On a class of nonsingular matrices containing B -matrices *

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Abstract

A real matrix with positive row sums and all its off-diagonal elements bounded above by their corresponding row means was called in [12] a B -matrix. The classes of doubly B -matrices and of B_π^R -matrices were introduced in [13] and [11] as generalizations of the previous class. We present characterizations and properties of B_π^R -matrices and we consider the problem of when a subdirect sum of two matrices in this class lies in the same class.

Key words: B -matrix, B_π^R -matrix, P -matrix, subdirect sum

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1 Introduction

A square real matrix $A = (a_{ij})_{i,j=1}^n$ with positive row sums is a B -matrix if all its off-diagonal elements are bounded above by the corresponding row means (see [12]), that is, for all $i \in \{1, \dots, n\}$,

$$\sum_{j=1}^n a_{ij} > 0$$

and

$$a_{ij} < \frac{1}{n} \sum_{k=1}^n a_{ik}, \forall j \neq i.$$

In [5] it was proved that these matrices have positive determinants and the author provided a first application to the localization of the real eigenvalues of a real matrix. In [12] the author proved that the class of B -matrices is a subset of the class of P -matrices (for definition and fundamental properties, see [6]) and applied this property to the localization of the real parts of all eigenvalues of a real matrix.

Let $\mathcal{Z}_n$ stand for the set of real square matrices of order n whose off-diagonal entries are nonpositive, that is $\mathcal{Z}_n = \{A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R}) : a_{ij} \leq 0 \text{ if } i \neq j, i, j = 1, \dots, n\}$. The following characterization of B -matrices was presented by Peña in [12].

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Proposition 1.1. *Let A be a matrix in $\mathcal{Z}_n$. Then the following properties are equivalent:*

- (1.) *A is a B -matrix.*
- (2.) *The row sums of A are positive.*
- (3.) *A is strictly diagonally dominant by rows with positive diagonal entries.*

In [13], the author defined another class of matrices, the *doubly B -matrices*, containing the B -matrices: a square real matrix $A = (a_{ij})_{i,j=1}^n$ is a doubly B -matrix if, for all i ,

$$a_{ii} > r_i$$

and, for all $i \neq j$ in $\{1, \dots, n\}$,

$$(a_{ii} - r_i)(a_{jj} - r_j) > \sum_{k \neq i} (r_i - a_{ik}) \sum_{k \neq j} (r_j - a_{jk}),$$

where $r_i = \max\{0, a_{ij} \mid j \neq i\}$ ($i = 1, \dots, n$). In that same paper, it was shown that doubly B -matrices are also P -matrices.

In [11], Neumann *et al.*, motivated by conditions that arise from results on mean first passage times matrices in Markov chains, considered the so called class of B_π^R -matrices, which generalizes the class of B -matrices. Applications of B -matrices and other related classes of nonsingular matrices can be found in [9], [11], [12] and [13].

Given a vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying

$$0 < \sum_{j=1}^n \pi_j \leq 1, \tag{1}$$

we say that a square real matrix $A = (a_{ij})_{i,j=1}^n$, with positive row sums, is a B_π^R -matrix if, for all $i = 1, \dots, n$,

$$a_{ij} < \pi_j R_i, \forall j \neq i,$$

where $R = (R_1, \dots, R_n)$ is the vector formed by the corresponding row sums of A .

Notice that when $\pi_j = 1/n$, for all j , then the previous definition coincides with that of a B -matrix. However, not every B_π^R -matrix is a B -matrix. Take for instance matrix $A = (a_{ij})_{i,j=1}^3$, whose entries are all equal to 1 except for $a_{11} = 5$ and $a_{22} = 2$. A is not a B -matrix nor a doubly B -matrix, but A is a B_π^R -matrix for any real vector $\pi = (\pi_1, \pi_2, \pi_3)$ such that $\pi_1 > 1/3$, $\pi_2 > 1/4$, $\pi_3 > 1/3$ and $\pi_1 + \pi_2 + \pi_3 \leq 1$.

In [11], the authors present the following characterization of B_π^R -matrices.

Lemma 1.2. *Let $A = (a_{ij})_{i,j=1}^n$ be a square real matrix with positive row sums and let $R = (R_1, \dots, R_n)$ be the vector whose entries are the corresponding row sums of A . There exists a vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition (1) such that A is a B_π^R -matrix if and only if*

$$\sum_{j=1}^n \max_{i \neq j} \left(\frac{a_{ij}}{R_i} \right) < 1.$$

In the mentioned work by Neumann *et al.* it is also stated that every B_π^R -matrix is a P -matrix.

In this paper, given a vector $R = (R_1, \dots, R_n)$ of real entries, the notation $R > 0$ (respectively, $R \geq 0$) means that all entries R_i are positive (respectively, nonnegative). Given two vectors $R = (R_1, \dots, R_n)$ and $S = (S_1, \dots, S_n)$, we write $R > S$ (respectively, $R \geq S$) if $R - S > 0$ (respectively, $R - S \geq 0$).

