On the roots of coquaternions

M. Irene Falcão[a,c] Fernando Miranda[a,c] Ricardo Severino[c] M. Joana Soares[b,c]

[a] CMAT - Centre of Mathematics, University of Minho, Portugal
[b] NIPE - Economic Policies Research Unit, University of Minho, Portugal
[c] Department of Mathematics and Applications, University of Minho, Portugal

Information

Keywords:
Coquaternions, Roots, Polar form, Similarity and quasi-similarity classes of a coquaternion, Matrix equations.

Original publication:
Advances in Applied Clifford Algebras, (2018) 28:97
DOI:10.1007/s00006-018-0914-3
Springer Link

Abstract

In this paper we give a complete characterization of the nth roots of a coquaternion q. In particular, we show that the number and type of roots — isolated and/or *hyperboloidal* — depend on the nature of q, on the parity of n and (eventually) on the sign of the real part of q. We also show how the coquaternionic formalism can be used to obtain, in a simple manner, explicit expressions for the real nth roots of any 2×2 real matrix.

1 Introduction

Coquaternions, introduced by Sir James Cockle in 1849, [6], form a four-dimensional hypercomplex real algebra generalizing complex numbers. Although coquaternions, also known in the literature as split quaternions, are not as popular as the well-known Hamilton's quaternions, recent studies show an increasing interest in these hypercomplex numbers, not only from the theoretical point of view, but also from the perspective of applications. Algebraic properties of coquaternions and coquaternionic polynomials are considered in e.g., [9, 10, 19] while [13] contains a study of coquaternionic analysis based on representation theory. Some geometric applications of coquaternions can be found in [16, 18, 20] and the relation between coquaternions and complexified mechanics is discussed in [3]. The use of coquaternions in dynamical systems was recently considered in [11, 12].

In [17], the author determined some of the roots of a coquaternion and presented a De Moivre's formula for coquaternions, under some implicit assumptions. In this paper we give a complete description of the roots of any coquaternion, extending in this way the work of [17]. Recent results on the structure of the zeros of coquaternionic unilateral polynomials [9, 10] allow to express the nth roots of a real number in terms of similarity classes.

The problem of finding the roots of a given square matrix A, i.e., of determining the matrices X such that $X^n = A$ has attracted the attention of many researchers over the years: see, e.g., [1, 2, 8, 14, 15], where the case of square roots, i.e., $n = 2$, is discussed (in some cases, for special types of matrices) and also [4, 5, 7], where only 2×2 matrices are considered, but more general nth roots are determined. In the seminal paper by A. Cayley [4], although not explicitly stated, only real roots of real 2×2 non-singular matrices satisfying certain conditions are found.[1] In [7] the author finds expressions, involving transcendental function, for the (complex) roots of 2×2 complex matrices with determinant equal to one (i.e., elements in $SL_2(\mathbb{C})$) and then, using those expressions, describes how to compute the roots of any non-singular 2×2 matrix (i.e., an element of $GL_2(\mathbb{C})$). In the more recent paper by Choudhry [5], the author describes a method for obtaining algebraic expressions for the (complex) nth roots of any 2×2 complex matrix; except in special cases, the method involves the determination of the roots of certain polynomials of degree n and as such, unless n is of moderate size, can be considered as quite elaborate.

In this paper we revisit the problem of the determination of the nth real roots of a matrix in $\mathcal{M}_2(\mathbb{R})$, the algebra of real 2×2 matrices, taking into account the well-known isomorphism between this algebra and the algebra of coquaternions. This allows us to discuss the number and nature of such roots and also to easily derive their explicit expressions.

2 Basic results on coquaternions

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathbf{i}^2 = -1, \ \mathbf{j}^2 = \mathbf{k}^2 = 1, \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

This non-commutative product generates the algebra of real coquaternions, which we will denote by $\mathbb{H}_{\rm coq}$.

We will embed the space $\mathbb{R}^4$ in $\mathbb{H}_{\rm coq}$ by identifying the element in $\mathbb{R}^4$ (q_0, q_1, q_2, q_3) with the coquaternion $\mathsf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. Given $\mathsf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}_{\rm coq}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$; the number q_0 is called the *real part* of q and denoted by $\operatorname{Re}\mathsf{q}$ and the *vector part* of q, denoted by $\underline{\mathsf{q}}$, is $\underline{\mathsf{q}} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. The quantity $\mathsf{q} + \overline{\mathsf{q}} = 2\operatorname{Re}\mathsf{q}$ is referred as the *trace* of q and denoted by $\operatorname{tr}\mathsf{q}$. We will identify the set of coquaternions with null vector part with the set $\mathbb{R}$ of real numbers. We call *determinant* of q and denote by $\det\mathsf{q}$ the quantity given by $\det\mathsf{q} = \mathsf{q}\,\overline{\mathsf{q}} = q_0^2 + q_1^2 - q_2^2 - q_3^2$ and we endow $\mathbb{H}_{\rm coq}$ with the *semi-norm* $\|\mathsf{q}\| = \sqrt{|\det\mathsf{q}|}$. We say that q is a *non-singular* coquaternion if $\det\mathsf{q} \neq 0$ and call q a *unit coquaternion* if $\|\mathsf{q}\| = 1$.

A coquaternion q is called *space-like, light-like* or *time-like* if $\det\mathsf{q} < 0$, $\det\mathsf{q} = 0$ or $\det\mathsf{q} > 0$, respectively; the sets of such coquaternions will be denoted by $\mathbb{S}$, $\mathbb{L}$ and $\mathbb{T}$, respectively. We will say that two coquaternions have the same nature if both belong to the same set $\mathbb{S}$, $\mathbb{L}$ or $\mathbb{T}$. We also adopt the notations

$$\mathbb{T}_{\mathbb{S}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{S}\}, \quad \mathbb{T}_{\mathbb{L}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{L}\}, \quad \mathbb{T}_{\mathbb{T}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{T}\},$$
$$\mathbb{L}_{\mathbb{S}} = \{\mathsf{q} \in \mathbb{L} : \underline{\mathsf{q}} \in \mathbb{S}\}, \quad \mathbb{L}_{\mathbb{L}} = \{\mathsf{q} \in \mathbb{L} : \underline{\mathsf{q}} \in \mathbb{L}\}.$$

Remark 2.1. *We observe that, since* $\det\mathsf{q} = q_0^2 + \det\underline{\mathsf{q}} \geq \det\underline{\mathsf{q}}$, *if* q *is space-like, then* $\underline{\mathsf{q}}$ *is of the same nature and a light-like coquaternion can not have a time-like vector part. Moreover, if* $q_0 = 0$ *then* $\mathsf{q} \in \mathbb{T}_{\mathbb{T}} \cup \mathbb{S} \cup \mathbb{L}_{\mathbb{L}}$*. We refer to [10] for other properties of* $\det\mathsf{q}$.

