ON NONLOCAL VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES WITH FRACTIONAL GRADIENT

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Abstract. We extend classical results on variational inequalities with convex sets with gradient constraint to a new class of fractional partial differential equations in a bounded domain with constraint on the distributional Riesz fractional gradient, the σ-gradient ($0 < \sigma < 1$). We establish continuous dependence results with respect to the data, including the threshold of the fractional σ-gradient. Using these properties we give new results on the existence to a class of quasi-variational variational inequalities with fractional gradient constraint via compactness and via contraction arguments. Using the approximation of the solutions with a family of quasilinear penalisation problems we show the existence of generalised Lagrange multipliers for the σ-gradient constrained problem, extending previous results for the classical gradient case, i.e., with $\sigma = 1$.

1. Introduction

In a series of two interesting papers [13] and [14], Shieh and Spector have considered a new class of fractional partial differential equations. Instead of using the well-known fractional Laplacian, their starting concept is the distributional Riesz fractional gradient of order $\sigma \in (0,1)$, which will be called here the σ-gradient D^σ, for brevity: for $u \in L^p(\mathbb{R}^N)$, $1 < p < \infty$, we set

$$(1.1)\qquad \big(D^\sigma u\big)_j = \frac{\partial^\sigma u}{\partial x_j^\sigma} = \frac{\partial}{\partial x_j} I_{1-\sigma} u, \qquad 0 < \sigma < 1, \quad j = 1, \dots, N,$$

where $\frac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v \in \mathscr{C}_0^\infty(\mathbb{R}^N)$,

$$\Big\langle \frac{\partial^\sigma u}{\partial x_j^\sigma}, v \Big\rangle = -\Big\langle I_{1-\sigma} u, \frac{\partial v}{\partial x_j} \Big\rangle = -\int_{\mathbb{R}^N} (I_{1-\sigma} u) \frac{\partial v}{\partial x_j}\, dx,$$

with I_α denoting the Riesz potential of order α, $0 < \alpha < 1$:

$$I_\alpha u(x) = (I_\alpha * u)(x) = \gamma_{N,\alpha} \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-\alpha}}\, dy, \qquad \text{with } \gamma_{N,\alpha} = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{\frac{N}{2}}\, 2^\alpha\, \Gamma(\frac{\alpha}{2})}.$$

As it was shown in [13], D^σ has nice properties for $u \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, namely

$$(1.2)\qquad D^\sigma u \equiv D(I_{1-\sigma} u) = I_{1-\sigma} * Du,$$

$$(1.3)\qquad (-\Delta)^\sigma u = -\sum_{j=1}^N \frac{\partial^\sigma}{\partial x_j^\sigma} \frac{\partial^\sigma}{\partial x_j^\sigma} u,$$

where the well-known fractional Laplacian may be given, for a suitable constant $C_{N,\sigma}$, by (see, for instance, [4]):

$$(-\Delta)^\sigma u \equiv C_{N,\sigma} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\sigma}}\, dy.$$

It was also observed in [14] that the σ-gradient is an example of the non-local gradients considered in [9], which can be also given by

$$D^\sigma u(x) = R(-\Delta)^{\frac{\sigma}{2}} u(x) = (1-\sigma-N)\gamma_{N,1-\sigma} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\sigma}}\, \frac{x-y}{|x-y|}\, dy, \tag{1.4}$$

in terms of the vector-valued Riesz transform (see [15], with $\rho_N = \Gamma\big(\frac{N+1}{2}\big)/\pi^{\frac{N+1}{2}}$):

$$Rf(x) = \rho_N \text{ P.V.} \int_{\mathbb{R}^N} f(y)\, \frac{x-y}{|x-y|^{N+1}}\, dy.$$

We observe that, from the properties of D^σ and a result of [7] on the Riesz kernel as approximation of the identity as $\alpha \to 0$, the σ-gradient approaches the standard gradient as $\sigma \to 1$: if $Du \in L^p(\mathbb{R}^N)^N \cap L^q(\mathbb{R}^N)^N$, $1 < q < p$, then $D^\sigma u \underset{\sigma\to 1}{\longrightarrow} Du$ in $L^p(\mathbb{R}^N)^N$.

Introducing the vector space of fractional differentiable functions as the closure of $\mathscr{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\sigma,p}^p = \|u\|_{L^p(\mathbb{R}^N)}^p + \|D^\sigma u\|_{(L^p(R^N))^m}^p, \quad 0<\sigma<1,\ p>1,$$

by [13, Theorem 1.7] it is exactly the Bessel potencial space $L^{\sigma,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$, $0 \le s < \sigma$, where $W^{s,p}(\mathbb{R}^N)$ denotes the usual fractional Sobolev space. In [13] the solvability of the fractional partial differential equations with variable coefficients and Dirichlet data was treated in the case $p = 2$, as well as the minimization of the integral functionals of the σ-gradient with p-growth, leading to the solvability of a fractional p-Laplace equation of a novel type.

In this work we are concerned with the Hilbertian case $p = 2$ in a bounded domain $\Omega \subset \mathbb{R}^N$, with Lipschitz boundary, where the homogeneous Dirichlet problem for a general linear PDE with measurable coefficients is considered under an additional constraint on the σ-gradient. We shall consider all solutions in the usual Sobolev space

$$H_0^\sigma(\Omega), \quad \text{with norm } \|u\|_{H_0^\sigma(\Omega)} = \|D^\sigma u\|_{L^2(\Omega)^N}, \quad 0<\sigma<1, \tag{1.5}$$

which, by the Sobolev-Poincaré inequality, is equivalent to the usual Hilbertian norm induced from $L^{\sigma,2}(\mathbb{R}^N) = W^{\sigma,2}(\mathbb{R}^N) = H^\sigma(\mathbb{R}^N)$, $0 < \sigma < 1$ in the closure of the Cauchy sequences of functions in $\mathscr{C}_0^\infty(\Omega)$ (see [13]).

For nonnegative functions $g \in L^\infty(\Omega)$, we consider the nonempty convex sets of the type

$$\mathbb{K}_g^\sigma = \big\{ v \in H_0^\sigma(\Omega) : |D^\sigma v| \le g \text{ a.e. in } \Omega \big\}. \tag{1.6}$$

Let $f \in L^1(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ be a measurable, bounded and positive definite matrix. We shall consider, in Section 2, the well-posedness of the variational inequality

$$u \in \mathbb{K}^\sigma_g : \qquad \int_\Omega AD^\sigma u \cdot D^\sigma(v-u) \geq \int_\Omega f(v-u), \qquad \forall v \in \mathbb{K}^\sigma_g. \tag{1.7}$$

In particular, we obtain precise estimates for the continuous dependence of the solution u with respect to f and g, and so we extend well-known results for the classical case $\sigma = 1$ (see [12] and its references).

