On generalized Vietoris' number sequences

Isabel Cação^{a,b}, M. Irene Falcão^{c,d}, Helmuth R. Malonek^{a,b,*}

^a*CIDMA - Center for Research and Development in Mathematics and Applications,
University of Aveiro, Portugal*

^b*Department of Mathematics, University of Aveiro, Portugal*

^c*CMAT - Centre of Mathematics, University of Minho, Portugal*

^d*Department of Mathematics and Applications, University of Minho, Portugal*

Abstract

Recently, by using methods of hypercomplex function theory, the authors have shown that a certain sequence $\mathcal{S}$ of rational numbers (Vietoris' sequence) combines seemingly disperse subjects in real, complex and hypercomplex analysis. This sequence appeared for the first time in a theorem by Vietoris (1958) with important applications in harmonic analysis (Askey/Steinig, 1974) and in the theory of stable holomorphic functions (Ruscheweyh/Salinas, 2004). A non-standard application of Clifford algebra tools for defining Clifford-holomorphic sequences of Appell polynomials was the hypercomplex context in which a one-parametric generalization $\mathcal{S}(n)$, $n \geq 1$, of $\mathcal{S}$ (corresponding to $n = 2$) surprisingly showed up. Without relying on hypercomplex methods this paper demonstrates how purely real methods also lead to $\mathcal{S}(n)$. For arbitrary $n \geq 1$ the generating function is determined and for $n = 2$ a particular case of a recurrence relation similar to that known for Catalan numbers is proved.

Dedicated to Domingos Cardoso on the occasion of his 65th birthday

1. Introduction

The motivation for this notes about a curious sequence $\mathcal{S}(n)$, $n \geq 1$, of rational numbers comes from a casual observation about its significant role for $n = 2$ in problems far away from those that were the target of our interest

*Corresponding author.

Email addresses: `isabel.cacao@ua.pt` (Isabel Cação), `mif@math.uminho.pt` (M. Irene Falcão), `hmalon@ua.pt` (Helmuth R. Malonek)

some time ago. It is a sequence that can be considered on the crossroad of positivity of trigonometric sums (see [2, 3, 4, 33]), stable behavior of some classes of holomorphic functions (see [31]), and a set of Appell polynomials in several hypercomplex variables (see e.g. [12, 15, 16, 20, 28]).

To be briefly, but a little bit more detailed, our main research was concerned with the application of function theoretic methods for solving partial differential equations in more than two variables by using non-commutative Clifford algebras. As such it is part of quaternionic analysis ([23]) or, more general, Clifford analysis ([18, 24, 25]) often simply named hypercomplex function theory. But naturally, analytical or, more concrete, numerical methods have to deal with the peculiarities of non-commutativity. For example, to find a, in some sense, universal set of suitable polynomials easy to handle for the approximation of solutions of PDEs (and, at least, being orthogonal) was already not a trivial task (see [6, 8, 13, 14, 26]). This even more if one would like it *without* borrowing in some sense tools from real higher dimensional analysis.

Therefore we introduced ([20, 28]) Clifford-holomorphic sequences of Appell polynomials ([1]) in n hypercomplex variables in the underlying real Euclidean space $\mathbb{R}^{n+1}$, $n \geq 1$. This process resembled the way of using the algebra of complex numbers in $\mathbb{R}^2$ -problems. Together with the embedding of the binary non-commutative multiplication into an n -ary symmetric product we were able to express hypercomplex (Clifford-) holomorphic polynomials by corresponding hypercomplex symmetric polynomials.

Noticing that a sequence $\mathcal{S}(n)$, $n \geq 1$, of rational numbers depending on the parameter n (the hypercomplex dimension in which we are working in) was dominating those Appell polynomials caused our curiosity. To our surprise, we found that in the case $n = 2$, $\mathcal{S}(2) = \mathcal{S}$ appeared already in 1958 in a theorem by Vietoris ([33]). Obviously, we saw that therefore $\mathcal{S}(n)$ for arbitrary $n \geq 2$ could be considered as a generalization of Vietoris' number sequence $\mathcal{S}$. The case $n = 1$ corresponding to the complex case $\mathbb{C} \cong \mathbb{R}^2$ in hypercomplex function theory fits naturally in the representation of $\mathcal{S}(n)$ and leads to the constant sequence of coefficients in the ordinary geometric series.

Due to the fact that generalized binomial coefficients appearing in $\mathcal{S}(n)$ play a central role in enumerative or other branches of combinatorics, we were challenged to determine its generating function as well as a recurrence relation for the elements of the ordinary $\mathcal{S}$. The latter was also motivated by the fact that Vietoris' numbers though being not integers and appearing in pairs have some similarity with the famous Catalan numbers and can be verbally described by properties of the ordinary Pascal triangle.