The polar form for a non-singular coquaternion q with a non-singular vector part can be found in [17] and reads as follows:

Lemma 2.2 (Polar forms of a coquaternion)**.** *Any coquaternion* $\mathsf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{S} \cup \mathbb{T}$*, such that* $\underline{\mathsf{q}} \notin \mathbb{L}$*, has a polar representation in one of the forms*

$$\mathsf{q} = \begin{cases} \|\mathsf{q}\|\big(\sinh\phi_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\cosh\phi_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{S}, \\ \|\mathsf{q}\|\big(\operatorname{sgn} q_0 \cosh\psi_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\sinh\psi_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{T}_{\mathbb{S}}, \\ \|\mathsf{q}\|\big(\cos\theta_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\sin\theta_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{T}_{\mathbb{T}}, \end{cases} \tag{2.1a}$$

where sgn *is the usual sign function,*

$$\sinh\phi_{\mathsf{q}} = \frac{q_0}{\|\mathsf{q}\|}, \quad \sinh\psi_{\mathsf{q}} = \frac{\|\underline{\mathsf{q}}\|}{\|\mathsf{q}\|}, \quad \cos\theta_{\mathsf{q}} = \frac{q_0}{\|\mathsf{q}\|}, \quad \sin\theta_{\mathsf{q}} = \frac{\|\underline{\mathsf{q}}\|}{\|\mathsf{q}\|}, \ \theta_{\mathsf{q}} \in (0, \pi) \tag{2.1b}$$

[1] In [4], the only matrix of order greater than two for which square roots are found is the identity matrix of third order.

and

$$\boldsymbol{\omega}_{\underline{\mathrm{q}}} = \frac{\underline{\mathrm{q}}}{||\underline{\mathrm{q}}||} \tag{2.1c}$$

is a unit coquaternion satisfying $\boldsymbol{\omega}_{\underline{\mathrm{q}}}^2 = 1$, *if* $\underline{\mathrm{q}} \in \mathbb{S}$ *and* $\boldsymbol{\omega}_{\underline{\mathrm{q}}}^2 = -1$, *if* $\underline{\mathrm{q}} \in \mathbb{T}$.

We observe that the case $\mathrm{q} \in \mathbb{T}_{\mathbb{S}}$ of formula (2.1a) was given in [17] under the implicit assumption that $q_0 = \operatorname{Re} \mathrm{q} > 0$. Using the results of the previous lemma, one can easily establish, by induction, De Moivre's type formulae for the nth power of q, when $n \in \mathbb{N}$, given in the following theorem (cf. Theorems 1-3 in [17]).

Theorem 2.3 (De Moivre's formulae for coquaternions)**.** *Under the conditions of the previous lemma, we have, for any* $n \in \mathbb{N}$*:*

$$\mathrm{q}^n = \begin{cases} \begin{cases} ||\mathrm{q}||^n \big(\sinh n\phi_{\mathrm{q}} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \cosh n\phi_{\mathrm{q}}\big), & n \text{ is odd}, \\ ||\mathrm{q}||^n \big(\cosh n\phi_{\mathrm{q}} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \sinh n\phi_{\mathrm{q}}\big), & n \text{ is even}, \end{cases} & \text{if } \mathrm{q} \in \mathbb{S}, \\ ||\mathrm{q}||^n (\operatorname{sgn} \mathrm{q}_0)^{\mathrm{n}-1} \big(\operatorname{sgn} \mathrm{q}_0 \cosh \mathrm{n}\psi_{\mathrm{q}} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \sinh \mathrm{n}\psi_{\mathrm{q}}\big), & \text{if } \mathrm{q} \in \mathbb{T}_{\mathbb{S}}, \\ ||\mathrm{q}||^n \big(\cos n\theta_{\mathrm{q}} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \sin n\theta_{\mathrm{q}}\big), & \text{if } q \in \mathbb{T}_{\mathbb{T}}. \end{cases} \tag{2.2}$$

Remark 2.4. *Relations* (2.2) *are trivially verified when* $n = 0$*; furthermore, since we are considering the case where* $\det \mathrm{q} \neq 0$*, we know that* q *is invertible and* $\mathrm{q}^{-1} = \frac{\bar{\mathrm{q}}}{\det \mathrm{q}}$*; from this, it is simple to verify that the relations are also valid when* $n = -1$ *and to conclude that* (2.2) *hold for any* $n \in \mathbb{Z}$.

3 Roots of a non-real coquaternion

We are now interested in determining the roots of index $n \in \mathbb{N}$, $n \geq 2$, of a given non-real coquaternion q, i.e., in determining $\mathrm{x} \in \mathbb{H}_{\mathrm{coq}}$ such that $\mathrm{x}^n = \mathrm{q}$. As was already pointed out, one can find in [17] a similar study, but with a different goal. In fact, the author of [17] considers only coquaternions q which are neither light-like nor have a light-like vector part and determines only roots which have the same nature as q. Here we address the problem of completely characterizing and determining all the roots of any non-real coquaternion.

We first consider the case where $\mathrm{q} \notin \mathbb{L}$ and $\underline{\mathrm{q}} \notin \mathbb{L}$. In the next theorem we give the number and type of the roots of index n of such a coquaternion q and also give explicit expressions for these roots. As the theorem shows, the number and type of roots will depend on the nature of q and $\underline{\mathrm{q}}$, on the parity of n and (eventually) on the sign of the real part of q.

Theorem 3.1. *Let* q *be a coquaternion admitting one of the representations in* (2.1a)*. For* $n \geq 2$*, the* n*th roots of* q *can be characterized as follows:*

1. *If* $\mathrm{q} \in \mathbb{S}$ *and*

   (a) n *is odd, then there is only one root, which is in* $\mathbb{S}$*:*

   $$\sqrt[n]{||\mathrm{q}||}\big(\sinh \tfrac{\phi_{\mathrm{q}}}{n} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \cosh \tfrac{\phi_{\mathrm{q}}}{n}\big);$$

   (b) n *is even, then there are no roots.*

2. *If* $\mathrm{q} \in \mathbb{T}_{\mathbb{S}}$ *and*

   (a) n *is odd, then there is only one root, which is in* $\mathbb{T}_{\mathbb{S}}$*:*

   $$\sqrt[n]{||\mathrm{q}||}\big(\operatorname{sgn} q_0 \cosh \tfrac{\psi_{\mathrm{q}}}{n} + \boldsymbol{\omega}_{\underline{\mathrm{q}}} \sinh \tfrac{\psi_{\mathrm{q}}}{n}\big);$$

   (b) n *is even and*

i. $q_0 > 0$, *then there are four roots, two of which are in* $\mathbb{T}_{\mathbb{S}}$*:*

$$\pm \sqrt[n]{||\mathsf{q}||}\left(\cosh \tfrac{\psi_{\mathsf{q}}}{n} + \boldsymbol{\omega}_{\underline{\mathsf{q}}} \sinh \tfrac{\psi_{\mathsf{q}}}{n}\right),$$

and the other two in $\mathbb{S}$*:*

$$\pm \sqrt[n]{||\mathsf{q}||}\left(\sinh \tfrac{\psi_{\mathsf{q}}}{n} + \boldsymbol{\omega}_{\underline{\mathsf{q}}} \cosh \tfrac{\psi_{\mathsf{q}}}{n}\right);$$

ii. $q_0 < 0$, *then there are no roots.*

3. *If* $\mathsf{q} \in \mathbb{T}_{\mathbb{T}}$ *then there are* n *roots, which are in* $\mathbb{T}_{\mathbb{T}}$*:*

$$\sqrt[n]{||\mathsf{q}||}\left(\cos \tfrac{\theta_{\mathsf{q}}+2k\pi}{n} + \boldsymbol{\omega}_{\underline{\mathsf{q}}} \sin \tfrac{\theta_{\mathsf{q}}+2k\pi}{n}\right);\ k = 0, \dots, n-1.$$