Extending the result of [2] for the gradient ($\sigma = 1$) case, we prove in Section 3 the existence of generalised Lagrange multipliers for the σ-gradient constrained problem. More precisely, we show the existence of $(\lambda, u) \in L^\infty(\Omega)' \times \Upsilon^\sigma_\infty(\Omega)$ such that

$$\langle \lambda D^\sigma u, D^\sigma v\rangle_{\left(L^\infty(\Omega)^N\right)' \times L^\infty(\Omega)^N} + \int_\Omega AD^\sigma u \cdot D^\sigma v = \int_\Omega fv, \qquad \forall v \in \Upsilon^\sigma_\infty(\Omega), \tag{1.8a}$$

$$|D^\sigma u| \leq g \ \text{ a.e. in } \Omega, \qquad \lambda \geq 0 \text{ and } \lambda(|D^\sigma u| - g) = 0 \ \text{ in } L^\infty(\Omega)' \tag{1.8b}$$

and, moreover, u solves (1.7).

Here, for each σ, we have set

$$\Upsilon^\sigma_\infty(\Omega) = \left\{ v \in H^\sigma_0(\Omega) : D^\sigma v \in L^\infty(\Omega)^N \right\}, \qquad 0 < \sigma < 1, \tag{1.9}$$

and

$$\langle \lambda \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_{(L^\infty(\Omega)^N)' \times L^\infty(\Omega)^N} = \langle \lambda, \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\rangle_{L^\infty(\Omega)'\times L^\infty(\Omega)} \quad \forall \lambda \in L^\infty(\Omega)' \ \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in L^\infty(\Omega)^N.$$

Finally, in the Section 4 we consider the solvability of solutions to quasi-variational inequalities corresponding to (1.7) when the threshold $g = G[u]$ and therefore also the convex set (1.6) depend on the solution $u \in \mathbb{K}^\sigma_{G[u]}$. We give sufficient conditions on the nonlinear and nonlocal operator $v \mapsto G[v]$ to obtain the existence of at least one solution u of (1.7) with $\mathbb{K}^\sigma_g$ replaced by $K^\sigma_{G[u]}$, by compactness methods, as in [6] for the case $\sigma = 1$. In a special case, when $G[u](x) = \Gamma(u)\varphi(x)$ is strictly positive and separates variables with a Lipschitz functional $\Gamma : L^2(\Omega) \to \mathbb{R}^+$, we adapt an idea of [5] (see also [12]) to obtain, by a contraction principle, the existence and uniqueness of the solution of the quasi-variational inequality under the "smallness" of the product of f with the Lipschitz constant of Γ and the inverse of its positive lower bound.

2. The variational inequality with σ-gradient constraint

For some $a_*, a^* > 0$, let $A = A(x) : \Omega \to \mathbb{R}^{N\times N}$ be a bounded and measurable matrix, not necessarily symmetric, such that, for a.e. $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$:

$$a_*|\xi|^2 \leq A(x)\xi\cdot\xi \leq a^*|\xi|^2. \tag{2.1}$$

Fixed $\nu > 0$, we define

$$L^\infty_\nu(\Omega) = \big\{ v \in L^\infty(\Omega) : v(x) \geq \nu > 0 \text{ a.e. } x \in \Omega \big\}. \tag{2.2}$$

For any $g \in L^\infty_\nu(\Omega)$ it is clear that the convex set $\mathbb{K}^\sigma_g$ defined in (1.6) is non-empty, closed and, by Sobolev embeddings, we have, using the notation (1.9), for all $0 < \beta < \sigma$:

$$\mathbb{K}^\sigma_g \subset \Upsilon^\sigma_\infty(\Omega) \subset \mathscr{C}^{0,\beta}(\overline{\Omega}) \subset L^\infty(\Omega), \tag{2.3}$$

where $\mathscr{C}^{0,\beta}(\overline{\Omega})$ is the space of Hölder continuous function with exponent β. Indeed, we recall (see for instance [3]) the embedding for the fractional Sobolev spaces $0 < \sigma \leq 1$, $1 < p < \infty$:

$$W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q \leq \tfrac{Np}{N-\sigma p}, \quad \text{if } \sigma p < N, \tag{2.4a}$$

$$W^{\sigma,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for every } q < \infty, \quad \text{if } \sigma p = N, \tag{2.4b}$$

$$W^{\sigma,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap \mathscr{C}^{0,\beta}(\overline{\Omega}), \quad \text{for every } 0 < \beta \leq \sigma - \tfrac{N}{p}, \quad \text{if } \sigma p > N, \tag{2.4c}$$

with continuous embeddings, which are also compact if also $q < \frac{Np}{N-\sigma p}$ in (2.4a) and $\beta < \sigma - \frac{N}{p}$ in (2.4c). In particular, we have

$$H^\sigma_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \quad \text{and} \quad L^{2^\#}(\Omega) \hookrightarrow H^{-\sigma}(\Omega) = \big(H^\sigma_0(\Omega)\big)', \ 0 < \sigma < 1, \tag{2.5}$$

where we set $2^* = \frac{2N}{N-2\sigma}$ and $2^\# = \frac{2N}{N+2\sigma}$ when $\sigma < \frac{N}{2}$, and if $N = 1$ we denote $2^* = q$, $2^\# = q' = \frac{q}{q-1}$ when $\sigma = \frac12$ and $2^* = \infty$, $2^\# = 1$ when $\sigma > \frac12$.

Here we are also assuming that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain with Lipschitz boundary, and we may conclude (2.3) from (2.4a)-(2.4c) by using a bootstrap argument.

Therefore, in the right hand side of the variational inequality (1.7), for $g_i \in L^\infty(\Omega)$, we can take $f_i \in L^1(\Omega)$, and the first two theorems give continuous dependence results with precise estimates for two different problems with $i = 1, 2$:

$$u_i \in \mathbb{K}^\sigma_{g_i} : \qquad \int_\Omega AD^\sigma u_i \cdot D^\sigma(v - u_i) \geq \int_\Omega f_i(v - u_i), \quad \forall v \in \mathbb{K}^\sigma_{g_i}. \tag{2.15$_i$}$$

Theorem 2.1. *Under the assumptions* (2.1)*, for each* $f_i \in L^1(\Omega)$ *and each* $g_i \in L^\infty(\Omega)$*,* $g_i \geq 0$*, there exists a unique solution* u_i *to* (2.15)$_i$ *such that*

$$u_i \in \mathbb{K}^\sigma_{g_i} \cap \mathscr{C}^{0,\beta}(\overline{\Omega}), \qquad \textit{for all } 0 < \beta < \sigma. \tag{2.16}$$

When $g_1 = g_2$*, the solution map* $L^1(\Omega) \ni f \mapsto u \in H^\sigma_0(\Omega)$ *is* $\frac12$*-Hölder continuous, i.e., for some* $C_1 > 0$*, we have*

$$\|u_1 - u_2\|_{H^\sigma_0(\Omega)} \leq C_1 \|f_1 - f_2\|^{\frac12}_{L^1(\Omega)}. \tag{2.17}$$

Moreover, if in addition $f_i \in L^{2^\#}(\Omega)$, $i = 1, 2$, $2^\#$ *defined in* (2.5) *and* $g_1 = g_2$, *then* $L^{2^\#}(\Omega) \ni f \mapsto u \in H^\sigma_0(\Omega)$ *is Lipschitz continuous:*

$$\|u_1 - u_2\|_{H^\sigma_0(\Omega)} \le C_\# \|f_1 - f_2\|_{L^{2^\#}(\Omega)}, \tag{2.18}$$

for $C_\# = C_*/a_* > 0$, *where* C_* *is the constant of the Sobolev embedding* $H^\sigma_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

Proof. Suppose that $f_i \in L^{2^\#}(\Omega) \subset H^{-\sigma}(\Omega)$. Since the assumption (2.1) implies that A defines a continuous bilinear and coercive form over $H^\sigma_0(\Omega)$, the existence and uniqueness of the solution $u_i \in \mathbb{K}^\sigma_i$ to $(2.15)_i$ is an immediate consequence of the Stampacchia Theorem (see, for instance, [11, p. 95]), and (2.16) follows from (2.3).