In [10] the authors presented several characterizations and properties of B -matrices and doubly B -matrices and analyzed the question whether subdirect sums (a general ‘sum’ of matrices introduced in [4] by S.M. Fallat and C.R. Johnson, of which the direct sum and ordinary sum are special cases) of B -matrices lie in the same class or not. The concept of subdirect sum arises naturally in several contexts, such as in matrix completion problems, overlapping subdomains in domain decomposition methods and global stiffness matrices in finite elements (see [1], [2], [3], [4], [7], [8]). Here we consider analogous corresponding questions for the class of B_π^R -matrices.

2 Some properties of B_π^R -matrices

As it was mentioned in the previous section, given real vectors π and R satisfying condition (1) and $R > 0$, all principal minors of a B_π^R -matrix are positive. In particular, all diagonal entries of a B_π^R -matrix are positive. In the following result, we present other lower bounds for the diagonal entries of such matrices and relate the signs of the entry π_i of vector π and of the non diagonal entries of column i of matrix A .

Proposition 2.1. *Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1) and let $A = (a_{ij})_{i,j=1}^n$ be a B_π^R -matrix, where $R = (R_1, \dots, R_n)$ is the vector whose entries are the corresponding row sums of A . The following hold*

- (1.) $a_{ii} > \pi_i R_i$, for all $i \in \{1, \dots, n\}$;
- (2.) If $\pi_i \geq \pi_j$, then $a_{ii} > a_{ij}$, for all $i, j \in \{1, \dots, n\}$, $i \neq j$;
- (3.) $a_{kk} > a_{kj}$, for all $j \neq k$, where $\pi_k = \max_i \{\pi_i\}$;
- (4.) If $\pi_i \leq 0$, then $a_{ji} < 0$, for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

Proof. (1.) For each $i \in \{1, \dots, n\}$,

$$\begin{aligned} (a_{i1} - \pi_1 R_i) + \dots + (a_{ii} - \pi_i R_i) + \dots + (a_{in} - \pi_n R_i) &= (a_{i1} + \dots + a_{ii} + \dots + a_{in}) - \left(\sum_{j=1}^n \pi_j \right) R_i \\ &= \left(1 - \sum_{j=1}^n \pi_j \right) R_i \\ &\geq 0. \end{aligned}$$

Since, for all $j \neq i$,

$$a_{ij} - \pi_j R_i < 0,$$

it follows that $a_{ii} - \pi_i R_i > 0$.

(2.) Assume that $\pi_i \geq \pi_j$. Given that $R_i > 0$, $\pi_i R_i \geq \pi_j R_i$. Being A a B_π^R -matrix, it follows that $a_{ij} < \pi_j R_i \leq \pi_i R_i < a_{ii}$.

(3.) Immediate from (2.).

(4.) Suppose $\pi_i \leq 0$ for some $i \in \{1, \dots, n\}$. We know, by the definition of B_π^R -matrix, that $a_{ji} < \pi_i R_j \leq 0$, for all $j \in \{1, \dots, n\}$, $j \neq i$. Therefore, if $\pi_i \leq 0$, column i of A has all non diagonal entries negative. $\square$

From the previous result, we can easily derive the following.

Corollary 2.2. *Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1), with $\pi_1 = \pi_2 = \dots = \pi_n = r$, and let $A = (a_{ij})_{i,j=1}^n$ be a B_π^R -matrix, where $R = (R_1, \dots, R_n)$ is the vector whose entries are the corresponding row sums of A . Then the following hold:*

(1.) $a_{ii} > a_{ij}$, for all $i, j \in \{1, \dots, n\}$, $i \neq j$;

(2.) $\text{tr } A > r \sum_{i=1}^n R_i$

As it was mentioned in the previous section, a B -matrix of order n is a B_π^R matrix for the corresponding vector R of row sums and for vector $\pi = (1/n, \dots, 1/n)$. Taking this and Proposition 1.1 into account, what happens in the context of $\mathcal{Z}_n$ is the question that arises naturally.

Theorem 2.3. *Let A be a matrix in $\mathcal{Z}_n$ and let $\pi = (\pi_1, \dots, \pi_n) > 0$ be a real vector satisfying condition (1). Then the following properties are equivalent:*

(1.) A is a B_π^R -matrix.

(2.) The row sums of A are positive.

(3.) A is strictly diagonally dominant by rows with positive diagonal entries.

(4.) A is a B -matrix.

Proof. Let $A = (a_{ij})_{i,j=1}^n$ and let $R = (R_1, \dots, R_n)$ be the vector formed by the corresponding row sums of A . Since $A \in \mathcal{Z}_n$, $a_{i,j} \leq 0$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. If the row sums of A are positive, it is obvious that, for all $i, j \in \{1, \dots, n\}$, $i \neq j$, $\pi_j R_i > 0 \geq a_{ij}$. Hence, A is a B_π^R -matrix. The rest of the equivalences follow from Proposition 1.1. $\square$

Corollary 2.4. *Let D be a diagonal matrix of order n whose i th diagonal entry is d_i , $i = 1, \dots, n$. Let $R = (d_1, \dots, d_n)$ and let $\pi = (\pi_1, \dots, \pi_n) > 0$ be a real vector satisfying condition (1). D is a B_π^R -matrix if and only if $R > 0$.*

The definition of B_π^R -matrices and the characterization given in Lemma 1.2 leads us to the question of what conditions on a vector α guarantee that a given B_π^R -matrix A is also a B_α^R -matrix.