Proof. Given a coquaternion $\mathsf{q} \in \mathbb{S} \cup \mathbb{T}$, with $\underline{\mathsf{q}} \notin \mathbb{L}$ and $n \geq 2$, we aim to determine coquaternions x such that $\mathsf{x}^n = \mathsf{q}$. We first observe that, as is well known, any coquaternion $\mathsf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ satisfies its *characteristic equation*

$$\mathsf{x}^2 - 2x_0\,\mathsf{x} + \det \mathsf{x} = 0$$

or $\mathsf{x}^2 = 2x_0\,\mathsf{x} - \det \mathsf{x}$. Using this, we can easily prove, by induction, that $\mathsf{x}^n \in \operatorname{span}_{\mathbb{R}}(1, \mathsf{x})$ and, naturally, $\operatorname{span}_{\mathbb{R}}(1, \mathsf{x}) = \operatorname{span}_{\mathbb{R}}(1, \underline{\mathsf{x}})$. In other words, there exists $\alpha, \beta \in \mathbb{R}$ such that $\mathsf{x}^n = \alpha + \beta\underline{\mathsf{x}}$. If x is such that $\mathsf{x}^n = \mathsf{q}$, we must have $\alpha + \beta\underline{\mathsf{x}} = q_0 + \underline{\mathsf{q}}$ and so we get $\beta\underline{\mathsf{x}} = \underline{\mathsf{q}}$, $\beta \neq 0$, $\underline{\mathsf{x}} \neq 0$, which in turn implies that $\det(\beta\underline{\mathsf{x}}) = \beta^2 \det \underline{\mathsf{x}} = \det \underline{\mathsf{q}}$. Therefore

$$\operatorname{sgn}(\det \underline{\mathsf{x}}) = \operatorname{sgn}(\det \underline{\mathsf{q}}). \tag{3.1}$$

Since $\det(\mathsf{x}^n) = (\det \mathsf{x})^n$, we also have

$$(\det \mathsf{x})^n = \det \mathsf{q}. \tag{3.2}$$

Relations (3.1)-(3.2) lead at once to the following remarks on the set $R^n_{\mathsf{q}} = \{\mathsf{x} \in \mathbb{H}_{\mathrm{coq}} : \mathsf{x}^n = \mathsf{q}\}$:

i. if $\mathsf{q} \in \mathbb{S}$, then $R^n_{\mathsf{q}} \subset \mathbb{S}$, when n is odd, while for n even, $R^n_{\mathsf{q}} = \emptyset$;

ii. if $\mathsf{q} \in \mathbb{T}_{\mathbb{S}}$, then $R^n_{\mathsf{q}} \subset \mathbb{T}_{\mathbb{S}}$, when n is odd and $R^n_{\mathsf{q}} \subset \mathbb{T}_{\mathbb{S}} \cup \mathbb{S}$, for even n;

iii. if $\mathsf{q} \in \mathbb{T}_{\mathbb{T}}$, then $R^n_{\mathsf{q}} \subset \mathbb{T}_{\mathbb{T}}$.

Taking into account the above considerations, the proof follows easily from the use of De Moivre's formulae (2.2) for x^n and of the appropriate formula (2.1) for q. ☐

We focus now on the expression of the nth roots of non-real coquaternions q which are light-like or with a light-like vector part.

Theorem 3.2. *Consider a non-real coquaternion* q *such that* $\mathsf{q} \in \mathbb{L}$ *or* $\underline{\mathsf{q}} \in \mathbb{L}$*. For* $n \geq 2$*, the* n*th roots of* q *can be characterized as follows:*

1. *If* $\mathsf{q} \in \mathbb{T}_{\mathbb{L}}$ *and*

   (a) n *is odd, then there is only one root, which is in* $\mathbb{T}_{\mathbb{L}}$*:*

$$\sqrt[n]{q_0} + \frac{\underline{\mathsf{q}}}{n(\sqrt[n]{q_0})^{n-1}} = \frac{\mathsf{q} + (n-1)q_0}{n(\sqrt[n]{q_0})^{n-1}};$$

   (b) n *is even and*

   i. $q_0 > 0$, *then there are two roots, both in* $\mathbb{T}_{\mathbb{L}}$*:*

$$\pm\left(\sqrt[n]{q_0} + \frac{\underline{\mathsf{q}}}{n(\sqrt[n]{q_0})^{n-1}}\right) = \pm\frac{\mathsf{q} + (n-1)q_0}{n(\sqrt[n]{q_0})^{n-1}};$$

   ii. $q_0 < 0$, *then there are no roots.*

2. *If* $\mathsf{q} \in \mathbb{L}_{\mathbb{S}}$ *and*

(a) n is odd, then there is only one root, which is in $\mathbb{L}_{\mathbb{S}}$: $\dfrac{\mathsf{q}}{(\sqrt[n]{2q_0})^{n-1}}$;

(b) n is even and

i. $q_0 > 0$, then there are two roots, both in $\mathbb{L}_{\mathbb{S}}$: $\pm\dfrac{\mathsf{q}}{(\sqrt[n]{2q_0})^{n-1}}$;

ii. $q_0 < 0$, then there are no roots.

3. If $\mathsf{q} \in \mathbb{L}_{\mathbb{L}}$, then there are no roots.

Proof. Observe that we are considering coquaternions q such that $\mathsf{q} \in \mathbb{L}_{\mathbb{L}}$ or $\mathsf{q} \in \mathbb{T}_{\mathbb{L}}$ or $\mathsf{q} \in \mathbb{L}_{\mathbb{S}}$ (cf. Remark 2.1). Then it follows easily that x and $\underline{\mathsf{x}}$ have the same nature of q and $\underline{\mathsf{q}}$ respectively (relations (3.1)-(3.2) are also valid here). The proof follows taking into account that the powers of a coquaternion under the assumptions of the theorem can be written in one of the forms

$$\mathsf{x}^n = \begin{cases} x_0^n + n x_0^{n-1}\underline{\mathsf{x}}, & \text{if } \mathsf{x} \in \mathbb{T}_{\mathbb{L}}, \\ (2x_0)^{n-1}\mathsf{x}, & \text{if } \mathsf{x} \in \mathbb{L}_{\mathbb{S}}, \\ 0, & \text{if } \mathsf{x} \in \mathbb{L}_{\mathbb{L}} \text{ and } n \geq 2. \end{cases} \tag{3.3}$$

□

4 Roots of a real number

To discuss the coquaternionic roots of a real number, it is convenient to review some more results on coquaternions.

We first recall the concepts of similarity and quasi-similarity for coquaternions; see, e.g., [9, 10] and references therein. We say that a coquaternion q is *similar* to a coquaternion p if there exists an invertible coquaternion h such that $\mathsf{q} = \mathsf{h}^{-1}\mathsf{p}\mathsf{h}$. This is an equivalence relation in $\mathbb{H}_{\text{coq}}$, partitioning $\mathbb{H}_{\text{coq}}$ in the so-called *similarity classes*. We denote by $[\mathsf{q}]$ the similarity class of $\mathsf{q} \in \mathbb{H}_{\text{coq}}$. It is easy to show that $[\mathsf{q}] = \{\mathsf{q}\}$ if and only if $\mathsf{q} \in \mathbb{R}$.

We say that two elements $\mathsf{p}, \mathsf{q} \in \mathbb{H}_{\text{coq}}$ are *quasi-similar* if and only if $\operatorname{tr}\mathsf{p} = \operatorname{tr}\mathsf{q}$ and $\det\underline{\mathsf{p}} = \det\underline{\mathsf{q}}$. This is also an equivalence relation in $\mathbb{H}_{\text{coq}}$; the class of an element $\mathsf{q} \in \mathbb{H}_{\text{coq}}$ with respect to this relation will be denoted by $[\![\mathsf{q}]\!]$ and referred to as the *quasi-similarity class* of q. It can be shown that for two non-real coquaternions, the concepts of quasi-similarity and similarity coincide, i.e., two non-real coquaternions are similar if and only if they are quasi-similar. However, if $\mathsf{q} = q_0 \in \mathbb{R}$, then q is only similar to itself but quasi-similar to all the coquaternions of the form $q_0 + \underline{\mathsf{p}}$ with $\det\underline{\mathsf{p}} = 0$. Observe that

$$[\![\mathsf{q}]\!] = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = \det\underline{\mathsf{q}}\}$$

can be identified with an hyperboloid in the hyperplane $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = q_0\}$: an hyperboloid of two sheets, if $\underline{\mathsf{q}} \in \mathbb{T}$, an hyperboloid of one sheet, if $\underline{\mathsf{q}} \in \mathbb{S}$ and a degenerate hyperboloid, i.e., a cone, if $\underline{\mathsf{q}} \in \mathbb{L}$. Note that, if $\underline{\mathsf{q}} \notin \mathbb{L}$, then $[\![\mathsf{q}]\!] = [\mathsf{q}]$.

