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Error patterns in Portuguese students' addition and subtraction calculation tasks: Implications for teaching

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Implications for teaching

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Abstract

Purpose – The purpose of this descriptive study is to investigate why some elementary children have difficulties mastering addition and subtraction calculation tasks.

Design/methodology/approach – The researchers have examined error types in addition and subtraction calculation made by 697 Portuguese students in elementary grades. Each student completed a written assessment of mathematical knowledge. A system code (e.g. FR = failure to regroup) has been used to grade the tests. A reliability check has been performed on 65 per cent randomly selected exams.

Findings – Data frequency analyses reveal that the most common type of error was miscalculation for both addition (n = 164; 38.6 per cent) and subtraction (n = 180; 21.7 per cent). The second most common error type was related to failure to regroup in addition (n = 74; 17.5 per cent) and subtraction (n = 139; 16.3 per cent). Frequency of error types by grade level has been provided. Findings from the hierarchical regression analyses indicate that students' performance differences emerged as a function of error types which indicated students' types of difficulties.

Research limitations/implications – There are several limitations of this study: the use of a convenient sample; all schools were located in the northern region of Portugal; the limited number of problems; and the time of the year of assessment.

Practical implications – Students' errors suggested that their performance in calculation tasks is related to conceptual and procedural knowledge and skills. Error analysis allows teachers to better understand the individual performance of a diverse group and to tailor instruction to ensure that all students have an opportunity to succeed in mathematics.

Social implications – Error analysis helps teachers uncover individual students' difficulties and deliver meaningful instruction to all students.

Originality/value – This paper adds to the international literature on error analysis and reinforces its value in diagnosing students' type and severity of math difficulties.

Keywords Assessment, Error analysis, Learning, Mathematics calculation, Error patterns, Elementary grades

Paper type Research paper

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Mathematics achievement is a subject of international concern. There is compelling evidence that mathematics skills are fundamental to long-term academic success, provide life-long skills and are needed for obtaining employment in today's competitive market (Judge and Watson, 2011; Lembre *et al.*, 2012). Lack of mathematical literacy can seriously limit an individual's opportunities to succeed in school and life (Aud *et al.*, 2013; Duncan *et al.*, 2007). However, both in the USA and in Portugal, a significant percentage of K-12 students function below proficiency levels in mathematics (Aud *et al.*, 2013; Serrão *et al.*, 2010). The most recent assessment results from the Program for International Student Assessment from 2015 (OECD, 2016) show that the mathematics average score for Portugal is 492 and for the USA is 470. Similar average scores were shown in the 2011 Trends in International of Mathematics and Science Study. American fourth graders' average score was 541 and Portuguese fourth graders' average score was 532 (Provasnik *et al.*, 2012).

Given the fact that American and Portuguese elementary students perform similarly in mathematics, examining the mathematics skills of Portuguese students can contribute to our understanding of typical and atypical development of elementary students' mathematical skills. Given the growing number of Portuguese students in American schools and the similarities in their mathematical performance with regard to their American counterparts in the two international assessments, we decided to collaborate on an investigation with the purpose:

- to examine why some elementary children have difficulties mastering addition and subtraction calculation tasks;
- to determine the most common types of errors elementary Portuguese students in Grades 1 through 4 make when solving addition and subtraction calculation problems;
- to identify the most predominant errors at each grade level; and
- to investigate what type of errors predicts student performance in addition and subtraction.

Importance of error analysis

Improving mathematics education often is thought as increasing teacher content knowledge or the depth and breadth of knowledge of the mathematical concepts and skills they will be teaching (National Commission on Mathematics and Science Teaching for the 21st Century, 2000; National Mathematics Advisory Panel [NMAP], 2008). There is no doubt that teachers must know well the concepts and skills of the mathematics curriculum they will be teaching to students with diverse instructional needs. However, besides knowledge of content, mathematics teachers must understand the nature of errors students make to provide corrective feedback to students to eliminate those errors (Ashlock, 2010). It is advantageous that teachers identify what is affecting their students' progress as early as possible and provide explicit instruction that addresses their individual needs.

Diagnostic teaching, which includes the qualitatively analysis of students' error patterns, provides teachers the opportunity to identify students' types of errors and to determine the misconceptions and difficulties of their students (Ashlock, 2010; O'Connell, 1999). It is important that mathematics teachers examine students' erroneous mathematics concepts and/or procedures to provide effective instruction that addresses the diverse needs of individual students. In the elementary grades, many students make systematic and consistent errors when computing addition and subtraction problems. Often, teachers assume that these are careless mistakes or reflect a lack of fact fluency. However, this is not always the case. If those error patterns or misconceptions are not corrected early, the error patterns may persist and affect students' acquisition of higher mathematical skills such as algebra (Ashlock, 2010; Khan and Chishti, 2011). Accordingly, early identification of error patterns by means of error analysis is one way to afford students the equal opportunity to improve the mathematics outcomes of *all* students and narrow the gap between struggling and achieving students.

The National Council of Teachers of Mathematics (NCTM, 2000) considers computation to be one of the critical skills in the mathematics curriculum. *Number and operations* is first of the five content standards. This standard includes understanding numbers, knowing the meaning of operations and computing fluently. Several researchers (Riccomini, 2005; Tolar *et al.*, 2009) have asserted that if students do not acquire proficiency in computation skills, they will experience difficulties in the other areas of mathematics. For example, Tolar *et al.* (2009) found that computational fluency is the strongest influence factor on the algebra performance of undergraduate students. In addition, student computational skills are emphasized in the National Mathematics Advisory Panel's (NMAP)(2008) final report. The NMAP panel encouraged elementary, middle school and special education teachers to adequately prepare their students for algebra by making sure that students develop conceptual understanding of mathematical operations, computational fluency and problem-solving skills. Thus, it is important that teachers recognize students' misconceptions and/or lack of conceptual and procedural knowledge on solving computational problems to craft the most effective instructional intervention.