As mentioned in the Abstract, here we try to avoid any detail about the hypercomplex background of our discovery of $\mathcal{S}(n)$. Therefore the interested reader should consult [27]. Instead of this, we found a way to use exclusively real methods and at some point in this way we came close (but not identically) to a relation for the elements of $\mathcal{S}(2)$ in terms of *Jacobi polynomials* as observed in [2, 3, 4]. The extension from $n = 2$ to $n \geq 2$ in our real approach is just realized at this point.

The paper is structured in the following way. Starting in Section 2 from some reminiscence of Vietoris' theorem from 1958 and using the generating function of $\mathcal{S}$ already directly obtained in [15] we arrive at the end to the compact representation of the elements of $\mathcal{S}(n)$ in form of a quotient of two Pochhammer symbols. This leads immediately to the determination of the corresponding generating function in Section 3. The next section includes the proof of a recurrence relation for Vietoris' numbers in a form that resembles the well known main recurrence relation for Catalan numbers. By using forward differences we show also the effect of their pairwise appearance. The paper ends with some concluding remarks.

2. Generalizing Vietoris' sequence

In the center of our attention lies the sequence of rational numbers

$$1, \frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{35}{128}, \frac{35}{128}, \frac{63}{256}, \frac{63}{256}, \frac{231}{1024}, \frac{231}{1024}, \dots \quad (1)$$

which by means of the *generalized central binomial coefficient* $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ can be written in compact form (cf. [12]) as $\mathcal{S} = (c_k)_{k \geq 0}$, where

$$c_k = \frac{1}{2^k} \binom{k}{\lfloor \frac{k}{2} \rfloor} = \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{(1)_{\lfloor \frac{k+1}{2} \rfloor}}. \quad (2)$$

Here, as usual, $\lfloor \cdot \rfloor$ denotes the floor function and $(\cdot)_k$ is the raising factorial in the classical form of the Pochhammer symbol.

Seemingly this sequence appeared, for the first time, in the context of positive trigonometric sums in a celebrated paper of L. Vietoris [33]. Askey's version [3, p. 5] of Vietoris' theorem is the following:

Theorem 1 (L. Vietoris).

$$\sum_{k=1}^n a_k \sin k\theta > 0, \quad 0 < \theta < \pi, \quad \text{and} \quad \sum_{k=0}^n a_k \cos k\theta > 0, \quad 0 \leq \theta < \pi,$$

where

$$a_{2k} = a_{2k+1} = \frac{(\frac{1}{2})_k}{k!}, \quad k = 0, 1, \dots \quad (3)$$

We call attention to the fact that because of (3), the coefficients in the *sine* sum used in Askey's as well as in Vietoris' original version are exactly the elements of $\mathcal{S}$ in (2) or, explicitly, in (1). Obviously, demanding in (3) that a_{2k} and a_{2k+1} coincide, the sequence of coefficients in the *cosine* sum differs from (1) by the inclusion of $a_0 = 1$ and the shift of the indexes by one to the left, i.e. $a_0 = 1$ and $a_{k+1} = c_k$, $k \geq 0$. Even though this small difference, we call $\mathcal{S}$ in the sequel simply *Vietoris' number sequence*. Compared with the traditional way of defining the coefficient sequence by (3), the use of the properties of the *generalized central binomial coefficient* allows a unique representation (2) with consecutively running index k .

Remark 1. In *The On Line Encyclopedia of Integer Sequences* one can find the *Minimal Exponent Integer Sequence* A283208 associated with *Vietoris sequence* as a sequence of integers related to the divisibility of the central binomial coefficients, (<http://oeis.org/A283208>). For more details cf. [15].

In [4, p. 67], the author noticed that the *sine* series in Vietoris' theorem is related to the expansion of

$$\frac{1}{(1-t)^\gamma(1+t)^\delta} \quad (4)$$

by Jacobi polynomials $P_k^{(\alpha, \beta)}$ in the form

$$\frac{1}{(1-t)^\gamma(1+t)^\delta} = \sum_{k=0}^{\infty} a_k P_k^{(\alpha, \beta)}(t), \quad (5)$$

with values

$$\alpha = \beta = \frac{1}{2}, \quad \gamma = \frac{3}{4}, \quad \delta = \frac{1}{4}.$$

The generating function $F(t)$ of $\mathcal{S}$ obtained in [15] by the hypercomplex approach allows now to establish a different expansion of a rational function of type (4) with c_k , $k = 0, \dots$, as coefficients.

Therefore we recall the following theorem about a generating function of $\mathcal{S}$ which was proved in [15, p. 1069] by a direct elementary procedure.