We now present very briefly some results on the zeros of coquaternionic polynomials. We consider only monic unilateral left polynomials, i.e., polynomials of the form

$$P(x) = x^n + \mathsf{a}_{n-1}x^{n-1} + \cdots + \mathsf{a}_1 x + \mathsf{a}_0, \ \mathsf{a}_i \in \mathbb{H}_{\text{coq}}, \tag{4.1}$$

with addition and multiplication of such polynomials defined as in the commutative case where the variable is allowed to commute with the coefficients.

Given a quasi-similarity class $[\![\mathsf{q}]\!] = [\![q_0 + \underline{\mathsf{q}}]\!]$, the *characteristic polynomial* of $[\![\mathsf{q}]\!]$, denoted by $\Psi_{[\![\mathsf{q}]\!]}$, is the polynomial given by

$$\Psi_{[\![\mathsf{q}]\!]}(x) = x^2 - 2q_0\, x + \det\mathsf{q}.$$

This is a second degree monic polynomial with real coefficients whose discriminant is $-4\det\underline{\mathsf{q}}$. This means that $\Psi_{[\![q]\!]}$ is an irreducible polynomial (over the reals), if $\underline{\mathsf{q}} \in \mathbb{T}$ and a polynomial of the form $(x-\alpha_1)(x-\alpha_2)$, with $\alpha_1, \alpha_2 \in \mathbb{R}$, if $\underline{\mathsf{q}} \in \mathbb{S} \cup \mathbb{L}$. Reciprocally, any second degree monic polynomial $S(x)$ with real coefficients is the characteristic polynomial of a (uniquely) defined quasi-similarity class; if $S(x)$ is irreducible with two complex conjugate roots $\mathsf{z}, \bar{\mathsf{z}}$, then $S = \Psi_{[\![\mathsf{z}]\!]}$; if S has real roots α_1 and α_2 (with, eventually, $\alpha_1 = \alpha_2$), then $S = \Psi_{[\![\mathsf{q}]\!]}$ with $\mathsf{q} = \frac{\alpha_1+\alpha_2}{2} + \frac{\alpha_1-\alpha_2}{2}\mathbf{j}$.

Given a polynomial P of the form (4.1), its *conjugate polynomial* is the polynomial defined by $\overline{P}(x) = x^n + \bar{\mathsf{a}}_{n-1}x^{n-1} + \cdots + \bar{\mathsf{a}}_1 x + \bar{\mathsf{a}}_0$ and its *companion polynomial* is the polynomial given by $\mathcal{C}_P(x) = P(x)\overline{P}(x)$.

The following theorem contains an important result relating the characteristic polynomials of the quasi-similarity classes of zeros of a given polynomial P and the companion polynomial of P.

Theorem 4.1 ([10, Theorem 3.8])**.** *Let P be a polynomial of the form (4.1). If $\mathsf{z} \in \mathbb{H}_{\mathrm{coq}}$ is a zero of P, then $\Psi_{[\![\mathsf{z}]\!]}$ is a divisor of $\mathcal{C}_P$.*

It can be shown that $\mathcal{C}_P$ is a polynomial of degree $2n$ with real coefficients and, as such, considered as a polynomial in $\mathbb{C}$, has $2n$ roots. If these roots are $\mathsf{z}_1, \bar{\mathsf{z}}_1, \ldots, \mathsf{z}_\ell, \bar{\mathsf{z}}_\ell \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{R}$, where $s = 2(n-\ell)$, $(0 \le \ell \le n)$, then it follows from the above considerations that the characteristic polynomials which divide $\mathcal{C}_P$ are the ones associated with the following quasi-similarity classes:

$$[\![\mathsf{z}_k]\!];\ k = 1, \ldots, \ell, \tag{4.2a}$$

$$[\![\mathsf{r}_{ij}]\!];\ i = 1, \ldots, s-1,\ j = i+1, \ldots, s, \tag{4.2b}$$

with

$$\mathsf{r}_{ij} = \frac{\alpha_i + \alpha_j}{2} + \frac{\alpha_i - \alpha_j}{2}\mathbf{j}. \tag{4.2c}$$

We thus have that the zeros of the polynomial P belong to the union of the classes (4.2) which are called *admissible* classes for P.

Another known result on coquaternionic polynomials is the following: if the coefficients of P are real and $\mathsf{q} \in \mathbb{H}_{\mathrm{coq}} \setminus \mathbb{R}$ is a zero of P, then the whole class $[\![\mathsf{q}]\!]$ is made up of zeros of P. When $[\![\mathsf{q}]\!]$ contains no other zeros of P, we say that q is an isolated zero; if all coquaternions in $[\![\mathsf{q}]\!]$ are zeros of P, then q is called an *hyperboloidal zero* of P. When a zero q is hyperboloidal, we treat, for simplicity, the whole class as a single zero and talk about the hyperboloidal zero $[\![\mathsf{q}]\!]$. Since the nth roots of a coquaternion q are the roots of the equation $x^n = \mathsf{q}$, we will also adopt the same nomenclature for these roots.

Remark 4.2. *An admissible class of a general polynomial of the form (4.1) may also contain an infinite number of zeros forming a strict subset of the class. As proved in [10], this set of zeros (considered as points in $\mathbb{R}^4$) form a straight line, and are therefore called* linear zeros*. However, as we will see, linear zeros never occur in the case of polynomials of the form $x^n - \mathsf{q}$ that we are considering here.*

We have proved in the previous section that all the roots of a non-real coquaternion are isolated. The situation is quite different for the roots of real numbers, as the result contained in the next theorem reveals.

In what follows, for a given $n \in \mathbb{N}$, $n \ge 2$, we use ζ_k and η_k to denote the complex nth roots of 1 and -1, respectively, i.e.,

$$\zeta_k = \cos\tfrac{2k\pi}{n} + \mathbf{i}\sin\tfrac{2k\pi}{n};\ k = 0, 1, 2 \ldots, n-1 \tag{4.3a}$$

and

$$\eta_k = \cos\tfrac{(2k+1)\pi}{n} + \mathbf{i}\sin\tfrac{(2k+1)\pi}{n};\ k = 0, 1, 2 \ldots, n-1. \tag{4.3b}$$

Theorem 4.3. *Let $\alpha \in \mathbb{R}$. For $n \ge 2$, the nth coquaternionic roots of α can be characterized as follows.*

1. *If $\alpha > 0$ and*

   (a) *$n = 2m$, then there exist:*

   - *two real roots: $\pm\sqrt[n]{\alpha}$;*
   - *one hyperboloidal root of elements in $\mathbb{S}$: $[\![\sqrt[n]{\alpha}\,\mathbf{j}]\!]$;*
   - *if $m > 1$, $m-1$ hyperboloidal roots of elements in $\mathbb{T}_{\mathbb{T}}$:*

$$[\![\sqrt[n]{\alpha}\,\zeta_k]\!];\ k = 1, \ldots, m-1.$$

(b) $n = 2m + 1$, *then there exist:*

- *one real root:* $\sqrt[n]{\alpha}$;
- m *hyperboloidal roots with elements in* $\mathbb{T}_{\mathbb{T}}$:

$$[\![\sqrt[n]{\alpha}\,\zeta_k]\!];\ k = 1, \ldots, m.$$

2. *If* $\alpha < 0$, *then there exist:*

- $m = \lfloor \frac{n}{2} \rfloor$ *hyperboloidal roots of elements in* $\mathbb{T}_{\mathbb{T}}$:

$$[\![\sqrt[n]{|\alpha|}\eta_k]\!];\ k = 0, \ldots, m-1;$$

- *if* n *is odd, one extra real root :* $\sqrt[n]{\alpha}$.