Proposition 2.5. *Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1) and let A be a B_π^R -matrix, where $R = (R_1, \dots, R_n)$ is the vector whose entries are the corresponding row sums of A . If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a real vector satisfying*

$$0 < \sum_{j=1}^n \alpha_j \leq 1$$

and $\alpha \geq \pi$, then A is a B_α^R -matrix.

Next we show that permutation similarity preserves the property of existing a vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition (1) such that a matrix is a B_π^R -matrix, where R is the vector of the corresponding row sums. $P = \text{perm}(i_1, i_2, \dots, i_n)$, where $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, will stand for the permutation matrix $P = (p_{kj})_{k,j=1}^n$ of order n defined by $p_{kj} = 1$ if $k \in \{1, \dots, n\}$ and $j = i_k$.

Proposition 2.6. *Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1) and let A be a B_π^R -matrix, where $R = (R_1, \dots, R_n)$ is the vector whose entries are the corresponding row sums of A . Let $P = \text{perm}(i_1, i_2, \dots, i_n)$ be a permutation matrix of order n , $\alpha = (\pi_{i_1}, \dots, \pi_{i_n})$ and $Q = (R_{i_1}, \dots, R_{i_n})$. Then PAP^T is a B_α^Q -matrix.*

Proof. Let $\alpha_k = \pi_{i_k}$ and $Q_k = R_{i_k}$ for all $k \in \{1, \dots, n\}$, let $A = (a_{ij})_{i,j=1}^n$ and let $PAP^T = (b_{ij})_{i,j=1}^n$. It is easy to see that $b_{kj} = a_{i_k i_j}$. Therefore, Q is the vector of the corresponding row sums of PAP^T and $Q > 0$. Moreover, $b_{kj} = a_{i_k i_j} < \pi_{i_j} R_{i_k} = \alpha_j Q_k$, for all $k, j \in \{1, \dots, n\}$, $k \neq j$. We can then assert that PAP^T is a B_α^Q -matrix. $\square$

In [11], the authors prove that a positive linear combination $sA + tB$ (with s, t nonnegative numbers such that $s + t > 0$) of a B_π^R -matrix A and a B_ψ^R -matrix B (with the same row sums) is a $B_{\frac{s\pi+t\psi}{s+t}}^{(s+t)R}$ -matrix. Note that, in particular, the authors prove that, given a B_π^R -matrix A and a positive real number s , sA is a B_π^{sR} -matrix. Here we present an extension of these results.

Theorem 2.7. *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be real vectors satisfying*

$$0 < \sum_{j=1}^n \alpha_j \leq 1 \quad \text{and} \quad 0 < \sum_{j=1}^n \beta_j \leq 1$$

and let $\pi = (\pi_1, \dots, \pi_n)$ be the real vector defined by $\pi_i = \max\{\alpha_i, \beta_i\}$, $i = 1, \dots, n$. If A is an $n \times n$ B_α^R -matrix and B is an $n \times n$ B_β^Q -matrix, where R and Q are the vectors whose entries are the corresponding row sums of A and B , respectively, and if $\sum_{j=1}^n \pi_j \leq 1$, then $A + B$ is a B_π^{R+Q} -matrix.

Proof. Let $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$ and $A + B = (c_{ij})_{i,j=1}^n$. Let $R = (R_1, \dots, R_n)$, $Q = (Q_1, \dots, Q_n)$ and let $S = (S_1, \dots, S_n)$ be the vector of the corresponding row sums of $A + B$. It is obvious that $S = R + Q$. Hence, $S > 0$.

Since, for every $j \in \{1, \dots, n\}$, $\pi_j \geq \alpha_j$ and since $0 < \sum_{j=1}^n \alpha_j$, it follows that π satisfies condition (1). Moreover,

$$\begin{aligned} c_{ij} &= a_{ij} + b_{ij} < \alpha_j R_i + \beta_j Q_i \\ &\leq \pi_j R_i + \pi_j Q_i = \pi_j (R_i + Q_i) \\ &= \pi_j S_i, \end{aligned}$$

for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Therefore, $A + B$ is a B_π^S -matrix. $\square$

Corollary 2.8. *Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1). If A is an $n \times n$ B_π^R -matrix, where R is the vector whose entries are the corresponding row sums of A , and B is an $n \times n$ B_π^Q -matrix, where Q is the vector whose entries are the corresponding row sums of B , then $A + B$ is a B_π^{R+Q} -matrix.*

Corollary 2.9. Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1) and let s, t be nonnegative real numbers such that $s+t > 0$. If A is an $n \times n$ B_π^R -matrix and B is an $n \times n$ B_π^Q -matrix, where R and Q are the vectors whose entries are the corresponding row sums of A and B , respectively, then $sA + tB$ is a B_π^{sR+tQ} -matrix.

Proof. The result follows from the result by Neumann *et al.* mentioned above and from the previous corollary. $\square$

From Corollary 2.8 and Corollary 2.4, given a real vector $\pi = (\pi_1, \dots, \pi_n) > 0$ satisfying condition (1) and given an $n \times n$ B_π^R -matrix A , where R is the vector whose entries are the corresponding row sums of A , then any matrix obtained by increasing all diagonal entries of A is a B_π^Q -matrix, where Q is the vector whose entries are the corresponding row sums of this new matrix. In fact, we may look at the increasing of all diagonal entries of A as the sum of matrix A with a positive diagonal matrix D whose i th diagonal entry is the increase on the i th diagonal entry of A ($i = 1, \dots, n$). The following is a generalization of this result.