Although computation is an important academic skill, many students have difficulty with this basic mathematics skill, and, often, their error patterns are not acknowledged or addressed. Knowing the meaning of the basic arithmetic operations involves the understanding and the application of the commutative, associative and distributive properties of the operations (National Mathematics Advisory Panel [NMAP], 2008). The same seems to be true for students who have mathematics learning difficulties and/or disabilities (MD). It is well documented that students with MD have deficits in many areas of mathematics, especially computation fluency (Geary, 2004; Jordan *et al.*, 2010; Judge and Watson, 2011; Traff and Samuelson, 2013; Zheng *et al.*, 2012). To better meet students' needs, teachers should identify students' errors and analyze those through a systematic examination to pinpoint the cause(s) of the error types (Ashlock, 2010). For example, many students do not understand the concept of regrouping to solve addition or subtraction problems such as $46 + 17$ or $46 - 17$, and make systematic errors. This suggests that they do not have a good understanding of place value. In this case, intervention should focus on the students' conceptual knowledge of place value and not on the procedural knowledge of solving addition or subtraction problems. Conceptual understanding often promotes transfer of learning to new problems (Ashlock, 2010; Gilmore and Bryant, 2008).

Analysis of error patterns in computation problems is a valuable assessment tool for teachers. It shows what type of conceptual and/or procedural knowledge the student is having difficulty understanding. By analyzing the errors and identifying the patterns of mistakes students make when solving computation problems, teachers will be better able to plan appropriate and effective interventions (Luneta and Makonye, 2010; Riccomini, 2005) and provide students an *equal* opportunity to succeed in mathematics.

Brief review of studies on error analysis

Studies on student error analysis of computation problems were mostly conducted in the late 1960s, 1970s and 1980s. For example, Roberts (1968) investigated third graders and found that 36 per cent of their errors were because of defective algorithms (e. g. regrouping).

He classified students' errors according to four categories: wrong operation, obvious computational error, defective algorithm and random responses. Roberts noticed that those students who made careless or fact calculation errors made them consistently throughout the four operations. In a two-year study, Cox (1975a, 1975b) analyzed the errors of second through sixth graders. Results indicated that 5 to 6 per cent of the students made systematic errors in addition, multiplication and division; whereas, 13 per cent made errors in subtraction. Follow-up data showed that 25 per cent of the students either were making the same type of errors or another systematic type of error on the same algorithm. Several years later, Engelhardt (1977) studied the computation errors of third- and sixth-grade students on an 84-item test and extended Roberts' (1968) classification of errors. He identified eight error types: basic fact, grouping, inappropriate inversion, incorrect operation, defective algorithm, incomplete algorithm, identity and concept of zero. Results showed that basic fact errors were the most common type. Furthermore, over 40 per cent of the total errors were made by students in the lowest quartile. On the basis of the data, Engelhardt (1977) concluded that defective algorithm error type distinguished high performers from low performers. Brown and Burton (1978) reported similar findings in their analysis of procedural "bugs" (errors) of fourth, fifth and sixth graders. Using a computer program to analyze students' computational errors, they found that approximately 40 per cent of the students consistently made incorrect algorithm type of errors, especially in subtraction calculation problems where the top digit was a zero.

There have not been many studies on error analysis of computation problems in the twenty-first century. McIntosh (2002) described the common errors in mental computation of students in Grades 3-10. Riccomini (2005) identified systematic subtraction error patterns of students with learning disabilities. Raghubar *et al.*(2009) and colleagues investigated the types of errors in multi-digit computation in 291 third- and fourth-grade students with and without learning disabilities and the role of attention in math performance. They reported that math fact errors were related to the severity of math difficulties. Students with math learning disabilities also made more procedural errors than those without math disabilities. One of the few studies published is the investigation of Traff and Samuelson (2013). They analyzed errors of multi-digit computation and problem-solving in 142 elementary students with MD and compared them to 112 students without MD. They reported that a higher number of students with MD made errors when subtracting the smaller from the larger number as well as in regrouping and failure to regroup than students without MD. Their findings were similar to those reported by Raghubar *et al.* (2009). Finally, Traff and Samuelson (2013) suggested that the errors of students with MD were because of deficits related to both conceptual and procedural knowledge.

In sum, research on error analysis of students' basic computation problems revealed that basic fact error type was the most common type of error. One of the most prevalent error patterns was in subtraction problems in which the students subtracted a smaller number from a larger number (e.g. 53 – 28 =). Regrouping and difficulty with the concept of zero also were systematic errors among students.

In the present study, we provide information on the types of errors Portuguese elementary students who had completed grades one through four made when solving addition and subtraction computation problems. We examined students' addition and subtraction computation errors by grade level and compared them with error patterns of students among grade levels. In this article, our goal is to offer a diagnostic analysis to guide teachers in the evaluation of their students' error patterns to plan effective interventions that address the specific needs of each student.

Method

Participants

Participants were 697 students from 42 classrooms of seven elementary schools in Northern Portugal. Students had completed the school year as first through fourth graders (Table I). In total 62 percent of participants came from schools where 31 per cent of students received free lunch. The other participants (38 per cent) were from lower socioeconomic schools, with 35-51 per cent of students receiving free lunch. There were 339 (48.6 per cent) girls and 358 (51.6 per cent) boys aged 6-13 years old (age average was 8.78). All students were receiving instruction in the general education curriculum for their particular grade level. In Portugal, learning disabilities is not a recognized disability. This means that students who may have a learning disability in mathematics have not been identified as having MD although their teachers might have identified them as underachievers. However, some Portuguese schools do provide assistance to those students struggling in mathematics.