Theorem 2. *Vietoris' number sequence $\mathcal{S} = (c_k)_{k \geq 0}$ is generated by the function*

$$F(t) = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}}, \quad 0 < |t| < 1. \quad (6)$$

From (6) it follows that

$$tF(t) = \frac{\sqrt{1+t} - \sqrt{1-t}}{\sqrt{1-t}} = \sum_{k=0}^{\infty} c_k t^{k+1}. \quad (7)$$

Observe that a simple derivation of (7), i.e. the determination of $(tF(t))'$ results in the expansion of a rational function of the same type as in (4), namely

$$\frac{1}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}} = \sum_{k=0}^{\infty} (k+1)c_k t^k, \quad (8)$$

but now with exponents

$$\gamma = \frac{3}{2} \text{ resp. } \delta = \frac{1}{2}$$

and without relying on Jacobi polynomials.

Comparing both rational functions in (5) and (8) more carefully we see that in (8) appears the square of the rational function in (5) since

$$\gamma + \delta = \frac{3}{2} + \frac{1}{2} = 2 \text{ while in Askey's case } \frac{3}{4} + \frac{1}{4} = 1. \quad (9)$$

Analogously, the differences of both types of exponents are

$$\gamma - \delta = \frac{3}{2} - \frac{1}{2} = 1 \text{ while in Askey's case } \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \quad (10)$$

respectively.

This is the point, where the hypercomplex approach to generalized Appell polynomials in $\mathbb{R}^{n+1}$ plays a decisive role and offers an idea for a generalization of Vietoris' sequence from $\mathcal{S} = \mathcal{S}(\mathbf{2})$ in $\mathbb{R}^{2+1}$ to $\mathcal{S}(\mathbf{n})$ in $\mathbb{R}^{n+1}$ (see [15]).

Taking into account that the coefficients c_k in (2), as explained before, are associated with the dimension $n = 2$, it seems plausible to look for a generalization $c_k(n)$ of $c_k = c_k(2)$ by considering now the system

$$\begin{cases} \gamma + \delta = n \\ \gamma - \delta = 1 \end{cases} \quad (11)$$

as defining the exponents of a corresponding rational function for values of $n \geq 2$. In (11) the first equation corresponds with the general value n on the right hand side to the first equation in (9) whereas the second equation

keeps the value of the first equation in (10). Obviously, this leads to the values $\gamma = \frac{n+1}{2}$ resp. $\delta = \frac{n-1}{2}$.

But how to understand the role of the multiplier $(k+1)$ in the right hand side of (8)? In the following second part of this section we will see that it is nothing else than the quotient of two well justified Pochhammer symbols.

After these general remarks let us now pass to the expansion of the rational function

$$g(t) = \frac{1}{(1-t)^\gamma(1+t)^\delta}, \quad (12)$$

with the general values $\gamma = \frac{n+1}{2}$ resp. $\delta = \frac{n-1}{2}$ instead of $\gamma = \frac{3}{2}$ resp. $\delta = \frac{1}{2}$ as in (8).

We expand each factor in the right-hand side of (12) and obtain

$$\frac{1}{(1-t)^{\frac{n+1}{2}}} = \sum_{k=0}^{+\infty} \frac{(\frac{n+1}{2})_k}{k!} t^k \quad \text{and} \quad \frac{1}{(1+t)^{\frac{n-1}{2}}} = \sum_{k=0}^{+\infty} (-1)^k \frac{(\frac{n-1}{2})_k}{k!} t^k.$$

Using the Cauchy product we get

$$g(t) = \sum_{k=0}^{+\infty} d_k(n) t^k, \quad (13)$$

where

$$d_k(n) := \sum_{s=0}^k (-1)^s \frac{(\frac{n+1}{2})_{k-s} (\frac{n-1}{2})_s}{(k-s)! s!}.$$

Observe that by a little rearrangement the function $g(t)$ can also be written as

$$g(t) = \frac{1}{(1-t)(1-t^2)^{\frac{n-1}{2}}} = \sum_{k=0}^{+\infty} \sum_{s=0}^k \frac{(\frac{n-1}{2})_{k-s}}{(k-s)!} t^{2k-s}, \quad (14)$$

since

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \quad \text{and} \quad \frac{1}{(1-t^2)^{\frac{n-1}{2}}} = \sum_{k=0}^{\infty} \frac{(\frac{n-1}{2})_k}{k!} t^{2k}.$$

Comparing (13) and (14) the coefficient $d_k(n)$ can be determined by an inductive process as

$$d_k(n) = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(\frac{n-1}{2})_s}{s!}. \quad (15)$$

Applying the formulas

$$\sum_{s=0}^k \frac{(a)_s}{s!} = \frac{(a+1)_k}{k!} \quad \text{and} \quad (a)_k = 2^k \left(\frac{a}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor} \left(\frac{1+a}{2}\right)_{\lfloor \frac{k}{2} \rfloor},$$

the sum (15) can be calculated and lead to the explicit expression

$$d_k(n) = \frac{\left(\frac{n+1}{2}\right)_{\lfloor \frac{k}{2} \rfloor}}{\lfloor \frac{k}{2} \rfloor!} = \frac{(n)_k}{k!} \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}. \quad (16)$$