Proposition 2.10. Let $\pi = (\pi_1, \dots, \pi_n) > 0$ be a real vector satisfying condition (1) and let A be an $n \times n$ B_π^R -matrix, where R is the vector whose entries are the corresponding row sums of A . If we increase diagonal entries of A we get a B_π^Q -matrix, where Q is the vector whose entries are the corresponding row sums of the obtained matrix.

Proof. Let $A = (a_{ij})_{i,j=1}^n$ and let $B = (b_{ij})_{i,j=1}^n$ be the matrix obtained from A by increasing some diagonal entries of A . There exists $D = \text{diag}(d_1, \dots, d_n)$ such that $d_i \geq 0$ for all $i \in \{1, \dots, n\}$ and $B = A + D$. Let $R = (R_1, \dots, R_n)$ and $Q = (Q_1, \dots, Q_n)$. Observe that $Q_i = R_i + d_i$ and $b_{ij} = a_{ij}$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Therefore, $Q_i > 0$ and $b_{ij} < \pi_j R_i \leq \pi_j Q_i$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Hence, B is a B_π^Q -matrix. $\square$

The following example shows that we can not generalize the previous result for real vectors $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition 1 with some $\pi_j < 0$.

Example 2.11. Consider matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$$

It is easy to see that A is a B_π^R -matrix, where $R = (1, 1, 1)$ and $\pi = (-1.5, 1.2, 1.1)$. However, there is no real vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $0 < \alpha_1 + \alpha_2 + \alpha_3 \leq 1$, such that $A + D$, where $D = \text{diag}(0, 0, 2)$, is a B_α^Q -matrix for $Q = (1, 1, 3)$.

The result that follows shows that, for a given real vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition (1), the class of all B_π^R -matrices is closed under positive diagonal left multiplication.

Proposition 2.12. Let $\pi = (\pi_1, \dots, \pi_n)$ be a real vector satisfying condition (1) and let A be an $n \times n$ B_π^R -matrix, where $R = (R_1, \dots, R_n)$ is the vector whose entries are the corresponding row sums of A . Given a positive diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$, DA is a B_π^Q -matrix, where $Q = (d_1 R_1, \dots, d_n R_n)$.

Proof. Let $A = (a_{ij})_{i,j=1}^n$ and $DA = (b_{ij})_{i,j=1}^n$. Observe that Q is the vector whose entries are the corresponding row sums of DA and $b_{ij} = d_i a_{ij} < d_i \pi_j R_i$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Therefore, $Q_i > 0$ and $b_{ij} < \pi_j Q_i$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$ and, consequently, DA is a B_π^Q -matrix. $\square$

Not surprisingly, however, positive diagonal right multiplication does not preserve the property of existing a vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition (1) such that a matrix is a B_π^R -matrix, where R is the vector of the corresponding row sums, as the next example shows.

Example 2.13. Consider matrix

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}.$$

It is easy to see that A is a B_π^R -matrix, where $R = (4, 1)$ and $\pi = (1/2, 1/2)$. However, there is no real vector $\alpha = (\alpha_1, \alpha_2)$, where $0 < \alpha_1 + \alpha_2 \leq 1$, such that AD , where $D = \text{diag}(1, 10)$, is a B_α^Q -matrix, where Q is the vector whose entries are the corresponding row sums of AD .

3 Sub-direct sums of B_π^R -matrices

It is well known that a direct sum is a P -matrix if and only if each of the direct summands is a P -matrix. This statement remains true for many other classes of matrices, including positive semidefinite, doubly negative, completely positive, totally nonnegative and M -matrices. This is not true for the case of B_π^R -matrices, as the following example illustrates.

Example 3.1. Consider matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & 1 \\ -2 & -1 & 4 \end{bmatrix}.$$

As we have mentioned before, A is a B_π^R -matrix, where $R = (1, 1, 1)$ and $\pi = (-1.5, 1.2, 1.1)$. However, there is no real vector $\alpha = (\alpha_1, \dots, \alpha_6)$, with $0 < \alpha_1 + \dots + \alpha_6 \leq 1$, such that $A \oplus A$ is a B_α^Q -matrix, where $Q = (1, 1, 1, 1, 1, 1)$ is the vector of the corresponding row sums of $A \oplus A$.

Under certain conditions, however, we can guarantee that the direct sum preserves the property of existing a vector $\pi = (\pi_1, \dots, \pi_n)$ satisfying condition (1) such that a matrix is a B_π^R -matrix, where R is the vector of the corresponding row sums. We can easily derive the following result from Lemma 1.2.