Materials

Each student completed a written assessment test of mathematical knowledge developed by Lospes and Bueno (2014). The instrument reflects the Portuguese mathematics curriculum of first through fourth grade. It is divided into three main sections, number knowledge, calculation and problem-solving, and it has a total of 46 items. In this article, we report on the ten items of the second part of the test (i.e. calculation) that involve addition and subtraction computation tasks. A principal component analysis, based on a tetracoric correlation matrix, revealed an initial six-factor solution for the test. A subsequent principal component analysis with an oblique rotation was conducted to test the hypothesis of a second-order factor related to the math general achievement. A 42-item solution representing five primary factors that cluster in a single second-order factor related to general math achievement was found. The Cronbach α for the general factor was 0.94. A psychometric analysis based on Rasch's model of the item response theory showed that the items cover the full extension of the subjects' skills.

Procedures

Before the school year started, two researchers met with the teachers from the seven schools involved in the project. In those meetings, the researchers described the purpose of the project, the development of the assessment tool and how students' answers would be evaluated. They explained to the teachers that their students were to complete the problems relevant to their particular grade level. Oral and written instructions on how to administer the test were provided and teachers' questions were answered during the meeting. Each classroom teacher received a package with the tests containing the written directions on how to administer the test to their students. Teachers were asked to instruct their students to complete the problems relevant to their specific grade level, but they could encourage the

Grade	No. of participants	Gender M	Gender F	Percentage of participants	Age range of participants
1st	117	63	54	16.8	6-10
2nd	139	73	66	19.9	7-10
3rd	144	71	73	20.7	8-10
4th	297	151	146	42.6	9-13
Total	697	358	339	100.0	

Table I. Descriptive analysis of participants students to answer other problems they knew how to solve. Whole-class assessment occurred in the students' own classrooms during the first two weeks of the school year. Our first graders were just starting second grade, our second graders were starting third grade, etc. The exams were returned to the researchers in their original packages.

Two researchers graded all tests three times. The first time, they identified correct/incorrect items. The second time, they used a code system to identify the types of errors (e.g. MC = miscalculation and FR = failure to regroup). The system code used was based on Engelhardt's (1977) and Traff and Samuelson's(2013) work. In addition to those types of errors, we found other errors that did not fit the coding system. We noted all different types of errors found in students' answers and added them to the original coding system. We re-examined all the problems using the new coding system (Table II). Errors were only counted when the student made the error solving his/her grade-level type of problem. We also noticed that a student could have more than one error in the same problem. For example, in the problem $347 + 236 = 572$, there are two different types of errors: 1) $7 + 6$ is 13, not 12 and 2) $4 + 3$ is 7, but the student should have added 1 to regroup and the answer should be 8. The first error type is a miscalculation error and the latter is an example of failure to regroup type of error. Thus, in the answer for this addition problem, there are two different types of errors, a miscalculation and a failure to regroup error.

After grading all tests for the third time, a reliability check was performed on 65 per cent ($n = 456$) of randomly selected exams from different grade levels. Inter-observer agreement was calculated by reporting agreements on occurrences divided by agreements plus disagreement (A / [A + D]). The percentage of agreement was 0.98. The two raters discussed the few disagreements and the disagreements were resolved by one of the other two researchers.

Results

The three most common error types

In this study, we investigated the types of errors Portuguese elementary students make when performing addition and subtraction computation problems. There were 424 addition errors and 854 subtraction errors. Data frequency analyses showed that the most common type of error was miscalculation for both addition ($n = 164$; 38.6 per cent) and subtraction ($n = 180$; 21.7 per cent) among the 697 students. In subtraction calculation problems, fourth graders had the highest percentage (35 per cent) of all miscalculation errors. Correlation analysis showed that there is a small significant negative correlation between grade-level and miscalculation errors in addition problems, $r\ (695) = -0.2$, $p < 0.005$. However, no

Table II.
Error types coding system

Addition	Subtraction
Miscalculation = MC	Miscalculation = MC
Wrong operation = WO	Wrong operation = WO
Failure to regroup = FR	Failure to regroup = FR
Concept of cero = CZ	Concept of zero = CZ
Regrouping error = RE	Regrouping error = RE
Decimal error = Ea	Decimal error = Ra
Copied number incorrectly = Rb	Copied number incorrectly = Rb
Incorrect answer w/o explanation = Rc	Incorrect answer w/o explanation = Rc
Omission (no attempt) = Rd	Omission (no attempt) = Rd
Procedure error = Re	Procedure error = Re
	Subtraction failure smaller from larger = SFL

significant correlation was found between grade and subtraction miscalculation errors. A significant positive correlation was observed between addition and subtraction calculation errors, r (695) = 0.15, $p < 0.005$.

The second most common error type among the 697 students was related to failure to regroup in addition (n = 74; 17.5 per cent) and subtraction (n = 139; 16.3 per cent). Failure to regroup in addition was positively correlated to the concept of zero error, r (695) = 0.22, $p <$ 0.005; decimal error, r (696) = 0.23, $p < 0.005$; and miscalculation error, r (696) = 0.15, $p <$ 0.005. Significant positive correlations were found between subtraction failure to regroup error and the concept of zero error, r (696) = 0.26, $p < 0.005$ and decimal error, r (696) = 0.24, $p < 0.005$. Grade level was positively correlated to failure to regroup error in addition, r (696) = 0.09, $p < 0.05$ and failure to regroup in subtraction, r (696) = 0.28, $p < 0.005$.