With our notation (1.5), the estimate (2.18) follows easily from $(2.15)_i$ with $g_1 = g_2$ and $v = u_j$ $(i, j = 1, 2,\ i \neq j)$ from

$$a_* \|\overline{u}\|^2_{H^\sigma_0(\Omega)} \le \int_\Omega A D^\sigma \overline{u} \cdot D^\sigma \overline{u} \le \|\overline{f}\|_{L^{2^\#}(\Omega)} \|\overline{u}\|_{L^{2^*}(\Omega)} \le C_* \|\overline{f}\|_{L^{2^\#}(\Omega)} \|\overline{u}\|_{H^\sigma_0(\Omega)},$$

where we have set $\overline{u} = u_1 - u_2$ and $\overline{f} = f_1 - f_2$.

By (2.3), letting κ be such that

$$\|v\|_{L^\infty(\Omega)} \le \kappa, \qquad \forall v \in \mathbb{K}^\sigma_{g_1}, \tag{2.19}$$

we may easily conclude the estimate (2.17) with $C_1 = \sqrt{2\kappa/a_*}$ for $f_1, f_2 \in L^{2^\#}(\Omega) \subset L^1(\Omega)$ from $(1.5)_i$ and

$$a_* \|\overline{u}\|^2_{H^\sigma_0(\Omega)} \le \|\overline{f}\|_{L^1(\Omega)} \|\overline{u}\|_{L^\infty(\Omega)} \le 2\kappa \|\overline{f}\|_{L^1(\Omega)}.$$

Finally, the solvability of $(2.15)_i$ for f_i only in $L^1(\Omega)$ can be easily obtained by taking an approximating sequence of $f^n_i \in L^{2^\#}(\Omega)$ such that $f^n_i \underset{n}{\to} f_i$ in $L^1(\Omega)$ and using (2.17) for that (Cauchy) sequence. The proof is complete. □

Remark 2.1. *As in* [13] *it is possible to extend the variational inequality with* σ*-gradient to arbitrary open domains* $\Omega \subset \mathbb{R}^N$ *with a generalised Dirichlet data* $\varphi \in H^\sigma(\mathbb{R}^N)$ *such that* $I_{1-\sigma} * \varphi$ *is well-defined and* $D^\sigma \varphi \in L^\infty(\mathbb{R}^N)$. *This would require in the definition* (1.6) *of* $\mathbb{K}^\sigma_g$ *to replace* $H^\sigma_0(\Omega)$ *by the space*

$$H^\sigma_\varphi = \left\{ v \in H^\sigma(\mathbb{R}^N) : v = \varphi \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

and, in addition, technical compatibility assumptions on φ *and* g *to guarantee that the new* $\mathbb{K}^\sigma_g \neq \emptyset$.

Remark 2.2. *It is well-known that if, in addition,* A *is symmetric, i.e.* $A = A^T$, *the variational inequality* (1.7) *corresponds (and is equivalent) to the optimisation problem (see, for instance,* [11]*)*

$$u \in \mathbb{K}^\sigma_g : \qquad \mathcal{J}(u) \le \mathcal{J}(v), \qquad \forall v \in \mathbb{K}^\sigma_g,$$

where $\mathcal{J} : \mathbb{K}^\sigma_g \to \mathbb{R}$ *is the convex functional*

$$\mathcal{J}(v) = \frac{1}{2}\int_\Omega AD^\sigma v \cdot D^\sigma v - \int_\Omega fv.$$

Theorem 2.2. *Under the framework of the previous theorem, when* $f_1 = f_2 \in L^1(\Omega)$*, the solution map*

$$L^\infty_\nu(\Omega) \ni g \mapsto u \in H^\sigma_0(\Omega)$$

is also $\frac{1}{2}$*-Hölder continuous, i.e., there exists* $C_\nu > 0$ *such that*

$$\|u_1 - u_2\|_{H^\sigma_0(\Omega)} \leq C_\nu \|g_1 - g_2\|^{\frac{1}{2}}_{L^\infty(\Omega)}. \tag{2.20}$$

Proof. Let $\eta = \|g_1 - g_2\|_{L^\infty(\Omega)}$ and, for $i, j = 1, 2$, $i \neq j$, notice that

$$u_{i_j} = \frac{\nu}{\nu + \eta} u_i \in \mathbb{K}^\sigma_{g_j},$$

if u_i denotes the unique solution of $(2.15)_i$ to g_i and f_i.

Denote by $\kappa = \max\limits_{i=1,2}\{\|g_i\|_{L^\infty(\Omega)}, \|u_i\|_{L^\infty(\Omega)}\}$ and observe that for $i = 1, 2$,

$$|u_i - u_{i_j}| + |D^\sigma(u_i - u_{i_j})| \leq \frac{\eta}{\nu + \eta}\big(|u_i| + |D^\sigma u_i|\big) \leq 2\kappa\,\frac{\eta}{\nu}.$$

Hence, letting $v = u_{i_j}$ in $(2.15)_j$ and using (2.1) we get

$$\begin{aligned} &a_*\|u_1 - u_2\|^2_{H^\sigma_0(\Omega)} \leq \int_\Omega AD^\sigma(u_1 - u_2)\cdot D^\sigma(u_1 - u_2)\\ &\leq \int_\Omega AD^\sigma u_1 \cdot D^\sigma(u_{2_1} - u_2) + \int_\Omega AD^\sigma u_2 \cdot D^\sigma(u_{1_2} - u_1) + \int_\Omega f\big((u_1 - u_{1_2}) + (u_2 - u_{2_1})\big)\\ &\qquad \leq 2\kappa\,\frac{\eta}{\nu}\big(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)}\big) = C^2_\nu\|g_1 - g_2\|_{L^\infty(\Omega)}, \end{aligned}$$

with $C_\nu = \sqrt{2\kappa\big(M\|g_1\|_{L^1(\Omega)} + M\|g_2\|_{L^1(\Omega)} + 2\|f\|_{L^1(\Omega)}\big)/a_*\nu} > 0$,where $M = \|A\|_{L^\infty(\Omega)^{N^2}}$ which yields (2.20). □

