2} \quad (17)$$

As U^0 and G are functions of u only, this equation implies that either $U^0(u) = 0$ or $U^0(u) \neq 0$ and $G(u) = 0$.

Suppose that $U^0(u) = 0$. Then, from (17), $\partial^2(\log f)/\partial(\zeta + \bar{\zeta})^2 = 0$ and $\log f(u, \zeta, \bar{\zeta}) = (\zeta + \bar{\zeta})f_1(u) + f_2(u)$, where f_1 and f_2 are real functions of u . Substituting into (2) yields

$$\log P(u, \zeta, \bar{\zeta}) = \zeta h_1(u) + \bar{\zeta} \bar{h}_1(u) \quad (18)$$

the transformation (15) having been used to set $f_2(u) = 0$. The freedom left in the transformation is now $u' = u + \text{const}$, $\zeta' = \zeta + \text{const}$. With this form for P , it is found that $\psi_4^0 = -8P^2 h_1 \bar{h}_1$. Therefore the condition for the space time to be nonflat is $\dot{h}_1(u) \neq 0$. With the previous results equation (13) is found to be identically satisfied so that (18) gives a canonical form for the function P generating the metric of type- N Robinson-Trautman space-time admitting at least one curvature collineation. It is important to stress that P satisfies the empty space field equations. The null tetrad components of the vector field ξ are obtained from (10), (11), and (12), are given by

$$\xi_1 = 0, \quad \xi_2 = 4i(\bar{h}_1 - h_1)r, \quad \xi_3 = ir/P$$

Equations (9h) and (9i) then give

$$\phi = 8i(\bar{h}_1 - h_1) \quad \text{and} \quad \alpha = 4i(\dot{\bar{h}}_1 - \dot{h}_1)r \quad (19)$$

The condition for α to be nonzero, i.e., for the collineation to be a proper one, is $\dot{\bar{h}}_1(u) - \dot{h}_1(u) \neq 0$.

Suppose that $U^0(u) \neq 0$. Then $G(u) = 0$ and from (16) the coordinate $\zeta - \bar{\zeta}$ is cyclic. Therefore ξ is a Killing vector, i.e., no proper collineation exists.

Case 4. Suppose that $\xi_1(u_1) \neq 0$ and $F^0(\xi_1) \neq 0$ for some value u_1 and ξ_1 of u and ζ , respectively. As in the previous cases it is possible to find neighborhoods of u_1 and ξ_1 in which ξ_1 and F^0 do not vanish. Considering the points

with coordinates u and ξ in such neighborhoods the transformation (7) can be used to set $\xi_1(u) = 1$ and $F^0(\xi) = \frac{1}{2}$. The freedom left in the transformation is $u' = u + \text{const}$, $\xi' = \xi + \text{const}$. Equations (12) and (14) yield

$$\frac{\partial^2(\log P)}{\partial u \partial \bar{\xi}} - \frac{\partial^2(\log P)}{\partial \xi \partial \bar{\xi}} - \frac{\partial^2(\log P)}{\partial \bar{\xi}^2} = 0$$

The general form for the function P satisfying this equation is

$$P(u, \xi, \bar{\xi}) = e^{G(u)} f(u + \xi, u + \bar{\xi})$$

with G and f real functions. The field equation (6) is satisfied provided that

$$f^2 \frac{\partial^2(\log f)}{\partial(u + \xi) \partial(u + \bar{\xi})} = k$$

where k is a real constant. Therefore f is given by

$$f(u + \xi, u + \bar{\xi}) = \frac{ah(u + \xi)\bar{h}(u + \bar{\xi}) + \bar{b}h(u + \xi) + b\bar{h}(u + \bar{\xi}) + k}{[h'(u + \xi)\bar{h}'(u + \bar{\xi})]^{1/2}}$$

With the previous results the remaining equation (13) is found to be identically satisfied, $k \neq 0$ being the condition for the space-time to be nonflat. Therefore the function P given by

$$P(u, \xi, \bar{\xi}) = e^{G(u)} \frac{ah(u + \xi)\bar{h}(u + \bar{\xi}) + \bar{b}h(u + \xi) + b\bar{h}(u + \bar{\xi}) + k}{|h'(u + \xi)\bar{h}'(u + \bar{\xi})|^{1/2}}$$

generates a subclass of type- N Robinson-Trautman space-times admitting at least one curvature collineation. The vector field ξ associated with the collineation has null tetrad components given by