Lemma 3.2. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^m$ be square real matrices with positive row sums and let $R = (R_1, \dots, R_n)$ and $Q = (Q_1, \dots, Q_m)$ be the vectors whose entries are the corresponding row sums of A and B , respectively. There exists a vector $\pi = (\pi_1, \dots, \pi_{n+m})$ satisfying $0 < \sum_{j=1}^{n+m} \pi_j \leq 1$ such that $A \oplus B$ is a B_π^S -matrix, where $S = (R_1, \dots, R_n, Q_1, \dots, Q_m)$, if and only if

$$\sum_{j=1}^n \max_{i \neq j} \left\{ \frac{a_{ij}}{R_i}, 0 \right\} + \sum_{j=1}^m \max_{i \neq j} \left\{ \frac{b_{ij}}{Q_i}, 0 \right\} < 1.$$

The following theorem states that if a direct sum is a B_π^R -matrix then its direct summands are also in that class of matrices (for suitable vectors π and R).

Theorem 3.3. Let A and B be square real matrices of order n and m , respectively, and let $\pi = (\pi_1, \dots, \pi_{n+m})$ be a real vector satisfying $0 < \sum_{j=1}^{n+m} \pi_j \leq 1$. If $A \oplus B$ is a B_π^S -matrix, where $S = (S_1, \dots, S_{n+m})$, then $\pi > 0$, A is a B_α^R -matrix, where $\alpha = (\pi_1, \dots, \pi_n)$ and $R = (S_1, \dots, S_n)$, and B is a B_β^Q -matrix, where $\beta = (\pi_{n+1}, \dots, \pi_{n+m})$ and $Q = (S_{n+1}, \dots, S_{n+m})$.

Proof. Observe that for every column j of $A \oplus B$, there is at least one null non diagonal entry, which implies that there exists $i \neq j$ such that $0 < \pi_j S_i$. Given that $S_i > 0$, it follows that $\pi_j > 0$. Therefore, $\pi > 0$.

Let $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^m$ and $A \oplus B = (c_{ij})_{i,j=1}^{n+m}$. Notice that, for every $i \in \{1, \dots, n\}$, $S_i = \sum_{j=1}^{n+m} c_{ij} = \sum_{j=1}^n a_{ij} + \sum_{j=n+1}^{n+m} 0 = \sum_{j=1}^n a_{ij}$. Hence, the vector whose entries are the corresponding row sums of A is $R = (S_1, \dots, S_n) > 0$. Moreover, for every $i, j \in \{1, \dots, n\}$, $i \neq j$, $a_{ij} = c_{ij} < \pi_j S_i$. Being $\pi > 0$ and $\sum_{j=1}^{n+m} \pi_j \leq 1$, it is obvious that $0 < \sum_{j=1}^n \pi_j \leq 1$. We can then assert that A is a B_α^R -matrix, where $\alpha = (\pi_1, \dots, \pi_n)$ and $R = (S_1, \dots, S_n)$. For every $i \in \{n+1, \dots, n+m\}$, $S_i = \sum_{j=1}^{n+m} c_{ij} = \sum_{j=1}^n 0 + \sum_{j=n+1}^{n+m} b_{i-n, j-n} = \sum_{j=1}^m b_{ij}$. So, the vector whose entries are the corresponding row sums of B is $Q = (S_{n+1}, \dots, S_{n+m})$. Besides, for every $i, j \in \{1, \dots, m\}$, $i \neq j$, $b_{ij} = c_{n+i, n+j} < \pi_{n+j} S_{n+i}$. Since $\pi > 0$ and $\sum_{j=1}^{n+m} \pi_j \leq 1$, it follows that $0 < \sum_{j=n+1}^{n+m} \pi_j \leq 1$. Therefore, B is a B_β^Q -matrix, where $\beta = (\pi_{n+1}, \dots, \pi_{n+m})$ and $Q = (S_{n+1}, \dots, S_{n+m})$. $\square$

The next proposition is a consequence of Theorem 2.3.

Proposition 3.4. *The direct sum of matrices $A \in \mathcal{Z}_n$ and $B \in \mathcal{Z}_m$ with positive vectors of corresponding row sums $R = (R_1, \dots, R_n)$ and $Q = (Q_1, \dots, Q_m)$, respectively, is a B_π^S -matrix for any real vector $\pi = (\pi_1, \dots, \pi_{n+m}) > 0$ satisfying $\sum_{j=1}^{n+m} \pi_j \leq 1$ and vector $S = (R_1, \dots, R_n, Q_1, \dots, Q_m)$.*

Assuming that the direct summands are a B_α^R -matrix and a B_β^Q -matrix with both vectors α and β positive, we set necessary and sufficient conditions to have the direct sum in the same class of matrices (for a specific vector π defined from α and β).

Theorem 3.5. *Let A be an $n \times n$ B_α^R -matrix, where $\alpha = (\alpha_1, \dots, \alpha_n) > 0$ and $R = (R_1, \dots, R_n)$, and let B be an $m \times m$ B_β^Q -matrix, where $\beta = (\beta_1, \dots, \beta_m) > 0$ and $Q = (Q_1, \dots, Q_m)$. Consider the real vector π defined by $\pi = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$. Then, $A \oplus B$ is a B_π^S -matrix, where $S = (R_1, \dots, R_n, Q_1, \dots, Q_m)$, if and only if $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i \leq 1$.*

Proof. Let $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^m$ and $A \oplus B = (c_{ij})_{i,j=1}^{n+m}$. Write $\pi = (\pi_1, \dots, \pi_{n+m})$ and $S = (S_1, \dots, S_{n+m})$. It is clear that S_i is the sum of the i th row of $A \oplus B$.