Omission and procedure errors together (n = 242; 18.9 per cent) were the third most common type of errors among all students. Omission and procedure errors indicate the student did not even attempt to solve the problem (i.e. omission) or they started to align the numbers but did not know how to proceed. We concluded that several students (n = 179; 25.7 per cent) did not know how to add (n = 84; 34.7 per cent) and/or subtract (n =158; 65.3 per cent) two-digit and multi-digit numbers. Those types of problems were part of the students' grade-level curriculum. If the problem was not part of the student's grade-level curriculum, it was not counted as an error of any type. There were 51 (61 per cent) omission types of errors in addition of two-digit and multi-digit problems and 44 (27.8 per cent) subtraction of two-digit and multi-digit problems were left blank. We coded them as omission errors. A total of 29 students (4.2 per cent) started to solve the addition of two-digit and multi-digit problems, but they did not know how to solve them. A total of 70 students (10 per cent) attempted to answer the subtraction of two-digit and multi-digit problems, but they did not know how to unravel the problems and left them incomplete. We coded those as procedure errors in addition (n= 33; 3.9 per cent) and subtraction (n = 114; 72 per cent). Procedure errors in addition had a small significant positive correlation to decimals, r (695) = 0.12, $p < 0.005$, and a small significant negative correlation to decimals in subtraction, r (695) = −0.9, $p < 0.05$.

The most predominant errors at each grade level

We used frequency to depict the number of times each error type occurred at each grade level. Miscalculation in addition was the most frequent type of error in all four grade levels (first = 53.3 per cent, second = 47.3 per cent, third = 62.3 per cent, and fourth = 22.7 per cent); adding decimals was the second most predominant type of error in Grade 4 (22.2 per cent). In subtraction, omission (36.4 per cent) and miscalculation (32.5 per cent) were the most frequent type of errors in first grade, subtracting a larger from a smaller number (32.9 per cent) was the number one type of error among second graders, failure to regroup (26.5 per cent) occurred more often in Grade 3, and errors involving the concept of zero (22.8 per cent) and failure to regroup (21.3 per cent) were repeatedly found among fourth graders. The number and percentage of the most recurrent types of errors in each grade level are displayed in Table III for addition and Table IV for subtraction.

Predictors of student performance

A series of hierarchical multiple regression analyses were performed to determine whether each independent variable (i.e. grade level and the most common type of errors) has a role in explaining the variance of student performance in addition and subtraction problems. In each analysis, the grade level of each student was entered into the regression equation, first as a control variable, followed by the six most common types of addition or subtraction

Table III.
Most predominant errors in addition per grade level

Grade	Type of error	No. of errors	Percentage
1st	Miscalculation	49	53.3
	Procedure	21	22.8
	Omission	13	14.1
2nd	Miscalculation	43	47.3
	Failure to regroup	26	28.6
	Omission	7	7.7
	Procedure	7	7.7
3rd	Miscalculation	27	62.3
	Failure to regroup	7	16.3
	Procedure	4	9.3
4th	Miscalculation	45	22.7
	Decimals	44	22.2
	Failure to regroup	41	20.7

Table IV.
Most predominant errors in subtraction per grade level

Grade	Type of error	No. of errors	Percentage
1st	Omission	28	36.4
	Miscalculation	25	32.5
	Procedure	10	13.0
2nd	Larger from small	69	32.9
	Miscalculation	52	24.8
	Omission	41	19.5
3rd	Failure to regroup	26	26.5
	Miscalculation	25	25.5
	Larger from smaller	24	24.5
4th	Concept of zero	107	22.8
	Failure to regroup	100	21.3
	Miscalculation	78	16.6

errors. These models were used to predict the total student performance in addition calculation problems and subtraction calculation problems. Intercorrelations of predictor and outcome variables are presented in Table V. This methodological approach permitted isolation of the relative contributions of the grade level first, followed by the individual contributions of each of the effects of error pattern variables on match calculation measures (Cohen and Cohen, 1983). In each of the analyses, the increments (I) in R^2 were determined to assess whether the different independent measures accounted for a significant proportion of variance in the dependent measures.

Table VI presents results of the hierarchical regression analysis predicting the addition calculation performance score. As can be seen in Table VI, this regression was significant ($F = 76.07$, $p < 0.001$). The addition of each predictor variable led to a statistically significant increase in R^2. After controlling for the grade-level variable, all five error patterns accounted for a statistically significant 36 per cent of the variance in the addition calculation performance score. Miscalculation error pattern was the strongest statistical predictor for addition calculation performance, $\Delta R^2 = 0.22$, $\Delta F\ (1{,}694) = 491.81$, $p < 0.001$. In addition, the procedure error pattern accounted for 7 per cent of the variance, $\Delta F\ (1{,}693) = 187.86$, $p <$ 0.001. Thus, these two error patterns accounted for 29 per cent of the variance in addition calculation performance.

Table V. Addition and subtraction errors, addition performance and subtraction performance: correlations and descriptions statistics (N = 697)

Variables	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1. Grade	–														
2. Addition miscalculation	−0.19***	–													
3. Addition failure to regroup	0.09**	0.15***	–												
4. Addition concept of zero	0.20***	0.01	0.22***	–											
5. Addition decimal	0.25***	−0.01	0.23***	0.70***	–										
6. Addition omission	−0.10**	−0.02	−0.01	0.04	0.12**	–									
7. Addition procedure	−0.10**	0.07	−0.06	−0.05	−0.06	0.01	–								
8. Subtraction miscalculation	−0.01	−0.16***	−0.09*	−0.02	0.04	0.02	0.02	–							
9. Subtraction failure to regroup	0.28***	−0.04	0.10**	0.13***	0.15***	0.02	−0.05	0.03	–						
10. Subtraction failure	−0.03	0.09*	0.10**	0.02	0.02	−0.05	−0.04	−0.00	−0.07	–					
11. Concept of zero	0.42***	−0.06	0.12**	0.35***	0.32***	−0.02	−0.08*	0.04	0.26***	0.09*	–				
12. Subtraction omission	−0.12**	0.18**	−0.03	−0.01	−0.01	0.07	0.47***	−0.01	−0.09*	−0.06	−0.11**	–			
13. Subtraction procedure	−0.05	−0.00	0.01	0.08*	0.10**	0.33***	0.04	0.03	0.01	−0.02	0.07	−0.03	–		
14. Addition performance	0.68***	−0.59***	−0.17***	−0.02	−0.01	−0.21***	−0.35***	−0.13**	0.16***	−0.02	0.22***	−0.27***	−0.10*	–	
15. Subtraction performance	0.67***	−0.29***	−0.00	0.06	0.07	−0.13**	−0.25***	−0.32***	−0.04	−0.15***	0.06	−0.40***	−0.18***	0.69***	–
M	2.89	0.24	0.11	0.05	0.06	0.05	0.07	0.26	0.20	0.20	0.15	0.16	0.06	3.85	3.12
SD	1.13	0.52	0.31	0.27	0.25	0.24	0.32	0.56	0.46	0.43	0.36	0.57	0.31	1.15	1.35