Therefore the expansion of the function (14) we were looking for is given in the form

$$\frac{1}{(1-t)^{\frac{n+1}{2}}(1+t)^{\frac{n-1}{2}}} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}} t^k. \quad (17)$$

Observe that for $n = 2$ the expression (16) reduces to the product $(k+1)c_k$ appearing in (8). In fact, taking into account (2), we can write

$$d_k(2) = \frac{(2)_k}{k!} \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(1\right)_{\lfloor \frac{k+1}{2} \rfloor}} = \frac{(2)_k}{k!} c_k = (k+1)c_k.$$

Of course, the factorization of (16) by separating the factor $\frac{(n)_k}{k!}$ which is nothing else than $(k+1)$ for $n = 2$ was the decisive step for recognizing (17) as the adequate generalization of (8) in the case $n > 2$.

The choice of the function $g(t)$ led to the following definition:

Definition 1. Let $n \in \mathbb{N}$. The generalized Vietoris' number sequence is defined by $\mathcal{S}(n) := (c_k(n))_{k \geq 0}$ with

$$c_k(n) := \frac{\left(\frac{1}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}{\left(\frac{n}{2}\right)_{\lfloor \frac{k+1}{2} \rfloor}}. \quad (18)$$

Remark 2. It is easy to recognize that the representation of $c_k(n)$ in (18) coincides with the one first obtained and frequently used in hypercomplex context (see e.g. [21])

$$c_k(n) = \begin{cases} \frac{k!!(n-2)!!}{(n+k-1)!!}, & \text{if } k \text{ is odd} \\ c_{k-1}(n), & \text{if } k \text{ is even.} \end{cases} \quad (19)$$

3. Generating functions and series involving $\mathcal{S}(n)$

This section is devoted to the study of generating functions for generalized Vietoris' number sequences $\mathcal{S}(n)$, $n \geq 1$, and some series build by them. We start by observing that the representation (18) of their elements in terms of quotients of numbers represented by the Pochhammer symbol suggests the use of the well known Gauss' hypergeometric function, which is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, \quad (20)$$

where $a, b \in \mathbb{C}$, $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

We are now ready to formulate the main result of this section.

Theorem 3. *Let $G(.; n)$ be the following real-valued function depending on a parameter $n \in \mathbb{N}$:*

$$G(t; n) = \begin{cases} \frac{1}{t} [(1+t) {}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; t^2) - 1], & \text{if } t \in]-1, 0[\cup]0, 1[\\ 1, & \text{if } t = 0. \end{cases} \quad (21)$$

Then, for any fixed $n \in \mathbb{N}$, $G(.; n)$ is a one-parameter generating function of the sequence $\mathcal{S}(n)$.

Proof. For each fixed $n \in \mathbb{N}$, consider the sequence $\mathcal{S}(n)$ whose terms are defined by (18). The one-parameter generating function for this sequence can be written as the following formal power series in the real variable t

$$\begin{aligned} G(t; n) &= \sum_{k=0}^{+\infty} c_k(n) t^k = \sum_{s=0}^{+\infty} c_{2s}(n) t^{2s} + \sum_{s=1}^{+\infty} c_{2s-1}(n) t^{2s-1} \\ &= \frac{1}{t} \left((1+t) \sum_{s=0}^{+\infty} c_{2s}(n) t^{2s} - 1 \right) \\ &= \frac{1+t}{t} \sum_{s=0}^{+\infty} \frac{(\frac{1}{2})_s}{(\frac{n}{2})_s} t^{2s} - \frac{1}{t} \end{aligned} \quad (22)$$

for $t \neq 0$, because $c_{2s}(n) = c_{2s-1}(n)$.

Taking $a = \frac{1}{2}$, $b = 1$, $c = \frac{n}{2}$ ($n \in \mathbb{N}$) and $z = t^2$ ($t \in]-1, 1[\setminus \{0\}$) in (20), we obtain from (22),

$$G(t; n) = \frac{1}{t} [(1+t) {}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; t^2) - 1].$$

Since ${}_2F_1(\frac{1}{2}, 1; \frac{n}{2}; 0) = 1$, the function (21) is continuous for $|t| < 1$. $\square$

It is clear that we can obtain a closed formula for the generating function (21) of the sequence $\mathcal{S}(n)$ as long as a closed formula for the corresponding hypergeometric series is known. As examples we list some cases where such closed formulae are well known

$${}_2F_1(a, b; b; z) = (1 - z)^{-a} \quad (23)$$

$${}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{1}{2z} \ln \frac{1+z}{1-z}. \quad (24)$$

$${}_2F_1\left(a, a + \frac{1}{2}; 1 + 2a; z\right) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-z}\right)^{-2a}. \quad (25)$$

and, consequently, closed formulae for $G(., n)$ can be easily obtained.