If $A \oplus B$ is a B_π^S -matrix, it follows that $\sum_{i=1}^{n+m} \pi_i \leq 1$, i.e. $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i \leq 1$.

Conversely, assume that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i \leq 1$. Given that $S_i = R_i$ for $i \in \{1, \dots, n\}$ and $S_i = Q_{i-n}$ for $i \in \{n+1, \dots, n+m\}$, we can assert that $S > 0$. Since $\alpha > 0$, $\beta > 0$ and $\sum_{i=1}^n \alpha_i + \sum_{i=1}^m \beta_i \leq 1$, it follows that $\pi > 0$ and $0 < \sum_{i=1}^{n+m} \pi_i \leq 1$. Moreover, for $i, j \in \{1, \dots, n\}$, $i \neq j$, $c_{ij} = a_{ij} < \alpha_j R_i = \pi_j S_i$. For $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$, and for $i \in \{n+1, \dots, n+m\}$ and $j \in \{1, \dots, n\}$, $c_{ij} = 0 < \pi_j S_i$. Finally, for $i, j \in \{n+1, \dots, n+m\}$, $i \neq j$, $c_{ij} = b_{i-n, j-n} < \beta_{j-n} Q_{i-n} = \pi_j S_i$. Therefore, $A \oplus B$ is a B_π^S -matrix, where $\pi = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ and $S = (R_1, \dots, R_n, Q_1, \dots, Q_m)$. $\square$

Corollary 3.6. *Let A be an $n \times n$ B_α^R -matrix, where $\alpha = (\alpha_1, \dots, \alpha_n) > 0$ and $R = (R_1, \dots, R_n)$, and let B be an $n \times n$ B_α^Q -matrix, where $Q = (Q_1, \dots, Q_n)$. Consider the real vector π defined by $\pi = (\alpha_1, \dots, \alpha_n, \alpha_1, \dots, \alpha_n)$. Then, $A \oplus B$ is a B_π^S -matrix, where $S = (R_1, \dots, R_n, Q_1, \dots, Q_n)$, if and only if $\sum_{i=1}^n \alpha_i \leq \frac{1}{2}$.*

Note that for matrix A given in Example 3.1 there is no positive real vector α such that A is a B_α^R -matrix, where R is the vector whose entries are the corresponding row sums of A . Therefore, no direct sum with A as one of the direct summands is a B_π^S -matrix.

Fallat and Johnson considered, for several classes of matrices, corresponding questions for a more general ‘sum’ of two matrices, of which direct sum and ordinary sum are special cases - the subdirect sum ([4]).

Let $0 \leq k \leq n, m$ and suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{M}_n(\mathbb{C}) \text{ and } B = \begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} \in \mathcal{M}_m(\mathbb{C}),$$

in which $A_{22}, B_{22} \in \mathcal{M}_k(\mathbb{C})$. Then

$$A \oplus_k B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{bmatrix}$$

is called the k -subdirect sum of A and B . We simply refer to a subdirect sum when the value of k is irrelevant or unambiguous. When $k = 0$ we have the familiar direct sum and we abbreviate $\oplus_0$ to $\oplus$. In many key positive classes of matrices, we have that the direct sum lies in the class if and only if each direct summand lies in the class. As we mentioned above, this does not hold, in general, for B_π^R -matrices. On the other hand, when $k = n = m$, we have the ordinary sum of a B_π^R -matrix and a B_π^Q -matrix, which we know to be a B_π^{R+Q} -matrix. In their paper, Fallat and Johnson address the following natural question: If A and B lie in the class must a k -subdirect sum C lie in the class? Naturally, one can also pose the reverse question: if $C = A \oplus_k B$ lies in the class, must A and B also lie in the class?

As we have mentioned above, regarding B_π^R -matrices, the first question has affirmative answer when $k = n = m$. However, in general, this question has negative answer. Consider, for example, matrix A given in Example 3.1 and let $B = A$. It is easy to see that $A \oplus_1 B$ is not a B_π^R -matrix for any π and R .

The next result follows from Lemma 1.2.

Lemma 3.7. *Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^m$ be square real matrices with positive row sums and let $R = (R_1, \dots, R_n)$ and $Q = (Q_1, \dots, Q_m)$ be the vectors whose entries are the corresponding row sums of A and B , respectively. Assume $n > k$ and $m > k$. There exists a vector $\pi = (\pi_1, \dots, \pi_{n+m-k})$ satisfying $0 < \sum_{j=1}^{n+m-k} \pi_j \leq 1$ such that $A \oplus_k B$ is a B_π^S -matrix, where $S = (R_1, \dots, R_{n-k}, R_{n-k+1} + Q_1, R_{n-k+2} + Q_2, \dots, R_n + Q_k, Q_{k+1}, \dots, Q_m)$, if and only if*