$$\xi_1 = 1$$

$$\xi_2 = 4ke^{2G} + \dot{G} - \frac{a\bar{h}h' + ah\bar{h}' + \bar{b}h' + b\bar{h}'}{ah\bar{h} + \bar{b}h + b\bar{h} + k} + \frac{1}{2} \frac{h''}{h'} + \frac{1}{2} \frac{\bar{h}''}{\bar{h}'}$$

$$\xi_3 = \frac{r}{2P}$$

The functions ϕ and α are obtained from (9h) and (9i). After some simplification they become

$$\phi = 4\dot{G} \quad \text{and} \quad \alpha = 2G''r \quad (20)$$

Therefore, provided that $\dot{G}$ is not a constant function, the collineation is a proper one.

Under the assumption that one curvature collineation exists, from (19) and (20) we conclude that if $\alpha = 0$ then ϕ is necessarily constant. If more than one

collineation exists then the tetrad can be adapted arbitrarily to any of the collineations so that for the space-times being considered a conformal motion is necessarily homothetic, as stated in the corollary in Section 1.

§(3): *The Group of Curvature Collineations*

We now consider the subclass of space-times characterized by the existence of a coordinate system in which the function P generating the metric takes the form given in (18), namely,

$$\log P(u, \zeta, \bar{\zeta}) = \zeta h_1(u) + \bar{\zeta} \bar{h}_1(u)$$

Solving equations (9a-9g) with P given above all the collineations admitted by such space-times are obtained. The results of the previous sections are used to determine the null tetrad components of the curvature collineation vectors in the form

$$\xi_1 = 2a_1 h_1 |\dot{h}_1| \quad (21)$$

$$\xi_2 = \{\dot{\xi}_1 + \xi_1(\zeta \dot{h}_1 + \bar{\zeta} \dot{\bar{h}}_1) - 4\bar{F}^0 \bar{h}_1 - 4F^0 h_1 + 2F^{0'} + 2\bar{F}^{0'}\} r \quad (22)$$

$$\xi_3 = r F^0 |P| \quad (23)$$

where

$$F^0(\zeta) = a_1 \zeta + a_2 \quad (24)$$

Here a_1 and a_2 are complex constants of integration and a_1 satisfies the condition

$$\bar{a}_1 = \frac{h_1 \dot{\bar{h}}_1}{\dot{h}_1 \bar{h}_1} a_1 \quad (25)$$

This condition implies that either $h_1 \dot{\bar{h}}_1 / \dot{h}_1 \bar{h}_1$ is constant or $a_1 = 0$. The two corresponding cases are now considered separately.

Case I. Suppose that

$$\frac{h_1 \dot{\bar{h}}_1}{\dot{h}_1 \bar{h}_1} = e^{2ik} \quad (26)$$

where k is a real constant. Then, from (25) it follows that

$$\begin{aligned} a_1 &= r_1 e^{-ik} \\ h_1 |\dot{h}_1| &= e^{ik} R \end{aligned} \quad (27)$$

where r_1 is a real constant and R is a real function of u alone. Substituting these results into (21)-(24) three linearly independent curvature collineation vectors

generated by the real constants

$$r_1, \quad r_2 = \frac{1}{2}(a_2 + \bar{a}_2), \quad r_3 = \frac{i}{2}(\bar{a}_2 - a_2) \quad (28)$$

are obtained, namely,

$$\xi_1 = 2R, \quad \xi_2 = r[-2e^{-ik}h_1\dot{\xi} - 2e^{ik}\bar{h}_1\dot{\bar{\xi}} + 2\dot{R} + 2(e^{ik} + e^{-ik})], \quad \xi_3 = \frac{re^{-ik}\xi}{P}$$

$$\xi_1 = 0, \quad \xi_2 = 4i(\bar{h}_1 - h_1)r, \quad \xi_3 = \frac{ri}{P}$$

$$\xi_1 = 0, \quad \xi_2 = -4(h_1 + \bar{h}_1)r, \quad \xi_3 = \frac{r}{P}$$

From equations (9h, i) we then obtain

$$\phi = 4r_1(2\dot{R} + e^{ik} + e^{-ik}) - 8a_2h_1 - 8\bar{a}_2\bar{h}_1 \quad (29)$$

$$\alpha = 4(r_1\ddot{R} - a_2\dot{h}_1 - \bar{a}_2\dot{\bar{h}}_1)r \quad (30)$$