3. *If* $\alpha = 0$, *then there exists one hyperboloidal root of elements in* $\mathbb{L}_{\mathbb{L}}$: $[\![0]\!]$.

Proof. Consider first the case 1. (a), where $\alpha > 0$ and $n = 2m$. As observed before, the nth roots of α are the zeros of the real polynomial $P(x) = x^n - \alpha$. The companion polynomial of P is $\mathcal{C}_P(x) = (x^n - \alpha)^2$ whose roots in $\mathbb{C}$ are the nth roots of α (all double roots), i.e., are the real roots

$$-\sqrt[n]{\alpha},\ -\sqrt[n]{\alpha},\ \sqrt[n]{\alpha},\ \sqrt[n]{\alpha}$$

and, if $m > 1$, the roots in $\mathbb{C} \setminus \mathbb{R}$

$$z_k,\ z_k,\ \overline{z}_k,\ \overline{z}_k; k = 1, \ldots, m-1, \quad \text{where} \quad z_k = \sqrt[n]{\alpha}\,\zeta_k.$$

We thus have the following three admissible classes for P:

$$[\![-\sqrt[n]{\alpha}]\!],\ [\![\sqrt[n]{\alpha}]\!],\ [\![\sqrt[n]{\alpha}\,\mathbf{j}]\!],$$

and, if $m > 1$, also the $m - 1$ classes

$$[\![\sqrt[n]{\alpha}\,\zeta_k]\!];\ k = 1, \ldots, m-1.$$

Since n is even, $(\sqrt[n]{\alpha}\mathbf{j})^n = \alpha$, i.e., $\sqrt[n]{\alpha}\mathbf{j}$ is a nth root of α. We thus have a non-real zero of the real polynomial P, the zero $\sqrt[n]{\alpha}\mathbf{j}$, and so we may conclude that all the elements in its quasi-similarity class are also zeros of P, hence nth roots of α. The same type of reasoning applies to the classes

$$[\![\sqrt[n]{\alpha}\,\zeta_k]\!];\ k = 1, \ldots, m-1.$$

Let us now show that the only roots of P in the two classes of real elements are these real elements. Assume that $\mathsf{x} \in [\![\sqrt[n]{\alpha}]\!]$ (the other case being similarly proved). Then $\mathsf{x} = x_0 + \underline{\mathsf{x}}$ where $x_0 = \sqrt[n]{\alpha}$ and $\underline{\mathsf{x}} \in \mathbb{T}_{\mathbb{L}}$, i.e., $\mathsf{x} \in \mathbb{T}_{\mathbb{L}}$. Then, as already observed (cf. (3.3)), we will have

$$\mathsf{x}^n = x_0^n + n x_0^{n-1}\underline{\mathsf{x}} = \alpha + n(\sqrt[n]{\alpha})^{n-1}\underline{\mathsf{x}}.$$

This shows that $\mathsf{x}^n = \alpha$ if and only if $\underline{\mathsf{x}} = 0$, i.e., that $\sqrt[n]{\alpha}$ is the only nth root of α in its quasi-similarity class, which concludes the proof.

The proofs of cases 1. (b) and 2. are totally analogous to the proof of 1. (a). In case 3., naturally, there is only one admissible class, the class $[\![0]\!]$; recalling formula (3.3), we immediately conclude that any element in $\mathbb{L}_{\mathbb{L}}$ — i.e., any element in $[\![0]\!]$ — is a nth root of $\alpha = 0$, which concludes the proof. □

Remark 4.4. *It is important to observe that the quasi-similarity classes referred to in the previous theorem always coincide — with the exception of case 3. — with similarity classes.*

In [17, Theorem 6], the author was aware of the possibility of the existence of infinitely many roots of the equation $\mathsf{x}^n = \alpha$, but did not fully discuss this case. The key ingredient to solve this problem was the use of the recent results obtained in [9, 10].

Example 4.5. *We consider now examples illustrating the conclusions of the previous theorems. Table 1 contains several coquaternions with different natures and the corresponding roots.*

Table 1: Number and nature of nth roots of some coquaternions.

q	nature	n	$\sqrt[n]{\mathsf{q}}$
$\mathbf{i}+2\mathbf{j}$	$\mathbb{S}$	2	—
		3	$\sqrt[3]{\frac{1}{3}}(\mathbf{i}+2\mathbf{j})$
$3+2\mathbf{j}$	$\mathbb{T}_{\mathbb{S}}$	2	$\pm(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\mathbf{j});\ \pm(\frac{1-\sqrt{5}}{2}-\frac{1+\sqrt{5}}{2}\mathbf{j})$
		3	$\frac{1+\sqrt[3]{5}}{2}-\frac{1-\sqrt[3]{5}}{2}\mathbf{j}$
$-3+2\mathbf{j}$	$\mathbb{T}_{\mathbb{S}}$	2	—
		3	$-\frac{1+\sqrt[3]{5}}{2}-\frac{1-\sqrt[3]{5}}{2}\mathbf{j}$
$1+\mathbf{i}$	$\mathbb{T}_{\mathbb{T}}$	2	$\pm\frac{1}{\sqrt{2}}(\sqrt{1+\sqrt{2}}+\sqrt{-1+\sqrt{2}}\mathbf{i})$
		3	$\frac{1}{\sqrt[3]{2}}(-1+\mathbf{i});\ \frac{1}{2\sqrt[3]{2}}(1\pm\sqrt{3}+(-1\pm\sqrt{3})\mathbf{i});$
$1+\mathbf{i}+\mathbf{j}$	$\mathbb{T}_{\mathbb{L}}\backslash\mathbb{R}$	2	$\pm(1+\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j})$
		3	$1+\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}$
$-1+\mathbf{i}+\mathbf{j}$	$\mathbb{T}_{\mathbb{L}}\backslash\mathbb{R}$	2	—
		3	$-1+\frac{1}{3}\mathbf{i}+\frac{1}{3}\mathbf{j}$
$4+3\mathbf{i}+5\mathbf{j}$	$\mathbb{L}_{\mathbb{S}}$	2	$\pm\sqrt{2}(1+\frac{3}{4}\mathbf{i}+\frac{5}{4}\mathbf{j})$
		3	$1+\frac{3}{4}\mathbf{i}+\frac{5}{4}\mathbf{j}$
$-4+3\mathbf{i}+5\mathbf{j}$	$\mathbb{L}_{\mathbb{S}}$	2	—
		3	$-1+\frac{3}{4}\mathbf{i}+\frac{5}{4}\mathbf{j}$
$\mathbf{i}+\mathbf{j}$	$\mathbb{L}_{\mathbb{L}}$	2	—
		3	—
1	$\mathbb{R}$	2	$\pm 1;\ [\mathbf{j}]$
		3	$1;\ [-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathbf{i}]$
		4	$\pm 1;\ [\mathbf{i}];\ [\mathbf{j}]$
		5	$1;\ [\frac{-1+\sqrt{5}}{4}+\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\mathbf{i}];\ [\frac{-1-\sqrt{5}}{4}+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\mathbf{i}]$
-1	$\mathbb{R}$	2	$[\mathbf{i}]$
		3	$-1;\ [\frac{1}{2}+\frac{\sqrt{3}}{2}\mathbf{i}]$
		4	$[\frac{1+\mathbf{i}}{\sqrt{2}}];\ [\frac{-1+\mathbf{i}}{\sqrt{2}}]$
		5	$-1;\ [\frac{1-\sqrt{5}}{4}+\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\mathbf{i}];\ [\frac{1+\sqrt{5}}{4}+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\mathbf{i}]$