$$\begin{aligned} & \sum_{j=1}^{n-k} \max_{\substack{i \neq j \\ 1 \leq i \leq n-k}} \left\{ \frac{a_{ij}}{R_i}, 0, \frac{a_{n-k+1,j}}{R_{n-k+1} + Q_1}, \frac{a_{n-k+2,j}}{R_{n-k+2} + Q_2}, \dots, \frac{a_{nj}}{R_n + Q_k} \right\} + \\ & + \sum_{j=k+1}^m \max_{\substack{i \neq j \\ k+1 \leq i \leq m}} \left\{ \frac{b_{ij}}{Q_i}, 0, \frac{b_{1j}}{R_{n-k+1} + Q_1}, \frac{b_{2j}}{R_{n-k+2} + Q_2}, \dots, \frac{b_{kj}}{R_n + Q_k} \right\} + \\ & + \sum_{j=n-k+1}^n \max \left\{ \max_{1 \leq i \leq n-k} \left\{ \frac{a_{ij}}{R_i} \right\}, \max_{\substack{i \neq j \\ n-k+1 \leq i \leq n}} \left\{ \frac{a_{ij} + b_{i-(n-k),j-(n-k)}}{R_i + Q_{i-(n-k)}} \right\}, \max_{n+1 \leq i \leq n+m-k} \left\{ \frac{b_{i-(n-k),j-(n-k)}}{Q_{i-(n-k)}} \right\} \right\} < 1. \end{aligned}$$

Observe that, similarly to what happens in the case of the direct sum, if a matrix of the form $A \oplus_k B$ is a B_π^R -matrix, then there are, in general, restrictions on the signs of the entries of vector π .

Lemma 3.8. *Let A and B be square real matrices of order n and m , respectively, with $n, m > k$, and let $\pi = (\pi_1, \dots, \pi_{n+m-k})$ be a real vector satisfying $0 < \sum_{j=1}^{n+m-k} \pi_j \leq 1$. If $A \oplus_k B$ is a B_π^R -matrix, then $\pi_j > 0$ for $j = 1, \dots, n-k, n+1, \dots, n+m-k$.*

Proof. Let $A \oplus_k B = (c_{ij})_{i,j=1}^{n+m-k}$ and let $R = (R_1, \dots, R_{n+m-k})$ be the vector whose entries are the corresponding row sums of $A \oplus_k B$. For any $j = 1, \dots, n-k$, $0 = c_{n+1,j} < \pi_j R_{n+1}$. Since $R_{n+1} > 0$, it follows that $\pi_j > 0$ for $j = 1, \dots, n-k$. Analogously, for every $j = n+1, \dots, n+m-k$, $0 = c_{1j} < \pi_j R_1$ and, hence, $\pi_j > 0$ for $j = n+1, \dots, n+m-k$. $\square$

In the following example we show that when the 1-subdirect sum of two matrices A and B is a B_π^R -matrix, it is not necessarily true that both A and B are matrices in that class.

Example 3.9. *Consider matrices*

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 9 & 5 \\ -2 & 1 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 18 & 1 & 1 \\ -2 & 4 & -1 \\ -2 & -4 & 7 \end{bmatrix}.$$

$A \oplus_1 B$ is a B_π^R -matrix, where $R = (2, 15, 21, 1, 1)$ and $\pi = (\frac{71}{1050}, \frac{263}{525}, \frac{117}{350}, \frac{17}{350}, \frac{17}{350})$. However, there is no real vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, with $0 < \alpha_1 + \alpha_2 + \alpha_3 \leq 1$, such that A is a B_α^Q -matrix, where $Q = (2, 15, 1)$.

The following result is a consequence of Theorem 2.3.

Proposition 3.10. *The k -subdirect sum of matrices $A \in \mathcal{Z}_n$ and $B \in \mathcal{Z}_m$ with positive vectors of corresponding row sums $R = (R_1, \dots, R_n)$ and $Q = (Q_1, \dots, Q_m)$, respectively, is a B_π^S -matrix for any real vector $\pi = (\pi_1, \dots, \pi_{n+m-k}) > 0$ satisfying $\sum_{j=1}^{n+m-k} \pi_j \leq 1$ and vector $S = (R_1, \dots, R_{n-k}, R_{n-k+1} + Q_1, R_{n-k+2} + Q_2, \dots, R_n + Q_k, Q_{k+1}, \dots, Q_m)$.*

Assuming that the direct summands are a B_α^R -matrix and a B_β^Q -matrix, the k -subdirect sum lies in the same class of matrices under certain conditions.

Theorem 3.11. *Let n, m, k be positive integers such that $n > k$ and $m > k$. Let A be an $n \times n$ B_α^R -matrix, $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_i > 0$ for all $i \leq n-k$, and $R = (R_1, \dots, R_n)$, and let B be an $m \times m$ B_β^Q -matrix, where $\beta = (\beta_1, \dots, \beta_m)$, with $\beta_i > 0$, for all $i > k$, and $Q = (Q_1, \dots, Q_m)$. Consider the real vector $\pi = (\alpha_1, \dots, \alpha_{n-k}, \max\{\alpha_{n-k+1}, \beta_1\}, \max\{\alpha_{n-k+2}, \beta_2\}, \dots, \max\{\alpha_n, \beta_k\}, \beta_{k+1}, \dots, \beta_m)$. Then, $A \oplus_k B$ is a B_π^S -matrix, where $S = (R_1, \dots, R_{n-k}, R_{n-k+1} + Q_1, R_{n-k+2} + Q_2, \dots, R_n + Q_k, Q_{k+1}, \dots, Q_m)$ if and only if $\sum_{i=1}^{n-k} \alpha_i + \sum_{i=n-k+1}^n \max\{\alpha_i, \beta_{i-(n-k)}\} + \sum_{i=k+1}^m \beta_i \leq 1$.*

Proof. Let $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^m$ and $A \oplus_k B = (c_{ij})_{i,j=1}^{n+m-k}$. Write $\pi = (\pi_1, \dots, \pi_{n+m-k})$ and $S = (S_1, \dots, S_{n+m-k})$. It is obvious that S_i is the sum of the i th row of $A \oplus_k B$.