1. $n = 1$

In this case, $c_k(1) = 1$ ($k \geq 0$) and the corresponding generating function is given by

$$G(t; 1) = \frac{1}{t} [(1+t) {}_2F_1\left(\frac{1}{2}, 1; \frac{1}{2}; t^2\right) - 1] = \frac{1}{1-t},$$

because ${}_2F_1\left(\frac{1}{2}, 1; \frac{1}{2}; t^2\right)$ reduces to the geometric function.

2. $n = 2$

Recalling (2), we have $c_{2k}(2) = c_{2k-1}(2) = \frac{(\frac{1}{2})_k}{k!}$. The use of (23) allows to restore (6), since

$$G(t; 2) = \frac{1}{t} [(1+t) {}_2F_1\left(\frac{1}{2}, 1; 1; t^2\right) - 1] = \frac{\sqrt{1+t}-\sqrt{1-t}}{t\sqrt{1-t}}.$$

3. $n = 3$

The generalized Vietoris' numbers are $c_{2k}(3) = c_{2k-1}(3) = \frac{1}{2k+1}$ and taking into account (24), the corresponding generating function is given by

$$G(t; 3) = \frac{1}{t} [(1+t) {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; t^2\right) - 1] = \frac{1}{t} \left(\frac{t+1}{t} \ln \sqrt{\frac{1+t}{1-t}} - 1 \right).$$

4. $n = 4$

In this case, $c_{2k}(4) = c_{2k-1}(4) = \frac{(\frac{1}{2})_k}{(k+1)!}$. The use of (25) leads to

$$G(t; 4) = \frac{1}{t} [(1+t) {}_2F_1\left(\frac{1}{2}, 1; 2; t^2\right) - 1] = \frac{2t+1-\sqrt{1-t^2}}{t(1+\sqrt{1-t^2})}.$$

Using the one-parameter generating function $G(., n)$, we can now study the convergence of some non-trivial series that involve generalized Vietoris' number sequences.

For $z \in \mathbb{C}$ outside the circle of convergence, the hypergeometric function ${}_2F_1$ is defined by analytic continuation. On the unit disc $|z| = 1$, the series (20) converges absolutely for $\operatorname{Re}(c - a - b) > 0$ and

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad (26)$$

it converges conditionally for $-1 < \operatorname{Re}(c - a - b) \leq 0$, with $z \neq 1$, and if $\operatorname{Re}(c - a - b) \leq -1$ it diverges.

As a consequence of these properties $G(t; n)$ can also be defined for $t = \pm 1$, provided that $n > 3$.

In fact, for $t = 1$, $G(1; n)$ corresponds to the series whose general term is $c_k(n)$. Using (26), its sum is equal to

$$\sum_{k=0}^{+\infty} c_k(n) = 2 {}_2F_1\left(\frac{1}{2}, 1; \frac{n}{2}; 1\right) - 1 = \frac{n-1}{n-3}, \quad \text{for } n > 3.$$

Observe that the case $n = 1$ leads clearly to a divergent series. For $n = 2$, the well known lower bound of the central binomial coefficients $\binom{2k}{k} \geq \frac{2^{2k}}{2k+1}$ ensures that the corresponding series is divergent. The case $n = 3$ leads to the series of reciprocal odd numbers, which is clearly divergent.

On the other hand $G(-1; n)$ can also be defined for $n = 2$ and $n = 3$, since (19) is a pairwise coefficient's sequence, decreasing to zero. Therefore, we obtain

$$\sum_{k=0}^{+\infty} (-1)^k c_k(n) = 1, \quad \text{for } n \geq 2.$$

The case $n = 1$ leads to $\sum_{k=0}^{+\infty} (-1)^k$ which is a divergent series.

4. A recurrence relation for the Vietoris' sequence $\mathcal{S}(2)$

First of all we would like to call attention to some formal similarity between Catalan numbers $\mathcal{C}_k$ and Vietoris' numbers $c_{2k} = c_{2k}(2)$, i.e. between

$$\mathcal{C}_k := \frac{1}{k+1} \binom{2k}{k} \quad \text{and} \quad c_{2k} := \frac{1}{2^{2k}} \binom{2k}{k}.$$

Both are defined as weighted central binomial coefficients of the $2k$ -th line in the Pascal triangle. In the case of Catalan numbers the weight is equal to the *number of binomial coefficients* in the k -th line of the Pascal triangle, whereas in the case of Vietoris' numbers the weight is equal to the

sum of binomial coefficients in the k -th line. The interpretation of Catalan numbers by counting combinatorial constellations is overwhelming (see e.g. “Catalania” in [32]), but to the best of our knowledge some purely combinatorial interpretation of Vietoris’ numbers is still missing. Going further and considering also a Vietoris’ number with odd index equal to its preceding number with even index, it seems to be even more difficult to find for such a pairwise constructed sequence a purely combinatorial interpretation.