Remark 2.3. *Using the trick of the above proof, if* $g_n \underset{n}{\to} g$ *in* $L^\infty(\Omega)$ *for a sequence* $g_n \in L^\infty_\nu(\Omega)$*, it is clear that, for any* $w \in \mathbb{K}^\sigma_g$ *we can choose* $w_n \in \mathbb{K}^\sigma_{g_n}$ *such that* $w_n \underset{n}{\to} w$ *in* $H^\sigma_0(\Omega)$*. On the other hand, also for any sequence* $w_n \underset{n}{\rightharpoonup} w$ *in* $H^\sigma_0(\Omega)$*-weak, with each* $w_n \in \mathbb{K}^\sigma_{g_n}$*,* $g_n \underset{n}{\to} g$ *in* $L^\infty(\Omega)$ *implies that also* $w \in \mathbb{K}^\sigma_g$*. These two conditions determine that if* $g_n \underset{n}{\to} g$ *in* $L^\infty_\nu(\Omega)$ *then the respective convex sets* $\mathbb{K}^\sigma_{g_n}$ *converge in the Mosco sense to* $\mathbb{K}^\sigma_g$*. An open question is to extend this convergence to the case* $0 < \sigma < 1$*, by dropping the strict positivity condition on* g_n *and* g*, as in* [1] *for* $\sigma = 1$*.*

3. Existence of Lagrange multipliers

In this section we prove the existence of solution of the problem (1.8a)-(1.8b).

For $\varepsilon \in (0,1)$ and denoting $\widehat{k}_\varepsilon = \widehat{k}_\varepsilon(D^\sigma u^\varepsilon) = k_\varepsilon(|D^\sigma u^\varepsilon| - g)$ for simplicity, we define a family of approximated quasi-linear problems

$$\int_\Omega \left(\widehat{k}_\varepsilon(D^\sigma u^\varepsilon) D^\sigma u^\varepsilon + A D^\sigma u^\varepsilon\right) \cdot D^\sigma v = \int_\Omega f v \qquad \forall v \in H^\sigma_0(\Omega) \tag{3.1}$$

where $k_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined by

$$k_\varepsilon(s) = 0 \text{ for } s < 0, \qquad k_\varepsilon(s) = e^{\frac{s}{\varepsilon}} - 1 \text{ for } 0 \le s \le \tfrac{1}{\varepsilon} \qquad k_\varepsilon(s) = e^{\frac{1}{\varepsilon^2}} - 1 \text{ for } s > \tfrac{1}{\varepsilon}.$$

Proposition 3.1. *Suppose that $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix. Then the quasi-linear problem* (3.1) *has a unique solution $u^\varepsilon \in H^\sigma_0(\Omega)$.*

Proof. The operator $B_\varepsilon : H^\sigma_0(\Omega) \to H^{-\sigma}(\Omega)$ defined by

$$\langle B_\varepsilon v, w\rangle = \int_\Omega \left(\widehat{k}_\varepsilon(D^\sigma v) D^\sigma v + A D^\sigma v\right) \cdot D^\sigma w$$

is bounded, strongly monotone, coercive and hemicontinuous, so problem (3.1) has a unique solution (see, for instance, [8]). □

Lemma 3.1. *If $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$, $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix and $1 \le q < \infty$, there exist positive constants C and C_q such that, for $0 < \varepsilon < 1$, setting $\widehat{k}_\varepsilon = k_\varepsilon(|D^\sigma u^\varepsilon| - g)$, the solution u^ε of the approximated problem* (3.1) *satisfies*

$$\|\widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2\|_{L^1(\Omega)} \le C, \tag{3.2a}$$

$$\|\widehat{k}_\varepsilon\|_{L^1(\Omega)} \le C, \tag{3.2b}$$

$$\|\widehat{k}_\varepsilon D^\sigma u^\varepsilon\|_{(L^\infty(\Omega)^N)'} \le C, \tag{3.2c}$$

$$\|\widehat{k}_\varepsilon\|_{L^\infty(\Omega)'} \le C \tag{3.2d}$$

$$\|D^\sigma u^\varepsilon\|_{L^q(\Omega^N)} \le C_q. \tag{3.2e}$$

Proof. Using u^ε as test function in (3.1), we get

$$\begin{aligned}\int_\Omega \left(\widehat{k}_\varepsilon + a_*\right)|D^\sigma u^\varepsilon|^2 &\le \int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + A D^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon \\ &= \int_\Omega f u^\varepsilon \le \frac{C_\#^2}{2a_*}\|f\|^2_{L^{2^\#}(\Omega)} + \frac{a_*}{2}\|D^\sigma u^\varepsilon\|^2_{L^2(\Omega)^N},\end{aligned}$$

since $A\xi\cdot\xi \geq a_*|\xi|^2$ for any $\xi\in\mathbb{R}^N$ by the assumptions on A. But $\widehat{k}_\varepsilon \geq 0$ and so

$$\frac{a_*}{2}\int_\Omega |D^\sigma u^\varepsilon|^2 \leq \frac{C_\#^{\,2}}{2a_*}\|f\|^2_{L^{2^\#}(\Omega)},$$

concluding then (3.2a).

Observing that the function $\varphi_\varepsilon = \widehat{k}_\varepsilon\,(t^2-g^2)+g^2\widehat{k}_\varepsilon \geq \nu^2\widehat{k}_\varepsilon$ and using (3.2a), there exists a positive constant C independent of ε such that

$$\nu^2\int_\Omega \widehat{k}_\varepsilon \leq C.$$

This implies the uniform boundedness of $\widehat{k}_\varepsilon$ in $L^1(\Omega)$ and also in $L^\infty(\Omega)'$, i.e., (3.2b) and (3.2d) respectively.

To prove (3.2c), it is enough to notice that, for $\boldsymbol{\beta}\in L^\infty(\Omega)^N$,

$$\begin{aligned}\|\widehat{k}_\varepsilon D^\sigma u^\varepsilon\|_{(L^\infty(\Omega)^N)'} = \sup_{\boldsymbol{\beta}\in L^\infty(\Omega)^N}\int_\Omega \widehat{k}_\varepsilon D^\sigma u^\varepsilon\cdot\boldsymbol{\beta} \qquad &\leq \Big(\int_\Omega \widehat{k}_\varepsilon|D^\sigma u^\varepsilon|^2\Big)^{\frac12}\Big(\int_\Omega \widehat{k}_\varepsilon|\boldsymbol{\beta}|^2\Big)^{\frac12}\\ &\leq C\|\boldsymbol{\beta}\|_{L^\infty(\Omega)^N}.\end{aligned}$$

Because for $t-g>0$ we have $k_\varepsilon(t-g)\geq \frac{1}{m!}(t-g)^m$, for any $m\in\mathbb{N}$, then using (3.2b) we conclude (3.2e),(for details see, for instance [10]). □

Proposition 3.2. *For $g\in L^\infty_\nu(\Omega)$, $f\in L^{2^\#}(\Omega)$ and $A:\Omega\to\mathbb{R}^{N\times N}$ a measurable, bounded and positive definite matrix, the family $\{u^\varepsilon\}_\varepsilon$ of solutions of the approximated problems* (3.1) *converges weakly in $H^\sigma_0(\Omega)$ to the solution of the variational inequality* (1.7).