5 Real roots of 2×2 matrices

It is well-known that the algebra of coquaternions is isomorphic to $\mathcal{M}_2(\mathbb{R})$, the algebra of real 2×2 matrices, with the map

$$\Phi : \mathcal{M}_2(\mathbb{R}) \to \mathbb{H}_{\mathrm{coq}} \tag{5.1a}$$

defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \mathsf{a} = \frac{1}{2}\big((a+d)+(c-b)\mathbf{i}+(b+c)\mathbf{j}+(a-d)\mathbf{k}\big), \tag{5.1b}$$

establishing the isomorphism. The inverse of Φ is the map

$$\Psi : \mathbb{H}_{\mathrm{coq}} \to \mathcal{M}_2(\mathbb{R}) \tag{5.2a}$$

defined by

$$\mathsf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \;\mapsto Q = \begin{pmatrix} q_0+q_3 & q_1+q_2 \\ q_2-q_1 & q_0-q_3 \end{pmatrix}. \tag{5.2b}$$

This means that all the results contained in the previous sections can be interpreted in terms of 2×2 real matrices; in particular, this will allow us to fully discuss the problem of the determination of the (real) nth roots of a given matrix $A \in \mathcal{M}_2(\mathbb{R})$, i.e., the determination of all the matrices $X \in \mathcal{M}_2(\mathbb{R})$ such that $X^n = A$, by making use of the results for coquaternions that we derived in the previous sections.

Note that we have $\Psi(1) = I$, where I denotes the identity matrix of order 2. Let us now denote by J and L the images under Ψ of the coquaternions, $\mathbf{j}$ and $\mathbf{i}$, respectively, i.e.,

$$J = \Psi(\mathbf{j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad L = \Psi(\mathbf{i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In what follows, A will be a given 2×2 real matrix of the form considered in (5.1b) and a will denote its image under the map (5.1), i.e., $\mathsf{a} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \Phi(A)$. It is also convenient to introduce the following notation, associated with the matrix A:

$$\alpha_0 = \frac{\operatorname{tr} A}{2} \qquad \text{and} \quad \underline{A} = A - \alpha_0 I. \tag{5.3}$$

It is very simple to verify that $\det A = \det \mathsf{a}$ and $\operatorname{tr} A = \operatorname{tr} \mathsf{a}$ and therefore $\alpha_0 = \frac{\operatorname{tr} \mathsf{a}}{2} = \operatorname{Re} \mathsf{a} = a_0$. Finally, denoting by $\|A\|$ the semi-norm given by $\sqrt{|\det A|}$, we observe that $\|A\| = \|\mathsf{a}\|$.

The results contained in the following theorems — Theorem 5.1, Theorem 5.2 and Theorem 5.3 — are the matrix counterparts of the results on roots of coquaternions given in Theorem 3.1, Theorem 3.2 and Theorem 4.3, respectively, and are obtained easily having in mind the following observations:

- The nth roots of the matrix A are the images, under the map Ψ, defined by (5.2), of the nth roots of the coquaternion a;
- The relations in (2.1b) can be written as
$$\sinh\phi_{\mathsf{a}} = \frac{a_0}{\|\underline{\mathsf{a}}\|} = \frac{\alpha_0}{\|\underline{A}\|}, \quad \sinh\psi_{\mathsf{a}} = \frac{\|\mathsf{a}\|}{\|\underline{\mathsf{a}}\|} = \frac{\|A\|}{\|\underline{A}\|}, \quad \cos\theta_{\mathsf{a}} = \frac{a_0}{\|\mathsf{a}\|} = \frac{\alpha_0}{\|A\|}, \quad \sin\theta_{\mathsf{a}} = \frac{\|\underline{\mathsf{a}}\|}{\|\mathsf{a}\|} = \frac{\|\underline{A}\|}{\|A\|};$$
- $\Psi(\boldsymbol{\omega}_{\underline{\mathsf{a}}}) = \Psi(\frac{\underline{\mathsf{a}}}{\|\underline{\mathsf{a}}\|}) = \frac{1}{\|\underline{\mathsf{a}}\|}\Psi(\underline{\mathsf{a}}) = \frac{1}{\|\underline{A}\|}\underline{A}$, where $\boldsymbol{\omega}_{\underline{\mathsf{a}}}$ is given by (2.1c);
- $\Psi(\zeta_k) = \cos\frac{2k\pi}{n} I + \sin\frac{2k\pi}{n} L$, where ζ_k is given by (4.3a);
- $\Psi(\eta_k) = \cos\frac{(2k+1)\pi}{n} I + \sin\frac{(2k+1)\pi}{n} L$, where η_k is given by (4.3b);
- The images under Ψ of coquaternions belonging to the same quasi-similarity class are matrices characterized by having the same trace and determinant, which, in the case of non-scalar matrices, i.e., matrices not of the form αI, correspond to similar matrices.[2]

The following theorem concerns the roots of non-scalar, non-singular matrices A such that $\underline{A}$ is also non-singular (cf. Theorem 3.1).

Theorem 5.1. *Let $A \in \mathcal{M}_2(\mathbb{R})$ be a non-scalar matrix such that both A and $\underline{A}$ are non-singular. Also, let ϕ_A, ψ_A , θ_A be the values given by*

$$\sinh\phi_A = \frac{\alpha_0}{\|\underline{A}\|}, \quad \sinh\psi_A = \frac{\|A\|}{\|\underline{A}\|}, \quad \cos\theta_A = \frac{\alpha_0}{\|A\|}, \quad \sin\theta_A = \frac{\|\underline{A}\|}{\|A\|}, \;\; \theta_A \in (0,\pi),$$

where α_0 and $\underline{A}$ are given by (5.3) and define the matrix

$$\Omega_{\underline{A}} = \frac{1}{\|\underline{A}\|}\underline{A}.$$

For $n \geq 2$, the real nth roots of A can be characterized as follows:

[2] Since we are working in $\mathcal{M}_2(\mathbb{R})$, when writing that A and B are similar matrices, this must be understood as meaning that $B = S^{-1}AS$ with S a real invertible matrix.