It is worth to point out the well known recurrence relation for Catalan numbers

$$C_{k+1} = \sum_{s=0}^k C_{k-s} C_s, \quad k = 0, 1, \dots \quad (27)$$

$$C_0 = 1 \quad (28)$$

as well as its generating function

$$f(t) = \frac{1 - \sqrt{1 - 4t}}{2t}$$

which is the solution of the quadratic equation

$$f^2 - \frac{f}{t} + \frac{1}{t} = 0, \quad (29)$$

for which $\lim_{t \rightarrow 0} f(t) = f(0) = C_0 = 1$.

The corresponding quadratic equation for the generating function (6) of Vietoris’ numbers, i.e.

$$F(t) = \sum_{k=0}^{\infty} c_k t^k = \frac{\sqrt{1+t} - \sqrt{1-t}}{t\sqrt{1-t}}, \quad 0 < |t| < 1,$$

for which $\lim_{t \rightarrow 0} F(t) = F(0) = c_0 = 1$ can easily be obtained as

$$F^2 + \frac{2F}{t} - \frac{2}{t} - \frac{2}{1-t} = F^2 + \frac{2F}{t} - \frac{2}{t(1-t)} = 0. \quad (30)$$

The small but essential difference between (29) and (30) is clearly visible in the left hand side of both equations which in case of (30) reflects the pairwise appearance of each c_k in $\mathcal{S}$.

The recurrence relation for the Vietoris’ numbers c_k we are looking for can now be obtained by using again the generating function $F(t)$.

With $G(t) := tF(t)$ and applying (30), we arrive to formula

$$G(t)^2 - tG(t)^2 = 2tG(t) - 2G(t) + 2t. \quad (31)$$

Since

$$G(t) = \sum_{k=0}^{\infty} c_k t^{k+1} \quad \text{and} \quad G(t)^2 = \sum_{k=0}^{\infty} \left(\sum_{s=0}^k c_{k-s} c_s \right) t^{k+2},$$

the left hand side of (31) can be written as

$$\begin{aligned} (1-t)G(t)^2 &= \sum_{k=0}^{\infty} \left(\sum_{s=0}^k c_{k-s} c_s \right) t^{k+2} - \sum_{k=0}^{\infty} \left(\sum_{s=0}^k c_{k-s} c_s \right) t^{k+3} \\ &= \sum_{k=0}^{\infty} \left(\sum_{s=0}^k (c_{k-s} - c_{k-1-s}) c_s \right) t^{k+2} \\ &= \sum_{k=0}^{\infty} \left(\sum_{s=0}^k (c_s - c_{s-1}) c_{k-s} \right) t^{k+2}, \end{aligned} \quad (32)$$

where $c_{-1} := 0$, while the right hand side is

$$\begin{aligned} 2tG(t) - 2G(t) + 2t &= 2 \left(\sum_{k=0}^{\infty} (c_k t^{k+2} - c_k t^{k+1}) + t \right) \\ &= 2 \sum_{k=0}^{\infty} (c_k - c_{k+1}) t^{k+2}. \end{aligned} \quad (33)$$

Comparing (32) and (33) we conclude that

$$2(c_{k+1} - c_k) = - \sum_{s=0}^k (c_s - c_{s-1}) c_{k-s}, \quad k = 0, 1, \dots, \quad (34)$$

or equivalently

$$c_{k+1} = c_k - \frac{1}{2} \left(\sum_{s=0}^k c_s c_{k-s} + \sum_{s=0}^{k-1} c_{s-1} c_{k-s} \right), \quad k = 0, 1, \dots, \quad (35)$$

where $c_0 = 1$ and $c_{-1} := 0$.

With (35) we arrive to the recurrence relation for the elements c_k of $\mathcal{S} = \mathcal{S}(2)$ in a form which serves well for comparing it with the simpler recurrence relation of the Catalan numbers given by (27)-(28).

The principal structure of both recurrence relations, namely involving convolutions, is evident. It seems also evident that the two convolutions in (35) are a result of the pairwise appearance of each c_k in $\mathcal{S}$.

This situation becomes even more clear, if we transform (34) into a recurrence with respect to forward difference terms. Thereby formula (34) can be given the form of a theorem that we have just proved.