Notes: $^{*}p < 0.05$; $^{**}p < 0.01$; $^{***}p < 0.001$

Table VII presents the results of the hierarchical regression analyses for variables predicting the subtraction calculation performance score. Once more, the regression was significant ($F = 50.23$, $p < 0.001$). Similar to the previous results, all six error patterns accounted for a statistically significant 34 per cent of the variance in the subtraction calculation performance score after controlling for the grade-level variable. However, the omission (i.e. no attempt) error pattern was the strongest statistical predictor for subtraction calculation performance, $\Delta R^2 = 0.11$, ΔF (1,694), $p < 0.001$, and that miscalculation error, concept of zero error, failure to regroup, subtraction of larger from smaller, and procedure error also contribute to the prediction.

Table VI. Hierarchical regression analysis summary predicting addition calculation ($N = 697$)

Variables	Model 1			Model 2		
	B	*SE B*	*β*	*B*	*SE B*	*β*
Step 1						
Grade	0.69	0.03	0.68*	0.61	0.02	0.60*
Step 2: error type						
Miscalculation				−0.98	0.04	−0.44*
Procedure				−0.98	0.06	−0.27*
Decimal				−0.59	0.08	−0.13*
Failure to regroup				−0.54	0.06	−0.15*
Omission				−0.69	0.08	−0.14*
R^2		0.46			0.82	
Adjusted R^2		0.46			0.82	
F-value		597.28*			76.07*	
ΔR^2					0.36*	

Note: *$p < 0.001$

Table VII. Hierarchical regression analysis summary predicting subtraction calculation ($N = 697$)

Variables	Model 1			Model 2		
	B	*SE B*	*β*	*B*	*SE B*	*β*
Step 1						
Grade	0.80	0.03	0.67*	0.90	0.02	0.76*
Step 2: error type						
Omission				−0.87	0.04	−0.37*
Miscalculation				−0.71	0.04	−0.29*
Concept of zero				−0.75	0.07	−0.20*
Failure to regroup				−0.67	0.05	−0.23*
Subtraction failure				−0.49	0.06	−0.16*
Procedure				−0.55	0.08	−0.12*
R^2		0.45			0.79	
Adjusted R^2		0.45			0.79	
F-value		565.02*			50.23*	
ΔR^2					0.34*	

Note: * $p < 0.001$

Discussion

Miscalculation in both addition and subtraction was a common type of error in all grade levels (i.e. first through fourth). Miscalculation often is associated with poor retrieval strategy (Geary, 1996; Geary *et al.*, 2000), inattentive behavior (Hecht and Vagi, 2010) or inability to count (Cheng and Chan, 2005). We found that one group of first graders relied heavily on drawing and then counting to arrive at the answers. The drawing-counting strategy seemed to have contributed to the high number of miscalculations in first graders in both addition and subtraction. Some authorities (Cheng, 2012; Murata, 2004) suggest that counting is an immature strategy and may delay the development of other mathematical skills.

Other types of errors found among the participants in both addition and subtraction tasks suggest that some students had problems with the conceptual knowledge of decimals, the base-10 system and place value. Failure to regroup was the second most common type of error which may indicate students' misunderstanding of place value. Some researchers (Cauley, 1988; Gilmore and Bryant, 2008; Hiebert and Lafevre, 1986) have asserted that conceptual and procedural knowledge should be linked for students to become competent in mathematics. Thanheiser (2012) and Weber (2001) added *strategic knowledge* to achieving competence in mathematics. They explained that students must know in what situations to apply their conceptual and procedure knowledge. Many students do not recognize the connection between the base-10 system context and the standard algorithms when regrouping, especially when adding and subtracting multi-digit numbers (Thanheiser, 2012). Covariant analyses showed significant correlations between failure to regroup, decimals and the concept of zero errors. Identifying students' misunderstanding and/or lack of knowledge of a mathematical concept and providing interventions that address those areas may offer struggling students opportunities to grow as their same-age peers without mathematics difficulties.

Difficulty with the conceptual knowledge of addition was observed in the answers for problems in which the sum of the given numbers did not result in a larger number than the digits given. The misunderstanding of the concept of subtraction was revealed in responses in which students subtracted a larger number from a smaller one. The analyses of errors suggest a relationship between conceptual and procedural knowledge (Fuson and Kwon, 1992; Schneider and Stern, 2010; van Lehn, 1982) and strategic knowledge (Thanheiser, 2012). This information is helpful to teachers because they are able to better examine students' levels of understanding, ability to integrate procedural and conceptual knowledge, and application of strategies used to solve calculation problems. Using the data from students' performance, educators can make informed decisions about effective instruction that addresses the individual needs of their students (Baroody *et al.*, 2007).

Findings from the hierarchical regression analyses indicate that students' performance differences emerged as a function of error types which indicate students' type of difficulties. As shown in Tables VI and VII, error patterns in addition and subtraction calculation problems accounted for significant variance in students' scores in addition and subtraction calculation problems. These results underscore the need for analysis of students' type of errors. Certain types of errors (e.g. failure to regroup) indicate students' difficulties with a particular kind of conceptual and procedure knowledge (e.g. base-10). For that reason, educators must thoroughly assess each student's strengths and weaknesses before deciding which area or type of instruction a student needs.