Case II. Suppose that $h_1\dot{\bar{h}}_1/\bar{h}_1\dot{h}_1$ is not constant so that a_1 must be zero. From (21)–(24) it follows that

$$\xi_1 = 0, \quad \xi_2 = -4r(a_2h_1 + \bar{a}_2\bar{h}_1), \quad \xi_3 = a_2r|P$$

Two curvature collineations are generated by the complex constant a_2 , the corresponding vectors being ξ and $\bar{\xi}$ of case I. From equations (9h, i) we then obtain

$$\phi = -8a_2h_1 - 8\bar{a}_2\bar{h}_1 \quad (31)$$

$$\alpha = -4(a_2\dot{h}_1 + \bar{a}_2\dot{\bar{h}}_1)r \quad (32)$$

§(4): *Improper Subgroups*

The existence of improper subgroups of the group of curvature collineations referred to in the previous section depends on the functions ϕ and α , the conditions for a homothetic motion and for a Killing motion being $\alpha = 0$ and $\phi = 0$, respectively. The two cases considered above are now analyzed separately.

Case I. From (29) and (30) it follows that the conditions for the existence of homothetic and Killing motions are

$$r_1\ddot{R} - a_2\dot{h}_1 - \bar{a}_2\dot{\bar{h}}_1 = 0 \quad (33)$$

and

$$r_1(2\dot{R} + e^{ik} + e^{-ik}) - 2a_2h_1 - 2\bar{a}_2\bar{h}_1 = 0 \quad (34)$$

respectively. The functions h_1 and R for which these equations can be satisfied determine those space-times admitting improper subgroups of the group of curvature collineation. Two cases arise when (33) is solved:

(A) Suppose that $\ddot{R}(u) = 0$ so that $R(u) = k_1 u + k_2$ with k_1 and k_2 arbitrary real constants. Then, from (33), $a_2 = \bar{a}_2 \dot{h}_1 / \dot{h}_1$.

(A-1) Suppose that $(d/du)(\dot{h}_1/\dot{h}_1) \neq 0$ in which case it follows from (28) that we must have $k \neq 0$. Then $a_2 = 0$ and ξ is found to generate a one-parameter subgroup of homothetic motions. The additional condition $k_1 = -\frac{1}{2}(e^{ik} + e^{-ik})$ is the condition for ξ to generate a subgroup of Killing motions, as obtained using equation (34).

(A-2) Suppose that $(d/du)(\dot{h}_1/\dot{h}_1) = 0$, in which case $k = 0$. An integration with respect to u gives $\dot{h}_1/\dot{h}_1 = e^{2ik_3}$ (k_3 is an arbitrary real constant). Then $a_2 = e^{2ik_3} \bar{a}_2$ and the vectors ξ and $\xi \cos k_3 - \xi \sin k_3$ are found to generate a two-parameter subgroup of homothetic motions. From (34) it follows that if $k_1 = -1$ those two vectors generate a subgroup of Killing motions and that more generally $\xi \cos k_3 - \xi \sin k_3$ generates a one-parameter subgroup of Killing motions.³

(B) We now assume that $\ddot{R}(u) \neq 0$. From equation (33) we obtain

$$a_2 \frac{d}{du} \left(\frac{\dot{h}_1}{\ddot{R}} \right) + \bar{a}_2 \frac{d}{du} \left(\frac{\dot{\bar{h}}_1}{\ddot{R}} \right) = 0 \quad (35)$$

after dividing by $\ddot{R}(u)$ and differentiating with respect to u .

(B-1) Suppose that $(d/du)(\dot{h}_1/\ddot{R}) = 0$, in which case $k = 0$. Integrating twice gives $\dot{R} = k_1 + h_1/c_1$, with c_1 a complex constant of integration and k_1 a real constant of integration. Then $r_1 = c_1 a_2 + \bar{c}_1 \bar{a}_2$ and the vectors $i(c_1 - \bar{c}_1)\xi + \xi$ and $(c_1 + \bar{c}_1)\xi + \xi$ are found to generate a two parameter subgroup of homothetic motions. From (34) we can conclude that if $k_1 = -1$ those two vectors generate a two-parameter subgroup of Killing motions and that more generally $\frac{1}{2}(\bar{c}_1 - c_1)\xi + \frac{1}{2}(c_1 + \bar{c}_1)\xi$ generates a one-parameter subgroup of Killing motions.

(B-2) Suppose that $(d/du)(\dot{h}_1/\ddot{R}) \neq 0$. Then, from (35)

$$a_2 = - \frac{(d/du)(\dot{h}_1/\ddot{R})}{(d/du)(\dot{h}_1/\ddot{R})} \bar{a}_2 \quad (36)$$

Two cases can be considered separately.