Theorem 4. *The Vietoris' numbers forming the sequence $\mathcal{S} = \mathcal{S}(2)$ satisfy the recurrence relation*

$$\begin{aligned}\Delta c_k &= -\frac{1}{2} \sum_{s=0}^k c_{k-s} \Delta c_{s-1}, \quad k = 0, 1, \dots \\ c_0 &= 1\end{aligned}\tag{36}$$

where $\Delta c_k = c_{k+1} - c_k$ denotes the forward difference with $\Delta c_{-1} := 1$.

If $k = 2r + 1$, i.e. for odd indexes, we know that $\Delta c_k = 0$ and therefore (36) can be rewritten as

Corollary 1. *The Vietoris' numbers forming the sequence $\mathcal{S}(2)$ satisfy the following relations*

$$\begin{aligned}\Delta c_{2r} &= -\frac{1}{2} \sum_{s=0}^{r-1} c_{2r-2s-1} \Delta c_{2s}, \\ 0 &= \sum_{s=0}^r c_{2r-2s} \Delta c_{2s}, \quad r = 0, 1, \dots\end{aligned}$$

with $c_0 = 1$ and $\Delta c_{-1} := 1$.

5. Concluding remarks

We have briefly mentioned how Vietoris's numbers, i.e. the sequence $\mathcal{S} = \mathcal{S}(2)$ of rational numbers, crop up in a problem of positivity in harmonic analysis (Theorem 1). A dozen years ago we noticed for the first time its role in the context of hypercomplex function theory, particularly in the construction of sequences of multivariate generalized Appell polynomials. But the observation that they were at the same time also subject of research on special holomorphic functions deepened our interest in its properties as *object of combinatorics*. The opportunity of presenting the one-parametric generalization $\mathcal{S}(n)$, $n \geq 1$, (Definition 1) of Vietoris' number sequence on the occasion of a conference in honor of our colleague Domingos Cardoso forced us to look also for a purely real analytic approach to $\mathcal{S}(n)$ as shown in this paper (Sections 2 and 3).

For the interested reader it may be worth noticing that in the framework of hypercomplex function theory those generalized Appell polynomials have meanwhile received a lot of attention from several authors due

to their important role in theory ([6, 7, 26, 30]) and applications of polynomials ([9, 10, 11]), in elasticity [5], special functions ([20, 29]), or 3D-quasiconformal mapping problems ([17, 19, 22]).

6. Acknowledgments

The work of the first and third authors was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (“FCT-Fundação para a Ciência e Tecnologia”), within project PEst-OE/MAT/UI4106/2013. The work of the second author was supported by Portuguese funds through the CMAT - Centre of Mathematics and FCT within the Project UID/MAT/00013/2013.

- [1] P. Appell, Sur une classe de polynomes, *Ann. Sci. École Norm. Sup.* 9 (2) (1880) 119–144.
- [2] R. Askey, J. Steinig, Some positive trigonometric sums, *Transactions AMS* 187 (1) (1974) 295–307.
- [3] R. Askey, *Orthogonal polynomials and special functions*, Society for Industrial and Applied Mathematics, Philadelphia, 2nd ed. 1994.
- [4] R. Askey, Vietoris’s inequalities and hypergeometric series, In: *Recent progress in inequalities* (Niš, 1996), Kluwer Acad. Publ., Dordrecht, 1998, 63–76.
- [5] S. Bock, On Monogenic Series Expansions with Applications to Linear Elasticity, *Adv. Appl. Clifford Algebras* (2014) 24: 931–943. <https://doi.org/10.1007/s00006-014-0490-0>
- [6] S. Bock and K. Gürlebeck, On a Generalized Appell System and Monogenic Power Series, *Math. Methods Appl. Sci.* 33 (4) (2010) 394–411.
- [7] S. Bock, K. Gürlebeck, R. Lávisčka, and V. Souček, Gelfand-Tsetlin bases for spherical monogenics in dimension 3, *Rev. Mat. Iberoam.* 28 no. 4 (2012), 1165–1192.
- [8] I. Cação, Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in R^3 , *Numer. Algor.* 55, (2010), 191–203.