Results from this study add to the available literature on mathematics development in several ways. Grade level significantly affected students' performance, suggesting an association with pre-skills to calculation (e.g. place value, counting strategies). These findings indicate that early intervention on students' understanding, procedural and strategic knowledge of pre-skills may be an effective way to eliminate the occurrence of certain type of errors in the upper grades. Some researchers (Hill *et al.*, 2005; Thanheiser, 2009) assert that teachers must have deep and detailed knowledge of mathematics to be able to explain adequate mathematical concepts (e.g. algorithm) to their students. This type of teacher knowledge, *mathematical knowledge for teaching*, has been shown to predict student success in mathematics (Ball *et al.*, 2005; Hill *et al.*, 2005). Professionals who work to construct curriculum and programs that involve the acquisition and teaching of mathematics knowledge and skill may find this information useful to provide equal opportunities to students whose evidence varied in mathematics knowledge and skills.

Limitations

There are several limitations of this study that should be noted. First, is the use of a convenient sample. We assessed students from schools who welcomed our project. Second, all schools were located in the northern region of Portugal and only two of the seven schools were considered suburban schools. The other five were rural schools. Future research would be enhanced by including participants from different parts of Portugal and students from more diverse settings (e.g. urban). The same is true for the socioeconomic background of students; our participants were from medium to low socioeconomic upbringings.

A third limitation of the study was the administration of only ten calculation problems representing four different grade levels. Another possible limitation was the administration of the assessment instrument during the first two weeks of the beginning of the school year. Our first graders were just starting second grade, our second graders were starting third grade, etc. Although, we wanted to examine the maintenance of skills from one grade to another, students may have increased or decreased their mathematics performance during the summer (two-and-a half-month vacation). Administering the assessment tool at the end of the school year and again at the beginning of the following year would afford researchers the opportunity to measure students' maintenance of skills over the summer.

Implications for practice

The findings of this study have practical implications beyond statistical significance. First, teachers responsible for instruction of mathematics need to assess students' mathematical conceptual and procedural knowledge and skills. Error analysis can yield important information about students' thinking, understanding and misconceptions (Busi and Jacobbe, 2014; Silver *et al.*, 2009; Thanheiser, 2009). Second, misconceptions come from prior knowledge and errors are the result of their naïve concept (Luneta and Makonye, 2010). With students' possible varying background knowledge and the earlier identified differences, the greater was the opportunity to promote student success in mathematics. Third, error analysis allows teachers to give specific corrective feedback to students and address individual student's misconceptions (Riccomini, 2005). Errors should be viewed as learning opportunities and used to monitor student progress. Another critical implication for practice is the importance of diagnostic assessment provided by error analysis. Diagnostic assessment of student error patterns gives teachers the opportunity to deliver meaningful instruction to every student (Horn *et al.*, 2015) and promote equality among a diverse population of students.

Conclusion

An international study involving the analysis of mathematics errors in other countries serves to inform classroom assessment and intervention practices. By examining the addition and subtraction calculation errors of elementary-aged Portuguese students, we identified the specific type of errors students make when adding and subtracting numbers. Without a thorough evaluation of students' error types, teachers will be unable to develop effective instruction. Assessments of student learning should be translated into diagnostic information to support instructional decision-making and, in turn, student learning (Horn *et al.*, 2015).

In the USA, mathematics instruction has suffered because of the limited amount of information on the development of students' deep understanding of mathematical concepts (Silver *et al.*, 2009). Learning mathematics with understanding requires teachers' ability to identify specific students' type of errors that reflect students' lack of understanding of conceptual knowledge. Not surprisingly, some authorities (Ball, 2000; Ball *et al.*, 2008; Hill and Ball, 2004) have identified teachers' *knowledge of common student misconceptions and errors* as one of the six constructs of mathematics teacher effectiveness to foster student learning. Teachers should be aware of high probability errors at each grade level. Drawing from accumulated research and results of the present study affords the importance of quality instruction for all students to be successful in the important area of mathematics. Quality instruction leads to student learning, equality in education and mathematics success (Kersting *et al.*, 2012).

References

Ashlock, R.B. (2010), *Errors Patterns in Computation: Using Error Patterns to Help Each Student Learn*, 10th ed., Allyn & Bacon, Boston, MA.

Aud, S., Wilkinson-Flicker, S., Kristapovich, P., Rathbun, A., Wang, X. and Zhang, J. (2013), *The Condition of Education 2013 (NCES 2013-037)*, U.S. Department of Education, National Center for Education Statistics, Washington, DC, available at: http://nces.ed.gov/pubsearch

Ball, D.L. (2000), "Bridging practices: Intertwining content and pedagogy in teaching and learning to teach", *Journal of Teacher Education*, Vol. 51 No. 3, pp. 241-247, doi: 10.1177/0022487100051003013.

Ball, D.L., Hill, H.C. and Bass, H. (2005), "Knowing mathematics for teaching: who knows mathematics well enough to teach third grade, and how can we decide?", *American Educator*, Vol. 14, 16-17, 20-22, 43-46.

Ball, D.L., Thames, M.H. and Phelps, G. (2008), "Content knowledge for teaching: what makes it special?", *Journal of Teacher Education*, Vol. 59 No. 5, pp. 398-407, doi: 10.1177/0022487108324554.

Baroody, A.J., Feil, Y. and Johnson, A.R. (2007), "An alternative conceptualization of procedural and conceptual knowledge", *Journal for Research in Mathematics Education*, Vol. 38 No. 2, pp. 115-131.

Brown, J.S. and Burton, R.B. (1978), "Diagnostic models for procedural bugs in basic mathematical skills", *Cognitive Science*, Vol. 2 No. 2, pp. 155-192.

Busi, R. and Jacobbe, T. (2014), "Examining student work in the preparation of preservice elementary school teachers", *The Mathematics Educator*, Vol. 23 No. 2, pp. 23-29.