1. *If* $\det A < 0$ *and*

   (a) n *is odd, then there is only one root:*

$$\sqrt[n]{||A||}(\sinh \tfrac{\phi_A}{n} I + \cosh \tfrac{\phi_A}{n} \Omega_{\underline{A}});$$

   (b) n *is even, then there are no roots.*

2. *If* $\det A > 0$, $\det \underline{A} < 0$ *and*

   (a) n *is odd, then there is only one root:*

$$\sqrt[n]{||A||}(\operatorname{sgn} \alpha_0 \cosh \tfrac{\psi_A}{n} I + \sinh \tfrac{\psi_A}{n} \Omega_{\underline{A}});$$

   (b) n *is even and*

   i. $\alpha_0 > 0$, *then there are four roots:*

$$\pm \sqrt[n]{||A||}\big(\cosh \tfrac{\psi_A}{n} I + \sinh \tfrac{\psi_A}{n} \Omega_{\underline{A}}\big), \quad \pm \sqrt[n]{||A||}\big(\sinh \tfrac{\psi_A}{n} I + \cosh \tfrac{\psi_A}{n} \Omega_{\underline{A}}\big);$$

   ii. $\alpha_0 < 0$, *then there are no roots.*

3. *If* $\det \underline{A} > 0$, *then there are* n *roots:*

$$\sqrt[n]{||A||}\big(\cos \tfrac{\theta_A + 2k\pi}{n} I + \sin \tfrac{\theta_A + 2k\pi}{n} \Omega_{\underline{A}}\big), \ k = 0, \ldots, n-1. \tag{5.4}$$

The next result concerns the roots of a non-scalar matrix $A \in \mathcal{M}_2(\mathbb{R})$, in case A or $\underline{A}$ are singular (cf. Theorem 3.2). Note that, since $\det \underline{A} = \det A - \alpha_0^2$, the matrix A under consideration is such that $\det A = 0$ or $\det A = \alpha_0^2$.

Theorem 5.2. *Let* $A \in \mathcal{M}_2(\mathbb{R})$ *be a non-scalar matrix such that* $\det A = 0$ *or* $\det \underline{A} = 0$. *For* $n \geq 2$, *the real* n*th roots of* A *can be characterized as follows:*

1. *If* $\det A = \alpha_0^2$, $\alpha_0 \neq 0$ *and*

   (a) n *is odd, then there is only one root:*

$$\frac{1}{n(\sqrt[n]{\alpha_0})^{n-1}}(A + (n-1)\alpha_0 I). \tag{5.5}$$

   (b) n *is even and*

   i. $\alpha_0 > 0$, *then there are two roots:*

$$\pm \frac{1}{n(\sqrt[n]{\alpha_0})^{n-1}}(A + (n-1)\alpha_0 I); \tag{5.6}$$

   ii. $\alpha_0 < 0$, *then there are no roots.*

2. *If* $\det A = 0, \alpha_0 \neq 0$ *and*

   (a) n *is odd, then there is only one root:*

$$\frac{1}{(\sqrt[n]{2\alpha_0})^{n-1}} A. \tag{5.7}$$

   (b) n *is even and*

   i. $\alpha_0 > 0$, *then there are two roots:*

$$\pm \frac{1}{(\sqrt[n]{2\alpha_0})^{n-1}} A; \tag{5.8}$$

ii. $\alpha_0 < 0$, *then there are no roots.*

3. *If* $\det A = \alpha_0 = 0$, *then there are no roots.*

Our last theorem concerns the roots of scalar matrices (cf. Theorem 4.3).

Theorem 5.3. *Let* $A = \alpha I \in \mathcal{M}_2(\mathbb{R})$. *For* $n \geq 2$, *the real* n*th roots of* A *can be characterized as follows.*

1. *If* $\alpha > 0$ *and*

   (a) $n = 2m$, *then the roots are:*

   - *the two scalar matrices:* $\pm \sqrt[n]{\alpha} I$;
   - *all the matrices similar to the matrix* $\sqrt[n]{\alpha} J$, *i.e., the matrices*

$$\sqrt[n]{\alpha}\begin{pmatrix} u & v \\ w & -u \end{pmatrix}, \tag{5.9}$$

   with $u, v, w \in \mathbb{R}$ *such that* $vw = 1 - u^2$;

   - *if* $m > 1$, *all the matrices similar to each of the* $m-1$ *matrices* $\sqrt[n]{\alpha}\big(\cos\frac{k\pi}{m} I + \sin\frac{k\pi}{m} L\big)$; $k = 1, \ldots, m-1$, *i.e., the matrices*

$$\sqrt[n]{\alpha}\begin{pmatrix} \cos\frac{k\pi}{m} + u & v \\ w & \cos\frac{k\pi}{m} - u \end{pmatrix},$$

   with $u, v, w \in \mathbb{R}$ *such that* $vw = -\big(u^2 + \sin^2\frac{k\pi}{m}\big)$.

   (b) $n = 2m+1$, *then the roots are:*

   - *the scalar matrix* $\sqrt[n]{\alpha} I$;
   - *all the matrices similar to each of the* m *matrices*

$$\sqrt[n]{\alpha}\big(\cos\tfrac{2k\pi}{n} I + \sin\tfrac{2k\pi}{n} L\big);\ k = 1, \ldots, m,$$

   i.e., the matrices

$$\sqrt[n]{\alpha}\begin{pmatrix} \cos\frac{2k\pi}{n} + u & v \\ w & \cos\frac{2k\pi}{n} - u \end{pmatrix},$$

   with $u, v, w \in \mathbb{R}$ *such that* $vw = -\big(u^2 + \sin^2\frac{2k\pi}{n}\big)$.

2. *If* $\alpha < 0$, *then the roots are:*

   - *all the matrices similar to each of the* $m = \lfloor \frac{n}{2} \rfloor$ *matrices*

$$\sqrt[n]{|\alpha|}\big(\cos\frac{(2k+1)\pi}{n} I + \sin\tfrac{(2k+1)\pi}{n} L\big);\ k = 0, \ldots, m-1,$$

   i.e., the matrices

$$\sqrt[n]{|\alpha|}\begin{pmatrix} \cos\frac{(2k+1)\pi}{n} + u & v \\ w & \cos\frac{(2k+1)\pi}{n} - u \end{pmatrix},$$

   with $u, v, w \in \mathbb{R}$ *such that* $vw = -\big(u^2 + \sin^2\frac{(2k+1)\pi}{n}\big)$;

   - *if* n *is odd, the extra root* $\sqrt[n]{\alpha} I$.

3. *If* $\alpha = 0$, *then the roots are all the matrices with null trace and null determinant, i.e., are the matrices*

$$\begin{pmatrix} u & v \\ w & -u \end{pmatrix},\ \text{with } u, v, w \in \mathbb{R} \text{ such that } v\,w = -u^2.$$

Theorems 5.1–5.3 illustrate clearly the benefits from using the coquaternionic formalism in the derivation of the real roots of a 2×2 real matrix. This leads, in particular, to a systematic procedure for determining the roots, which is not present in the literature that we came across. However, some of the results here derived can be found in the work of other authors using a direct matrix approach, as we now emphasize:

- Formula (5.4) for the n roots of a non-scalar, non-singular matrix A such that $\underline{A}$ is also non-singular already appears in the paper by Cayley [4, Sec.49, p.34] (with a slight different expression). Although not explicitly stated, when determining the roots of a non-scalar matrix A, it is assumed by Cayley that $\det A > 0$ and $\det \underline{A} > 0$, i.e., the other two cases in Theorem 5.1 and the cases contained in Theorem 5.2 are not discussed in [4].

- The paper by Damphousse [7] contains results from which one can derive some of the results presented in Theorem 5.3. However, the roots given by formula (5.9) corresponding to case 1. and n even are left out of the study.[3] Expressions (5.5) and (5.6) are also given in [7]; this same paper also contains a general formula for the roots of a non-scalar matrix in $SL_2(\mathbb{C})$ for which $\det \underline{A} \neq 0$, explaining later how to deal with the more general case of $A \in GL_2(\mathbb{C})$. The referred formula allows complex matrices and involves the use of transcendental functions of complex variables, making it difficult to identify the number and expressions of the real solutions.

- Expressions (5.5) and (5.6) can be seen as a particular case of a more general formula (involving complex solutions) given in the paper by Choudhry [5]. In the same reference, formulae containing (5.7) and (5.8) for the case of a singular matrix such that $\det \underline{A} < 0$, and formulae for the complex roots of a scalar matrix are also given; in this last case, the real solutions written in the forms given in Theorem 5.3 are, however, not easily identifiable.