Proof. The uniform boundedness of $\{u^\varepsilon\}_\varepsilon$ in $H^\sigma_0(\Omega)$ implies that, at least for a subsequence,

$$u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \quad \text{in } H^\sigma_0(\Omega). \tag{3.3}$$

For $v\in\mathbb{K}^\sigma_g$ we have, since $\widehat{k}_\varepsilon>0$ when $|D^\sigma u_\varepsilon|>g\geq|D^\sigma v|$,

$$\widehat{k}_\varepsilon D^\sigma u^\varepsilon\cdot D^\sigma(v-u^\varepsilon)\leq \widehat{k}_\varepsilon|D^\sigma u^\varepsilon|(|D^\sigma v|-|D^\sigma u^\varepsilon|)\leq 0$$

and so, testing the first equation of (3.1) with $v-u^\varepsilon$, we get

$$\int_\Omega AD^\sigma u^\varepsilon\cdot D^\sigma(v-u^\varepsilon)\geq\int_\Omega f(v-u^\varepsilon).$$

But

$$\begin{aligned}\int_\Omega AD^\sigma u^\varepsilon\cdot D^\sigma(v-u^\varepsilon) &= \int_\Omega AD^\sigma(u^\varepsilon-v)\cdot D^\sigma(v-u^\varepsilon)+\int_\Omega AD^\sigma v\cdot D^\sigma(v-u^\varepsilon)\\ &\leq \int_\Omega AD^\sigma v\cdot D^\sigma(v-u^\varepsilon)\end{aligned}$$

So, utilizing the weak convergence $u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u$ in $H^\sigma_0(\Omega)$,

$$\int_\Omega AD^\sigma v \cdot D^\sigma(v-u) \geq \int_\Omega f(v-u).$$

Let $w \in \mathbb{K}^\sigma_g$ and setting $v = u + \theta(w-u)$, then $v \in \mathbb{K}^\sigma_g$ for any $\theta \in (0,1]$ and we get

$$\theta \int_\Omega AD^\sigma(u+\theta(w-u)) \cdot D^\sigma(w-u) \geq \theta \int_\Omega f(w-u).$$

Dividing this inequality by θ and letting $\theta \to 0$, we obtain (1.7). The proof is concluded if we show that $u \in \mathbb{K}^\sigma_g$. Indeed we split Ω in three subsets

$$U_\varepsilon = \big\{|D^\sigma u^\varepsilon| - g \leq \sqrt{\varepsilon}\big\}, \quad V_\varepsilon = \big\{\sqrt{\varepsilon} \leq |D^\sigma u^\varepsilon| - g \leq \tfrac{1}{\varepsilon}\big\}, \quad W_\varepsilon = \big\{|D^\sigma u^\varepsilon| - g > \tfrac{1}{\varepsilon}\big\}$$

and, following the steps in [10], we conclude that

$$\begin{aligned}\int_\Omega \big(|D^\sigma u| - g\big)^+ &\leq \varliminf_{\varepsilon\to 0} \int_\Omega \big(\big(|D^\sigma u^\varepsilon| - g\big) \vee 0\big) \wedge \frac{1}{\varepsilon} \\ &= \varliminf_{\varepsilon\to 0} \bigg(\int_{U_\varepsilon} \big(|D^\sigma u^\varepsilon| - g\big) \vee 0 + \int_{V_\varepsilon} \big(|D^\sigma u^\varepsilon| - g\big) + \int_{W_\varepsilon} \tfrac{1}{\varepsilon}\bigg) \\ &\leq \varliminf_{\varepsilon\to 0} \bigg(\sqrt{\varepsilon}|\Omega| + \||D^\sigma u^\varepsilon| - g\|_{L^2(\Omega)}\, |V_\varepsilon|^{\frac{1}{2}} + \int_{W_\varepsilon} \tfrac{1}{\varepsilon}\bigg) \underset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

because

$$|V_\varepsilon| \leq \int_{V_\varepsilon} \frac{\widehat{k}_\varepsilon + 1}{e^{\frac{1}{\sqrt{\varepsilon}}}} \leq Ce^{\frac{-1}{\sqrt{\varepsilon}}} \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{and} \quad \int_{W_\varepsilon} \tfrac{1}{\varepsilon} = \tfrac{1}{\varepsilon} \int_{W_\varepsilon} \frac{\widehat{k}_\varepsilon + 1}{e^{\frac{1}{\varepsilon^2}}} \leq \tfrac{C}{\varepsilon} e^{-\frac{1}{\varepsilon^2}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

So $|D^\sigma u| \leq g$ a.e. in Ω, which means that $u \in \mathbb{K}^\sigma_g$.

The uniqueness of solution of the variational inequality (1.7) implies that the whole sequence $\{u^\varepsilon\}_\varepsilon$ converges to u in $H^\sigma_0(\Omega)$. ☐

Theorem 3.1. *If $g \in L^\infty_\nu(\Omega)$, $f \in L^{2^\#}(\Omega)$ and $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable, bounded and positive definite matrix, then problem* (1.8a)-(1.8b) *has a solution*

$$(\lambda, u) \in L^\infty(\Omega)' \times \Upsilon^\sigma_\infty(\Omega).$$

Proof. By estimates (3.2c) and (3.2d) and the Banach-Alaoglu-Bourbaki theorem we have, at least for a subsequence,

$$\widehat{k}_\varepsilon D^\sigma u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \Lambda \text{ weak in } \big(L^\infty(\Omega)^N\big)'$$

and

$$\widehat{k}_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \lambda \text{ weak in } L^\infty(\Omega)'.$$

For $v \in H^\sigma_0(\Omega)$, since

$$\int_\Omega \big(\widehat{k}_\varepsilon D^\sigma u^\varepsilon + AD^\sigma u^\varepsilon\big) \cdot D^\sigma v = \int_\Omega fv, \tag{3.4}$$

we obtain, letting $\varepsilon \to 0$ with $v \in \Upsilon^\sigma_\infty(\Omega)$,

$$\langle \Lambda, D^\sigma v\rangle + \int_\Omega AD^\sigma u \cdot D^\sigma v = \int_\Omega fv. \tag{3.5}$$

Taking $v = u^\varepsilon$ in (3.4) we get

$$\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon = \int_\Omega fu^\varepsilon \tag{3.6}$$

Observe first that

$$\begin{aligned} \int_\Omega AD^\sigma(u^\varepsilon - u)\cdot D^\sigma u^\varepsilon &= \int_\Omega AD^\sigma(u^\varepsilon - u)\cdot D^\sigma(u^\varepsilon - u) \\ &\quad + \int_\Omega AD^\sigma(u^\varepsilon - u)\cdot D^\sigma u \geq \int_\Omega AD^\sigma(u^\varepsilon - u)\cdot D^\sigma u \end{aligned} \tag{3.7}$$

and therefore

$$\int_\Omega AD^\sigma u \cdot D^\sigma u \leq \varliminf_{\varepsilon\to 0} \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon.$$

So, from(3.6) and (3.5) with $v = u$,

$$\begin{aligned} \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u\cdot D^\sigma u &\leq \varliminf_{\varepsilon\to 0}\Big(\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 + \int_\Omega AD^\sigma u^\varepsilon \cdot D^\sigma u^\varepsilon\Big) \\ &= \int_\Omega fu = \langle \Lambda, D^\sigma u\rangle + \int_\Omega AD^\sigma u\cdot D^\sigma u \end{aligned}$$

and then

$$\varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 \leq \langle \Lambda, D^\sigma u\rangle.$$