Cauley, K.M. (1988), "Children's misconceptions about the multidigit subtraction algorithm", Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA, April 5-9). ED 294766.

Cheng, Z.J. (2012), "Teaching young children decomposition strategies to solve addition problems: an experimental study", *Journal of Mathematical Behavior*, Vol. 31 No. 1, pp. 29-47, doi: 10.1016/j.jmthb.2011.09.002.

Cheng, Z.J. and Chan, L.K.S. (2005), "Chinese number-naming advantages? Analyses of Chinese Pre-Schoolers' computational strategies and errors", *International Journal of Early Years Education*, Vol. 13 No. 2, pp. 179-192, doi: 10.1080/09669760500171279.

Cohen, J. and Cohen, P. (1983), *Applied Multiple Regression/Correlation Analysis for The Behavioral Sciences*, Erlbaum, Hillsdale, NJ.

Cox, L.S. (1975a), "Systematic errors in the four vertical algorithms in normal and handicapped populations", *Journal for Research in Mathematics Education*, Vol. 6 No. 4, pp. 202-220.

Cox, L.S. (1975b), "Diagnosing and remediating systematic errors in addition and subtraction", *The Arithmetic Teacher*, Vol. 22, pp. 151-157.

Duncan, G.J., Dowsett, C.J., Claessens, A., Magnuson, K., Huston, A.C., Klebanov, P., Pagani, L.S., Feinstein, L., Engel, M., Brooks-Gunn, J., Duckworth, K. and Japel, C. (2007), "School readiness and later achievement", *Developmental Psychology*, Vol. 43 No. 6, pp. 1428-1446, available at: www.ncbi.nlm.nih.gov/pubmed/18020822

Engelhardt, J.M. (1977), "Analysis of children's computational errors: a qualitative approach", *British Journal of Educational Psychology*, Vol. 47 No. 2, pp. 149-154, doi: 10.1111/j.2044-8279.1977.tb02340.x.

Fuson, K.C. and Kwon, Y. (1992), "Korean children's single digit addition and subtraction: Numbers structured by ten", *Journal for Research in Mathematics Education*, Vol. 23 No. 2, pp. 148-165.

Geary, D.C. (1996), *Children's Mathematical Development*, American Psychological Association, Easton, MD.

Geary, D.C. (2004), "Mathematics and learning disabilities", *Journal of Learning Disabilities*, Vol. 37 No. 1, pp. 4-15.

Geary, D.C., Hamson, C.O. and Hoard, M.K. (2000), "Numerical and arithmetical cognition: a longitudinal study of process and concept deficits in children with learning disabilities", *Journal of Experimental Child Psychology*, Vol. 77 No. 3, pp. 236-263, available at: www.ncbi.nlm.nih.gov/pubmed/11023658

Gilmore, C.K. and Bryant, P. (2008), "Can children construct inverse relations in arithmetic? evidence for individual differences in the development of conceptual understanding and computational skill", *British Journal of Developmental Psychology*, Vol. 26 No. 3, pp. 301-316, doi: 10.1348/026151007X236007.

Hecht, S.A. and Vagi, K.J. (2010), "Sources of group and individual differences in emerging fraction skills", *Journal of Educational Psychology*, Vol. 102 No. 4, pp. 843-859, doi: 10.1037/a0019824.

Hiebert, J. and Lafevre, P. (1986), "Conceptual and procedural knowledge in mathematics: an introductory analysis", in J., Hiebert (Ed.), *Conceptual and Procedural Knowledge*, Lawrence Erlbaum, Hillsdale, NJ, pp. 1-27.

Hill, H.C. and Ball, D.L. (2004), "Learning mathematics for teaching: results from California's mathematics professional development institutes", *Journal for Research in Mathematics Education*, Vol. 35 No. 5, pp. 330-351, doi: 10.2307/30034819.

Hill, H.C., Rowan, B. and Ball, D.L. (2005), "Effects of teachers' mathematical knowledge for teaching on student achievement", *American Educational Research Journal*, Vol. 42 No. 2, pp. 37-406, available at: http://deimos3.apple.com/WebObjects/Core.woa/FeedEnclosure/tamu-public.2117699024.02117699032.2276427372/enclosure.pdf

Horn, I.S., Kane, B.D. and Wilson, J. (2015), "Making sense of student performance data: data use logic and mathematics teachers' learning opportunities", *American Educational Research Journal*, Vol. 52 No. 2, pp. 208-242, doi: 10.3102/0002831215573773.

Jordan, N.C., Glutting, J. and Ramineni, C. (2010), "The importance of number sense to mathematics achievement in first and third grades", *Learning and Individual Differences*, Vol. 20 No. 2, pp. 82-88, doi: 10.1016/j.lindif.2009.07.004.

Judge, S. and Watson, S.M.R. (2011), "Longitudinal outcomes for mathematics achievement for students with learning disabilities", *Journal of Educational Research*, Vol. 104 No. 3, pp. 147-157, doi: 10.1080/00220671003636729.

Kersting, N.B., Givvin, K.B., Thompson, B.J., Santagata, R. and Stigler, J.W. (2012), "Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning", *American Educational Research Journal*, Vol. 49 No. 3, pp. 568-589, doi: 10.3102/0002831212437853.

Khan, S.B. and Chishti, S.H. (2011), "Learners' errors: supporting learners for participating in mathematics classroom", *International Journal of Academic Research*, Vol. 3, pp. 229-239, available at: www.researchgate.net/publication/258047531_LEARNERS'_ERRORS_SUPPORTING_LEARNERS_FOR_PARTICIPATING_IN_MATHEMATICS'_CLASSROOM

Lembre, E.S., Hampton, D. and Beyers, S.J. (2012), "Response to intervention in mathematics: Critical elements", *Psychology in the Schools*, Vol. 49 No. 3, pp. 257-272, doi: 10.1002/pits.21596.