- [9] I. Cação, M. I. Falcão, and H. R. Malonek, Laguerre derivative and monogenic Laguerre polynomials: An operational approach. *Math. Comput. Model.*, 53 (5-6) (2011), 1084–1094.
- [10] I. Cação, M. I. Falcão, and H. R. Malonek, On generalized hypercomplex laguerre-type exponentials and applications, In B. Murgante et al. (eds.) *Lecture Notes in Computer Science*, Vol. 6784, Springer-Verlag Berlin Heidelberg, (2011), 271–286.
- [11] I. Cação and H. R. Malonek, On an hypercomplex generalization of Gould-Hopper and related Chebyshev polynomials, In B. Murgante et al. (eds.) *Lecture Notes in Computer Science*, Vol. 6784, Springer-Verlag Berlin Heidelberg, (2011), 316–326.
- [12] I. Cação, M. I. Falcão, and H. R. Malonek, Matrix representations of a basic polynomial sequence in arbitrary dimension, *Comput. Methods Funct. Theory* 12 (2) (2012) 371–391.
- [13] I. Cação, M. I. Falcão, and H. R. Malonek, A Matrix Recurrence for Systems of Clifford Algebra-Valued Orthogonal Polynomials, *Adv. Appl. Clifford Algebr.* 24 (4) (2014) 981–994.
- [14] I. Cação, M. I. Falcão, and H. R. Malonek, Three-Term Recurrence Relations for Systems of Clifford Algebra-Valued Orthogonal Polynomials, *Adv. Appl. Clifford Algebr.* 27 (1) (2017) 71–85.
- [15] I. Cação, M. I. Falcão, and H. R. Malonek, Hypercomplex Polynomials, Vietoris’ Rational Numbers and a Related Integer Numbers Sequence, *Complex Anal. Oper. Theory* 11 (5) (2017) 1059–1076.
- [16] I. Cação, H. R. Malonek, and G. Tomaz, Shifted Generalized Pascal Matrices in the Context of Clifford Algebra-Valued Polynomial Sequences, In B. Murgante et al. (eds.) *Lecture Notes in Computer Science*, Vol. 10405, Springer-Verlag Berlin Heidelberg, (2017), 409–421.
- [17] C. Cruz, C., M. I. Falcão, and H. R. Malonek, 3D Mappings by Generalized Joukowski Transformations, In B. Murgante et al. (eds.) *Lecture Notes in Computer Science*, Vol. 6784, Springer-Verlag Berlin Heidelberg, (2011), 358–373.
- [18] R. Delanghe, F. Sommen, and V. Souček, *Clifford Algebra and Spinor-Valued Functions. A function theory for the Dirac operator*, Mathematics and its Applications (Dordrecht) 53 Kluwer Academic Publishers, 1992.

- [19] M. I. Falcão, J. Cruz, and H.R. Malonek, Remarks on the generation of monogenic functions, In K. Gürlebeck and C. Könke, (eds.), Proc. of the 17-th Inter. Conf. on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, Bauhaus-University Weimar, ISSN 1611-4086, (2006), 12–14
- [20] M. I. Falcão, H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. In: T. E. Simos, G. Psihoyios, C. Tsitouras (Eds.), AIP Conference Proceedings 936, 2007, 738–741.
- [21] M. I. Falcão, H. R. Malonek, A note on a one-parameter family of non-symmetric number triangles, *Opuscula Mathematica* 32 (4) (2012), 661–673.
- [22] M. I. Falcão, F. Miranda, Quaternions: A Mathematica package for quaternionic analysis, In B. Murgante et al. (eds.) *Lecture Notes in Computer Science*, Vol. 6784, Springer-Verlag Berlin Heidelberg, (2011), 200–214.
- [23] R. Fueter, Analytische Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.*, 4, (1932), 9–20.
- [24] K. Gürlebeck, K. Habetha, and W. Sprößig, *Holomorphic Functions in the Plane and n -Dimensional Space*, Translated from the 2006 German original. Birkhäuser Verlag, Basel, 2008.
- [25] K. Gürlebeck, H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{m+1}$ and its applications, *Complex Variables* 39 (1999) 199–228.
- [26] R. Lávička, Complete Orthogonal Appell Systems for Spherical Monogenics, *Complex Anal. Oper. Theory* 6 (2012) 477–489.
- [27] H. R. Malonek, Selected topics in hypercomplex function theory. In S.-L. Eriksson, ed., *Clifford algebras and potential theory*, University of Joensuu, 7, (2004) 111–150.
- [28] H. Malonek and M. Falcão, Special monogenic polynomials—properties and applications. In T. E. Simos, G. Psihoyios, and C. Tsitouras, editors, AIP Conference Proceedings, 936, 2007, 764–767.
- [29] H. R. Malonek, M. I. Falcão, On Special Functions in the Context of Clifford Analysis. In: T.E.Simos, G. Psihoyios, C. Tsitouras (Eds.), AIP Conference Proceedings 1281, (2010), 1492–1495.

- [30] J. Morais and H. T. Le. Orthogonal Appell systems of monogenic functions in the cylinder, *Math. Methods Appl. Sci.*, 34(12) (2011), 1472–1486.
- [31] St. Ruscheweyh, L. Salinas, Stable functions and Vietoris' theorem, *J. Math. Anal. Appl.* 291 (2004) 596–604.
- [32] R. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics 62 Cambridge University Press, Cambridge, 1999.
- [33] L. Vietoris, Über das Vorzeichen gewisser trigonometrischer Summen, *Sitzungsber. Österr. Akad. Wiss* 167, (1958) 125–135.