³ Any r -parameter group of homothetic motions must admit an $r - 1$ parameter subgroup of Killing motions.

(B-2-i) Suppose that

$$\frac{d}{du} \left[\frac{(d/du)(\dot{\bar{h}}_1/\ddot{R})}{(d/du)(\dot{h}_1/\ddot{R})} \right] = 0$$

Then

$$\frac{(d/du)(\dot{\bar{h}}_1/\ddot{R})}{(d/du)(\dot{h}_1/\ddot{R})} = e^{2ik_3}$$

where k_3 is a real constant, and $a_2 = ir_4 e^{ik_3}$ (r_4 is an arbitrary real constant). Integrating with respect to u twice yields

$$ik_4 \dot{R} = e^{-ik_3} \bar{h}_1 - e^{ik_3} h_1 + ik_5$$

where k_4 and k_5 are real constants of integration. Then $r_1 = k_4 r_4$ and the vector $-k_4 \xi + \xi \cos k_3 - \xi \sin k_3$ is found to generate a one-parameter subgroup of homothetic motions. From (34) we obtain that $k_5 = -\frac{1}{2} k_4 (e^{ik} + e^{-ik})$ is the condition for this vector to generate a one-parameter subgroup of Killing motions.

(B-2-ii) Finally, if

$$\frac{d}{du} \left[\frac{(d/du)(\dot{\bar{h}}_1/\ddot{R})}{(d/du)(\dot{h}_1/\ddot{R})} \right] \neq 0$$

from (36) it follows that $a_2 = \bar{a}_2 = 0$. This case is of interest in that no improper subgroup of the three-parameter group of curvature collineations exists.

Case II. From (31) and (32) it follows that, in this case, the equations giving the conditions for the existence of homothetic and Killing motions are

$$a_2 \dot{h}_1 + \bar{a}_2 \dot{\bar{h}}_1 = 0 \quad (37)$$

and

$$a_2 h_1 + \bar{a}_2 \bar{h}_1 = 0 \quad (38)$$

respectively. The first equation is used to write

$$a_2 = -\bar{a}_2 \dot{\bar{h}}_1 / \dot{h}_1 \quad (39)$$

Two cases can now arise.

(A) Suppose that $(d/du)(\dot{\bar{h}}_1/\dot{h}_1) \neq 0$. Then, from (39), $a_2 = \bar{a}_2 = 0$ and the group of curvature collineations does not admit any improper subgroup.

(B) Suppose that $(d/du)(\dot{\bar{h}}_1/\dot{h}_1) = 0$ so that $\dot{\bar{h}}_1/\dot{h}_1 = e^{2ik_1}$, k_1 being an arbitrary real constant. Substituting into (39) yields $a_2 = ir_4 e^{ik_1}$, with r_4 an arbitrary real constant. The real constants r_2 and r_3 , as defined in the previous section, become $r_2 = r_4 \cos k_1$ and $r_3 = -r_4 \sin k_1$. The vector $\xi \cos k_1 - \xi \sin k_1$ in general generates a one-parameter subgroup of homothetic motions. From these results,

together with (38), it is found that in this case no improper subgroup of Killing motions can exist.

References

1. Katzin, G. H., Levine, J., and Davis, W. R. (1969). *J. Math. Phys.*, **10**, 617.
2. Collinson, C. D. (1970). *J. Math. Phys.*, **11**, 818.
3. McIntosh, C. B. G., and Halford, W. D. (1981). *J. Phys. A*, **14**, 2331.
4. Vaz, E. G. L. R. (1982). Ph.D. thesis, University of Hull.
5. Hall, G. (1982). Preprint.
6. Collinson, C. D., and Vaz, E. G. L. R. (1982). *Gen. Rel. Grav.*, **14**, 5.
7. Robinson, I., and Trautman, A. (1962). *Proc. R. Soc. (London) Ser. A*, **265**, 463.
8. Newman, E. T., and Unti, T. W. J. (1962). *J. Math. Phys.*, **3**, 891.
9. Newman, E. T., and Tamburino, L. A. (1962). *J. Math. Phys.*, **3**, 902.
10. Collinson, C. D., and French, D. C. (1967). *J. Math. Phys.*, **8**, 701.
11. Katzin, G. H., and Levine, J. (1970). *J. Math. Phys.*, **11**, 1578.
12. Aichelburg, P. C. (1970). *J. Math. Phys.*, **11**, 2548.