If $A \oplus_k B$ is a B_π^S -matrix, it follows that $\sum_{i=1}^{n+m-k} \pi_i \leq 1$, i.e. $\sum_{i=1}^{n-k} \alpha_i + \sum_{i=n-k+1}^n \max\{\alpha_i, \beta_{i-(n-k)}\} + \sum_{i=k+1}^m \beta_i \leq 1$.

Conversely, assume that $\sum_{i=1}^{n-k} \alpha_i + \sum_{i=n-k+1}^n \max\{\alpha_i, \beta_{i-(n-k)}\} + \sum_{i=k+1}^m \beta_i \leq 1$.

Given that $S_i = R_i$ for $i \in \{1, \dots, n-k\}$, $S_i = R_i + Q_{i-(n-k)}$ for $i \in \{n-k+1, \dots, n\}$ and $S_i = Q_{i-(n-k)}$ for $i \in \{n+1, \dots, n+m-k\}$, we can assert that $S > 0$. Since $\sum_{i=1}^n \alpha_i > 0$, $\pi_i \geq \alpha_i$ for all $i \in \{n-k+1, \dots, n\}$, and $\beta_i > 0$ for all $i > k+1$, it follows that $0 < \sum_{i=1}^{n+m-k} \pi_i \leq 1$.

Observe that for $i, j \in \{1, \dots, n-k\}$, $i \neq j$, $c_{ij} = a_{ij} < \alpha_j R_i = \pi_j S_i$. For $i \in \{1, \dots, n-k\}$ and $j \in \{n-k+1, \dots, n\}$, $c_{ij} = a_{ij} < \alpha_j R_i \leq \pi_j R_i = \pi_j S_i$, since $\pi_j = \max\{\alpha_j, \beta_{j-(n-k)}\}$ and $R_i > 0$. For $i \in \{1, \dots, n-k\}$ and $j \in \{n+1, \dots, n+m-k\}$, $c_{ij} = 0 < \beta_{j-(n-k)} R_i = \pi_j S_i$, given that $R_i > 0$ and $\beta_{j-(n-k)} > 0$.

Moreover, for $i \in \{n-k+1, \dots, n\}$ and $j \in \{1, \dots, n-k\}$, $c_{ij} = a_{ij} < \alpha_j R_i < \alpha_j (R_i + Q_{i-(n-k)}) = \pi_j S_i$, since $\alpha_j > 0$, $R_i > 0$ and $Q_{i-(n-k)} > 0$. For $i, j \in \{n-k+1, \dots, n\}$, $i \neq j$, $c_{ij} = a_{ij} + b_{i-(n-k), j-(n-k)} < \alpha_j R_i + \beta_{j-(n-k)} Q_{i-(n-k)} \leq \pi_j R_i + \pi_j Q_{i-(n-k)} = \pi_j S_i$, given that $\pi_j = \max\{\alpha_j, \beta_{j-(n-k)}\}$, $R_i > 0$ and $Q_{i-(n-k)} > 0$. For $i \in \{n-k+1, \dots, n\}$ and $j \in \{n+1, \dots, n+m-k\}$, $c_{ij} = b_{i-(n-k), j-(n-k)} < \beta_{j-(n-k)} Q_{i-(n-k)} < \beta_{j-(n-k)} (R_i + Q_{i-(n-k)}) = \pi_j S_i$, since $\beta_{j-(n-k)} > 0$, $R_i > 0$ and $Q_{i-(n-k)} > 0$.

Finally, for $i \in \{n+1, \dots, n+m-k\}$ and $j \in \{1, \dots, n-k\}$, $c_{ij} = 0 < \alpha_j Q_{i-(n-k)} = \pi_j S_i$, since $Q_{i-(n-k)} > 0$ and $\alpha_j > 0$. For $i \in \{n+1, \dots, n+m-k\}$ and $j \in \{n-k+1, \dots, n\}$, $c_{ij} = b_{i-(n-k), j-(n-k)} < \beta_{j-(n-k)} Q_{i-(n-k)} \leq \pi_j Q_{i-(n-k)} = \pi_j S_i$, since $\pi_j = \max\{\alpha_j, \beta_{j-(n-k)}\}$. For $i, j \in \{n+1, \dots, n+m-k\}$, $i \neq j$, $c_{ij} = b_{i-(n-k), j-(n-k)} < \beta_{j-(n-k)} Q_{i-(n-k)} = \pi_j S_i$.

We have just proved that $c_{ij} < \pi_j S_i$, for all $i \neq j$. Therefore, we can conclude that $A \oplus_k B$ is a B_π^S -matrix. $\square$

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