Using $\widehat{k}_\varepsilon(|D^\sigma u^\varepsilon|^2 - g^2) \geq 0$, we obtain

$$\langle \Lambda, D^\sigma u\rangle \geq \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 \geq \lim_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon g^2 = \langle \lambda, g^2\rangle \geq \langle \lambda, |D^\sigma u|^2\rangle.$$

We also have

$$\begin{aligned} 0 \leq \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2 &= \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 - 2\lim_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon D^\sigma u^\varepsilon\cdot D^\sigma u + \lim_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u|^2 \\ &\leq \langle \Lambda, D^\sigma u\rangle - 2\langle \Lambda, D^\sigma u\rangle + \langle \lambda, |D^\sigma u|^2\rangle \\ &= -\langle \Lambda, D^\sigma u\rangle + \langle \lambda, |D^\sigma u|^2\rangle, \end{aligned}$$

and therefore we conclude

$$\langle \Lambda, D^\sigma u\rangle = \langle \lambda, |D^\sigma u|^2\rangle \quad \text{and} \quad \varliminf_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon - u)|^2 = 0.$$

Given $v \in \mathbb{K}_g$, we have

$$(3.8)\quad \varlimsup_{\varepsilon\to 0}\Big|\int_\Omega \widehat{k}_\varepsilon D^\sigma(u^\varepsilon - u)\cdot D^\sigma v\Big| \\ \leq \varlimsup_{\varepsilon\to 0}\left(\int_\Omega \widehat{k}_\varepsilon |D^\sigma(u^\varepsilon-u)|^2\right)^{\frac12}\|\widehat{k}_\varepsilon\|^{\frac12}_{L^1(\Omega)}\|D^\sigma v\|_{L^\infty(\Omega)} = 0,$$

because, by estimate (3.2b), $\widehat{k}_\varepsilon$ is uniformly bounded in $L^1(\Omega)$. So, for any $v \in \mathbb{K}_g$,

$$\int_\Omega fv = \varlimsup_{\varepsilon\to 0}\int_\Omega(\widehat{k}_\varepsilon + A)D^\sigma u^\varepsilon\cdot D^\sigma v = \varlimsup_{\varepsilon\to 0}\Big(\int_\Omega(\widehat{k}_\varepsilon + A)D^\sigma(u^\varepsilon-u)\cdot D^\sigma v \\ + \lim_{\varepsilon\to 0}\int_\Omega(\widehat{k}_\varepsilon + A)D^\sigma u\cdot D^\sigma v\Big) = \langle \lambda D^\sigma u, D^\sigma v\rangle + \int_\Omega A D^\sigma u\cdot D^\sigma v,$$

concluding the proof of (1.8a).

Since $\displaystyle\int_\Omega \widehat{k}_\varepsilon v \geq 0$ for all $v \in L^\infty(\Omega)$ such that $v \geq 0$ then, for such v, we also have $\langle\lambda, v\rangle \geq 0$, which means that $\lambda \geq 0$.

For $v \in L^\infty(\Omega)$ set $v^+ = \max\{v, 0\}$, $v^- = (-v)^+$. Since $\widehat{k}_\varepsilon(|D^\sigma u^\varepsilon|^2 - g^2) \geq 0$ then

$$\begin{aligned}\langle\lambda, g^2\, v^\pm\rangle &\leq \varlimsup_{\varepsilon\to 0}\int_\Omega \widehat{k}_\varepsilon |D^\sigma u^\varepsilon|^2 v^\pm \\ &= \varlimsup_{\varepsilon\to 0}\left(\int_\Omega \widehat{k}_\varepsilon|D^\sigma(u^\varepsilon-u)|^2 v^\pm - 2\int_\Omega \widehat{k}_\varepsilon D^\sigma(u^\varepsilon-u)\cdot D^\sigma u v^\pm + \int_\Omega \widehat{k}_\varepsilon |D^\sigma u|^2 v^\pm\right) \\ &= \langle\lambda, |D^\sigma u|^2\, v^\pm\rangle, \quad \text{using (3.8)},\end{aligned}$$

concluding that

$$\langle\lambda, (|D^\sigma u|^2 - g^2)\, v^\pm\rangle \geq 0.$$

The fact that $\widehat{k}_\varepsilon \geq 0$ and $u \in \mathbb{K}^\sigma_g$ imply $\widehat{k}_\varepsilon(|D^\sigma u|^2 - g^2)v^\pm \leq 0$ and, therefore, integrating and letting $\varepsilon \to 0$, $\langle\lambda, (D^\sigma u|^2 - g^2)\, v^\pm\rangle \leq 0$, and so

$$\langle\lambda, (|D^\sigma u|^2 - g^2)\, v\rangle = 0.$$

Writting $v = \frac{w}{|D^\sigma u| + g}$, for any $w \in L^\infty(\Omega)$, we conclude (1.8b). □

4. The quasi-variational inequality with σ-gradient constraint

In this section we consider a map G such that

$$(4.1)\qquad G : L^{2^*}(\Omega) \to L^\infty_\nu(\Omega)$$

is a continuous and bounded operator, where 2^* is the Sobolev exponent as in (2.5) for $0 < \sigma < 1$.

We set

$$(4.2)\qquad \mathbb{K}^\sigma_{G[u]} = \big\{v \in H^\sigma_0(\Omega) : |D^\sigma v| \leq G[u] \text{ a.e. in } \Omega\big\}$$

and we shall consider the quasi-variational inequality

$$(4.3)\qquad u\in\mathbb{K}^\sigma_{G[u]}:\qquad \int_\Omega AD^\sigma u\cdot D^\sigma(v-u)\geq\int_\Omega f(v-u),\qquad \forall v\in\mathbb{K}^\sigma_{G[u]}.$$

Generalising a compactness argument of [6] where quasi-variational inequalities of this type were considered for the gradient case $\sigma=1$, we may give a general existence theorem.

Theorem 4.1. *Under the assumptions* (2.1)*, for continuous and bounded operators G satisfying* (4.1) *and for any $f\in L^{2^\#}(\Omega)$, with $2^\#$ as in* (2.5)*, there exists at least one solution for the quasi-variational inequality* (4.3)*.*

Proof. Let $u=S(f,g)$ be the unique solution of the variational inequality (1.7) with $g=G[w]$ for any $w\in L^{2^*}(\Omega)$. If $C_*>0$ denotes the Sobolev constant as in Theorem 2.1, since $f_2=0$ corresponds always to the solution $u_2=0$, we have the a priori estimate

$$(4.4)\qquad \|u\|_{L^{2^*}(\Omega)}\leq C_*\|u\|_{H^\sigma_0(\Omega)}\leq \tfrac{C_*}{a_*}\|f\|_{L^{2^\#}(\Omega)}\equiv c_f,$$

independently of $g\in L^\infty_\nu(\Omega)$.