Example 5.4. *To illustrate the results obtained in this section, we reconsider here two examples studied in [5]. The first one consists on the determination of the 5th roots of the matrix*

$$A = \begin{pmatrix} 25 & 7 \\ -7 & 39 \end{pmatrix}.$$

Note that for this matrix, we have $\alpha_0 = 32$, $\det A = 1024 = \alpha_0^2$. Hence, we are in case 1. (a) of Theorem 5.2, with $\alpha_0 = 32$. We thus conclude that there exists only one real 5th root of A, given by formula (5.5):

$$\frac{1}{5(\sqrt[5]{32})^4}(A + 4 \times 32I) = \frac{1}{80}\begin{pmatrix} 153 & 7 \\ -7 & 167 \end{pmatrix},$$

which, as expected, is exactly the matrix found in [5].

The other example is concerned with finding the 4th roots of the matrix

$$A = \begin{pmatrix} -179 & 390 \\ -130 & 276 \end{pmatrix}.$$

In this case, $\alpha_0 = \frac{97}{2}$, $\underline{A} = \begin{pmatrix} -455/2 & 390 \\ -130 & 455/2 \end{pmatrix}$, $\det A = 1296 > 0$, $\det \underline{A} = -\frac{4255}{4} < 0$. Hence, we are in case 2. (b) i. of Theorem 5.1 and have $\|A\| = 36$, $\|\underline{A}\| = \frac{65}{2}$ and $\psi_A = \sinh^{-1}\frac{65}{72} = \log\frac{9}{4}$. By using the results of Theorem 5.1, we find that there are four different 4th roots of A, given by

$$\pm\sqrt{6}\big(\cosh(\log\tfrac{9}{4})I + \sinh(\log\tfrac{9}{4})\Omega_{\underline{A}}\big) = \pm\left(\frac{5}{2}I + \frac{1}{2}\begin{pmatrix} -7 & 12 \\ -4 & 7 \end{pmatrix}\right) = \pm\begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix}$$

and

$$\pm\sqrt{6}\big(\sinh(\log\tfrac{9}{4})I + \cosh(\log\tfrac{9}{4})\Omega_{\underline{A}}\big) = \pm\left(\frac{1}{2}I + \frac{5}{2}\begin{pmatrix} -7 & 12 \\ -4 & 7 \end{pmatrix}\right) = \pm\begin{pmatrix} -17 & 30 \\ -10 & 18 \end{pmatrix}.$$

These coincide with the four roots belonging to $\mathcal{M}_2(\mathbb{R})$ given in [5].[4]

It should be remarked that the computation of the nth roots of this last matrix, for values of $n > 4$, using the technique proposed in [5], would be rather involved and none of the previous work to obtain the fourth roots would be useful. However, the results of Theorem 5.1 would give us explicit expressions for any of these

[3] In particular, this leads the author of [7] to conclude that the identity matrix has only two square roots, which is clearly not true.

[4] We should observe that in [5], other 12 roots of A are found; these roots, however, have complex entries.

roots, using the already computed values of ψ_A, $\|A\|$ and $\Omega_{\underline{A}}$. For example, we would obtain the following four real 6th roots of this matrix:

$$\pm\begin{pmatrix} 4\times 2^{2/3}-3\times 3^{2/3} & -6(2^{2/3}-3^{2/3}) \\ 2(2^{2/3}-3^{2/3}) & -3\times 2^{2/3}+4\times 3^{2/3} \end{pmatrix}$$

and

$$\pm\begin{pmatrix} -4\times 2^{2/3}-3\times 3^{2/3} & 6\left(2^{2/3}+3^{2/3}\right) \\ -2\left(2^{2/3}+3^{2/3}\right) & 3\times 2^{2/3}+4\times 3^{2/3} \end{pmatrix},$$

as can be easily verified.

6 Conclusion

In this paper we derive and present in a systematic form, the nth roots of a coquaternion and consider, as an application, the determination of the real roots of a real 2×2 matrix, illustrating in this way the power and simplicity of the coquaternion language.

Mathematica codes to classify and determine the roots of any coquaternion are available at the webpage: http://w3.math.uminho.pt/CoquaternionsRoots.

References

[1] A. Alefeld and N. Scheider. On square roots of M-matrices. Lin. Alg. Appl., 42 (1982) 119–132.

[2] Å. Björck and S. Hammarling. A Schur method for the square roots of a matrix. Lin. Alg. Appl., 52/53 (1983) 127–140.

[3] D.C. Brody and E.M. Graefe. On complexified mechanics and coquaternions. J. Phys. A: Math. Theory, 44 (2011) 1–9.

[4] A. Cayley. A memoir on the theory of matrices. Philos. Trans. Roy. Soc. London, 148 (1858) 17–37.

[5] A. Choudhry. Extraction of nth roots of 2×2 matrices. Lin. Alg. Appl., 387(2004) 183–192.

[6] J. Cockle. On systems of algebra involving more than one imaginary; and on equations of the fifth degree. Philos. Mag., 35(3) (1849) 434–435.

[7] P. Damphousse. The arithmetic of powers and roots in $GL_2(\mathbb{C})$ and $SL_2(\mathbb{C})$. Fibonnacci Quart., 27 (1989) 386–401.

[8] E.D. Denman. Roots of real matrices. Lin. Alg. Appl., 36 (1981) 133–139.

[9] M.I. Falcão, F. Miranda, R. Severino, and M.J. Soares. Polynomials over quaternions and coquaternions: a unified approach. Lecture Notes in Comput. Sci., 10405 (2017) 379–393.

[10] M.I. Falcão, F. Miranda, R. Severino, and M.J. Soares. The number of zeros of unilateral polynomials over coquaternions revisited. Linear Multilinear Algebra, (2018) DOI: 10.1080/03081087.2018.1450828

[11] M.I. Falcão, F. Miranda, R. Severino, and M.J. Soares. Basins of attraction for a quadratic coquaternion map. Chaos Soliton. Fract., 104 (2017) 716–724.

[12] M.I. Falcão, F. Miranda, R. Severino, and M.J. Soares. The iteration of quadratic maps on coquaternions. Int. J. Bifurcat. Chaos, 27 (12) (2017) 1730039.

[13] I. Frenkel and M. Libine. Split quaternionic analysis and separation of the series for $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})/SL(2,\mathbb{R})$. Adv. Math., 228 (2) (2011) 678–763.

[14] N.H. Higham. Computing real square roots of a real matrix. Lin. Alg. Appl., 88/89 (1987) 405–429.

[15] W.D. Hoskins and D.J. Walton. A faster method for computing the square root of a matrix. IEEE Trans. Automat. Control AC, 23 (1978) 494–495.

[16] L. Kula and Y. Yayli. Split quaternions and rotations in semi Euclidean space E_2^4. J. Korean Math. Soc., 44 (2007) 1313–1327.

[17] M. Özdemir. The roots of a split quaternion. App. Math. Lett., 22(2) (2009) 258–263.

[18] M. Özdemir and A. Ergin. Rotations with unit timelike quaternions in Minkowski 3-space. J. Geometry Phys., 56(2) (2006) 322–336.

[19] A.A. Pogoruy and R.M. Rodriguez-Dagnino. Some algebraic and analytical properties of coquaternion algebra. Adv. Appl. Clifford Algebras, 20(1) (2010) 79–84.

[20] R. Serôdio, P.D. Beites and J. Vitória. Intersection of a double cone and a line in the split-quaternions context. Adv. Appl. Clifford Algebras, 27 (2017) 2795–2803.