Set $B_{c_f}=\big\{v\in L^{2^*}(\Omega):\|v\|_{L^{2^*}(\Omega)}\leq c_f\big\}$ and define the nonlinear map $T=S\circ G:L^{2^*}(\Omega)\ni w\mapsto u\in L^{2^*}(\Omega)$ where $u=S(f,G[w])\in\mathbb{K}^\sigma_{G[w]}\cap\mathscr{C}^{0,\beta}(\overline{\Omega})$, $0<\beta<\sigma$ by (2.16).

Clearly, (4.4) implies $T(B_{c_f})\subset B_{c_f}$ and, by the continuity of G and Theorem 2.2, T is also a continuous map. On the other hand, G is bounded, i.e. transforms bounded sets in $L^{2^*}(\Omega)$ into bounded sets of $L^\infty_\nu(\Omega)$ and $S\circ T$ is also a bounded operator. Therefore, by (2.16), $T(B_{c_f})$ is also a bounded set of $C^{0,\beta}(\overline{\Omega})$. Since the embedding $C^{0,\beta}(\overline{\Omega})\hookrightarrow L^{2^*}(\Omega)$ is compact, the Schauder fixed point theorem guarantees the existence of $u=Tu$, which solves (4.3). □

Example 4.1. *Consider the operator $G:L^{2^*}(\Omega)\to L^\infty_\nu(\Omega)$ defined as follows:*

$$(4.5)\qquad G[u](x)=F(x,w(x)),$$

where $F:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function bounded in $x\in\Omega$ and continuous in $w\in\mathbb{R}$, uniformly in $x\in\Omega$, satisfying, for some $\nu>0$,

$$(4.6)\qquad 0<\nu\leq F(x,w)\leq\varphi(|w|)\qquad a.e.\ x\in\Omega,$$

and for some monotone increasing function φ. We may choose

$$(4.7)\qquad w(x)=\int_\Omega\vartheta(x,y)u(y)\,dy,$$

where we give $\vartheta\in L^\infty\big(\Omega_x;L^{2^\#}(\Omega_y)\big)$. For $u_n\underset{n}{\to}u$ in $L^{2^*}(\Omega)$, from the estimate

$$\sup_{x\in\Omega}|w_n(x)-w(x)|=\sup_{x\in\Omega}\left|\int_\Omega\vartheta(x,y)(u_n(y)-u(y))dy\right|\leq\sup_{x\in\Omega}\|\vartheta(x,\cdot)\|_{L^{2\#}(\Omega)}\|u_n-u\|_{L^{2^*}(\Omega)}$$

and by the uniform continuity of F, we have

$$\|G[u_n] - G[u]\|_{L^\infty(\Omega)} = \|F(w_n) - F(w)\|_{L^\infty(\Omega)} \underset{n}{\to} 0,$$

implying the continuity of G.

The boundedness of G is a consequence of (4.6) *and therefore G satisfies the assumptions of Theorem 4.1.*

Example 4.2. *Consider now the operator $G : H_0^\sigma(\Omega) \to L_\nu^\infty(\Omega)$ given also by* (4.5) *with F under the same assumptions as in the previous example, but now with*

$$w(x) = \Phi(u)(x) = \int_\Omega \Theta(x, y) \cdot D^\sigma u(y) dy, \tag{4.8}$$

where $\Theta \in \mathscr{C}\big(\overline{\Omega}_x; L^2(\Omega_y)^N\big)$. Now G is not only bounded but also completely continuous, since $\Phi : H_0^\sigma(\Omega) \to \mathscr{C}^0(\overline{\Omega})$ is also completely continuous. Indeed, if $u_n \underset{n}{\rightharpoonup} u$ in $H_0^\sigma(\Omega)$-weak, then $w_n = \Phi(u_n) \underset{n}{\to} \Phi(u) = w$ in $\mathscr{C}(\overline{\Omega})$, because $\{D^\sigma u_n\}_n$, being bounded in $L^2(\Omega)^N$ implies $\{w_n\}_n$ uniformly bounded in $\mathscr{C}^0(\overline{\Omega})$,

$$|w_n(x)| \leq \|\Theta(x, \cdot)\|_{L^2(\Omega)^N} \|D^\sigma u_n\|_{L^2(\Omega)^N}, \qquad \forall x \in \overline{\Omega}$$

and also equicontinuous in $\overline{\Omega}$ by

$$|w_n(x) - w_n(z)| \leq C\|\Theta(x, \cdot) - \Theta(z, \cdot)\|_{L^2(\Omega)^N}.$$

But G is not defined in the whole $L^{2^*}(\Omega)$ and therefore we cannot apply Theorem 4.1 to solve (4.3). Nevertheless, the solvability of (4.3) in this example is an immediate consequence of the following theorem.

Theorem 4.2. *Assume* (2.1) *and let $f \in L^{2^\#}(\Omega)$ as previously. If the nonlinear and nonlocal operator G satisfies*

$$G : H_0^\sigma(\Omega) \to L_\nu^\infty(\Omega) \quad \text{is bounded and completely continuous} \tag{4.9}$$

then there exists a solution u to the quasi-variational inequality (4.3).

Proof. Due to the estimate (4.4) and the assumption (4.9), the proof is analogous by applying the Schauder fixed point theorem to the nonlinear completely continuous map

$$T = S \circ G : H_0^\sigma(\Omega) \ni w \mapsto u = S(f, G[w]) \in H_0^\sigma(\Omega).$$

□

Example 4.3. *By restricting the domain of G and using the same type of Carathéodory function F as in Example 4.1, we can introduce the superposition operator*

$$G[u](x) = F(x, u(x)), \qquad u \in \mathscr{C}^0(\overline{\Omega}),\ x \in \Omega. \tag{4.10}$$

In order to guarantee that $G : \mathscr{C}(\overline{\Omega}) \to L^\infty_\nu(\Omega)$ *is a continuous and bounded operator in an appropriate space to obtain a fixed point, we need to require that the function* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a bounded function in* $x \in \Omega$ *in each compact for the variable* u*, continuous in* $u \in \mathbb{R}$ *uniformly in* $x \in \Omega$*, and satisfying* (4.6)*, where the monotone increasing function* φ *satisfies*

$$0 < \nu \leq \varphi(t) \leq C_0 + C_1 t^{2^*/p}, \qquad t \in \mathbb{R}, \tag{4.11}$$

for some $p > \frac{N}{\sigma}$ *and* 2^* *the Sobolev exponent as in* (2.5)*.*

This situation is covered by the next theorem, since the assumption (4.11) *implies the condition* (4.13) *below.*

Theorem 4.3. *Assume* (2.1)*, let* $f \in L^{2^\#}(\Omega)$ *and the functional* G *be such that*

$$G : \mathscr{C}^0(\overline{\Omega}) \to L^\infty_\nu(\Omega) \qquad \text{is a continuous operator} \tag{4.12}$$

and satisfying, for some positive monotone increasing function η*,*

$$\|G[w]\|_{L^p(\Omega)} \leq \eta\big(\|w\|_{L^{2^*}(\Omega)}\big) \tag{4.13}$$

for some $p > \frac{N}{\sigma}$ *and* 2^* *the Sobolev exponent of* $H^\sigma_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$*. Then there exists a solution of the quasi-variational inequality* (4.3)*.*

Proof. As before, we set $T = S \circ G : \mathscr{C}^0(\overline{\Omega}) \to H^\sigma_0(\Omega)$, where $u = S(f, G[w])$, for $w \in \mathscr{C}^0(\overline{\Omega})$ solves (1.7) with $g = G[w]$.

In order to apply the Leray-Schauder principle, we set

$$\mathscr{S} = \big\{ w \in \mathscr{C}^0(\overline{\Omega}) : w = \theta T w,\, \theta \in [0,1] \big\}$$

and we show that $\mathscr{S}$ is a priori bounded. For any $w \in \mathscr{S}$, $u = Tw$ solves (1.7) with $g = G[w]$. Hence, by (2.4c) and the assumption (4.13) we have, noting that $w = \theta u$,

$$\begin{aligned} \|w\|_{\mathscr{C}^0(\overline{\Omega})} &\leq C_\sigma \|D^\sigma w\|_{L^p(\Omega)^N} \leq C_\sigma \theta \|G[w]\|_{L^p(\Omega)^N} \\ &\leq C_\sigma \eta\big(\|w\|_{L^{2^*}(\Omega)}\big) \leq C_\sigma \eta(c_f), \end{aligned}$$

by the a priori estimate (4.4).

Since, by (2.3), $T(\mathscr{C}^0(\overline{\Omega})) \hookrightarrow \mathscr{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow \mathscr{C}^0(\overline{\Omega})$ and this last embedding is compact, we may conclude that T is a completely continuous mapping into a closed ball of $\mathscr{C}^0(\overline{\Omega})$ and its fixed point $u = Tu$ solves (4.3). □

It is clear that in general we cannot expect the uniqueness of solution to quasi-variational inequalities of the type (4.3). However, the Lipschitz continuity of the solution map $f \mapsto u$ to the variational inequality (1.7), given by Theorem 2.1, allows us to obtain, via the strict contraction Banach fixed point principle, a uniqueness result in a special case of "small" and controlled variations of the convex sets for the quasi-variational situation with separation of variables in the nonlocal constraint G.

We denote, for $R > 0$,

$$B_R = \{v \in H^\sigma_0(\Omega) : \|v\|_{H^\sigma_0(\Omega)} \leq R\}.$$

Theorem 4.4. *Let $f \in L^{2^\#}(\Omega)$, $\varphi \in L^\infty_\nu(\Omega)$ and*

$$G[u](x) = \varphi(x)\Gamma(u), \qquad x \in \Omega, \tag{4.14}$$

where $\Gamma : H^\sigma_0(\Omega) \to \mathbb{R}^+$ is a functional satisfying

i) $0 < \eta(R) \leq \Gamma(u) \leq E(R), \quad \forall u \in B_R,$

ii) $|\Gamma(u_1) - \Gamma(u_2)| \leq \gamma(R)\|u_1 - u_2\|_{H^\sigma_0(\Omega)}, \quad \forall u_1, u_2 \in B_R,$

for sufficiently large $R \in \mathbb{R}^+$, with η, E and γ being monotone increasing positive functions of R.

Then the quasi-variational inequality (4.3) *has a unique solution, provided*

$$2C_\# \frac{\gamma(R_f)}{\eta(R_f)} \|f\|_{L^{2^\#}(\Omega)} < 1, \tag{4.15}$$

where $R_f \equiv C_\#\|f\|_{L^{2^\#}(\Omega)}$ with $C_\# = C_*/a_*$ and C_* is the constant of the Sobolev embedding as in (4.4).

Proof. Let $S : B_R \ni v \mapsto u \in H^\sigma_0(\Omega)$ be the solution map with $u = S(f, G[v])$ being the unique solution of the variational inequality (1.7) with $g = G[v]$.

The a priori estimate (4.4) implies $S(B_{R_f}) \subset B_{R_f}$.

Given $v_i \in B_R$, let $u_i = S(v_i) = S(f, \varphi\,\Gamma(v_i))$, $i = 1, 2$, and choose $\mu = \frac{\Gamma(v_2)}{\Gamma(v_1)} > 1$, without loss of generality.

Setting $g = \varphi\,\Gamma(v_1)$, we have $\mu\, g = \varphi\,\Gamma(v_2)$ and

$$S(\mu\, f, \mu\, g) = \mu S(f, g),$$

$$\mu - 1 = \frac{\Gamma(v_2) - \Gamma(v_1)}{\Gamma(v_1)} \leq \frac{\gamma(R_f)}{\eta(R_f)}\|v_1 - v_2\|_\sigma$$

by recalling the assumptions i) and ii) and denoting $\|w\|_\sigma = \|w\|_{H^\sigma_0(\Omega)}$ for simplicity.

Consequently, using (4.4) and (2.18) with $f_1 = f$ and $f_2 = \mu\, f$, we have

$$\begin{aligned}
\|S(v_1) - S(v_2)\|_\sigma &\leq \|S(f, g) - S(\mu f, \mu g)\|_\sigma + \|S(\mu f, \mu g) - S(f, \mu g)\|_\sigma \\
&\leq (\mu - 1)\|u_1\|_\sigma + (\mu - 1)C_\#\|f\|_{L^{2^\#}(\Omega)} \\
&\leq 2C_\#(\mu - 1)\|f\|_{L^{2^\#}(\Omega)} \\
&\leq 2C_\# \frac{\gamma(R_f)}{\eta(R_f)}\|v_1 - v_2\|_\sigma\|f\|_{L^{2^\#}(\Omega)}
\end{aligned}$$

and the conclusion of the theorem follows immediately. □

Example 4.4. *We can take* Γ *of the form*

$$\Gamma(u) = \int_\Omega e(y, u(y), D^\sigma u(y))\, dy, \quad u \in H_0^\sigma(\Omega),$$

with $e : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [\eta, \infty)$, *for some* $\eta > 0$, *under a local Lipschitz condition of the type*

$$|e(y, v, \xi) - e(y, w, \zeta)| \leq \gamma(R)\big(|v - w| + |\xi - \eta|\big)$$

for $|v|$, $|w|$, $|\xi|$ *and* $|\zeta|$ *less or equal to* R.

Remark 4.1. *Assumptions* i) *and* ii) *have been used in Appendiz B of* [5] *under the implicit assumptions of smallness of the term* f, *and in* [12] *in a simplified and more precise form in the case of gradient type (i.e.* $\sigma = 1$*) and for a class of general operators of p-Laplacian type.*

Remark 4.2. *The existence of solution of the quasi-variational inequality* (4.3) *is obtained in this section by finding a fixed point of the map* $w \mapsto S(f, G[w]) = u$, *under suitable assumptions. But when* $u = S(f, G[w])$ *is the solution of* (1.7) *then there exists* $\lambda \in L^\infty(\Omega)'$ *such that* (u, λ) *solves problem* (1.8a)-(1.8b) *with data* $(f, G[w])$. *In particular, when* u *is a fixed point* $u = S(f, G[u])$ *it solves the quasi-variational inequality, and we immediately get existence of a solution* (λ, u) *of problem* (1.8a)-(1.8b) *for the quasi-variational case.*

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