Lospes, J. and Bueno, M. (2014), "Construção e validação de uma prova de mathemática Para alunos do primeiro ao quarto ano de escolaridade (development and validation of a math test for first to fourth grade students)", *Psicologia: Reflexão e Crítica*, Vol. 27 No. 3, pp. 434-442, doi: 10.1590/1678-7153.201427303.

Luneta, K. and Makonye, P.J. (2010), "Learner errors and misconceptions in elementary analysis: a case study of a grade 12 class in South Africa", *Acta Didactica Napocensia*, Vol. 3 No. 3, pp. 35-47.

McIntosh, A. (2002). "Common errors in mental computation of students in grades 3-10", in Barton, B., Irwin, K.C., Pfannkuch, M. and Thomas, M.O.J. (Eds), *Mathematics Education in the South Pacific (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, MERGA*, *Auckland, Sydney*, pp. 457-464.

Murata, A. (2004), "Paths to learning ten-structured understanding of teen sums: Addition solution of Japanese grade 1 students", *Cognition and Instruction*, Vol. 22 No. 2, pp. 185-218, available at: www.jstor.org/stable/3233927

National Commission on Mathematics and Science Teaching for the 21st Century (2000), *Before it's too late: A report to the nation from the National Commission on Mathematics and Science Teaching for the 21st Century. U.S.* Department of Education, Washington, DC, available at: www.csu.edu/cerc/researchreports/documents/BeforeItsTooLate2000.pdf

National Council of Teachers of Mathematics (NSTM) (2000), *Principles and Standards for School Mathematics*, Author, Reston, VA.

National Mathematics Advisory Panel (NMAP) (2008), *Foundation for Success: The Final Report of the National Mathematics Advisory Panel*, US Department of Education, Washington, DC, available at: www.ed.gov/pubs/edpubs.html

O'Connell, A.A. (1999), "Understanding the nature of errors in probability problem-solving", *Educational Research and Evaluation: An International Journal on Theory and Practice*, Vol. 5 No. 1, pp. 1-21, available at: http://dx.doi.org/10.1076/edre.5.1.1.3887

OECD (2016), *PISA Results (Volume I): Excellence and Equity in Education*, OECD Publishing, Paris, available at: http://dx.doi.org/10.1787/978926466490-en

Provasnik, S., Kastberg, D., Ferraro, D., Lemanski, N., Roey, S. and Jenkins, F. (2012), *Highlights from TIMMS 2011: Mathematics and Science Achievement of U. S. fourth- and Eighth-Grade Students in an International Context (NCES 2013-009)*, National Center for Educational Statistics, Institute of Education Sciences, U. S. Department of Education. Washington, DC, available at: http://nces.ed.gov/pubs2013/2013009_1.pdf

Raghubar, K., Cirino, P., Barnes, M., Ewing-Cobbs, L., Fletcher, J. and Fuchs, L. (2009), "Errors in multi-digit arithmetic and behavior inattention in children with math difficulties", *Journal of Learning Disabilities*, Vol. 42 No. 4, pp. 356-371, doi: 10.1177/0022219409335211.

Riccomini, P.J. (2005), "Identification and remediation of systematic error patterns in subtraction", *Learning Disability Quarterly*, Vol. 28 No. 3, pp. 233-242.

Roberts, G.H. (1968), "The failure strategies of third grade arithmetic pupils", *The Arithmetic Teacher*, Vol. 15 No. 5, pp. 442-446.

Schneider, M. and Stern, E. (2010), "The developmental relations between conceptual and procedural knowledge: a multimethod approach", *Developmental Psychology*, Vol. 46 No. 1, pp. 178-192, doi: 10.1037/a0016701.

Serrão, A., Ferreira, C. and de Sousa, H.D. (2010), *PISA 2009: Competências Dos Alunos Portugueses: Síntese De Resultados. (Competencies of Portuguese Students: Synthesis of the Results)*, GAVE, Ministry of Education, Lisbon.

Silver, E.A., Mesa, V.M., Morris, K.A., Star, J.R. and Benken, B.M. (2009), "Teaching mathematics for understanding: an analysis of lessons submitted by teachers seeking NBPTS certification", *American Educational Research Journal*, Vol. 46 No. 2, pp. 501-531, doi: 10.3102/0002831208326559.

Thanheiser, E. (2009), "Preservice elementary school teachers' conceptions of multidigit whole numbers", *Journal for Research in Mathematics Education*, Vol. 40, pp. 251-281, available at: www.jstor.org/stable/40539337

Thanheiser, E. (2012), "Understanding multidigit whole numbers: the role of knowledge components connections, and context in understanding regrouping 3 +− digit numbers", *The Journal of Mathematical Behavior*, Vol. 31 No. 2, pp. 220-234, doi: 10.1016/j.jmathb.2011.12.007.

Tolar, T.D., Lederberg, A.R. and Fletcher, J.M. (2009), "A structural model of algebra achievement: computational fluency and spatial visualization as mediators of the effect of working memory on algebra achievement", *Educational Psychology: An International Journal of Experimental Educational Psychology*, Vol. 29 No. 2, pp. 239-266, doi: 10.1080/01443410802708903.

Traff, U. and Samuelson, J. (2013), "Analysis of errors in multi-digit arithmetic and arithmetic word problem solving in children with mathematics learning difficulties", *Special Education*, Vol. 1, pp. 121-132.

Van Lehn, K. (1982), "Bugs are not enough: Empirical studies of bugs, impasses and repairs in procedural kills", *Journal of Mathematics Behavior*, Vol. 3, pp. 3-71, available at: www.public.asu.edu/~kvanlehn/Stringent/PDF/BugsAreNotEnough1982.pdf

Weber, K. (2001), "Student difficulty in constructing proofs: the need for strategic knowledge", *Educational Studies in Mathematics*, Vol. 48 No. 1, pp. 101-119, doi: 10.1023/A:1015535614355.

Zheng, X., Flynn, L.J. and Swanson, H.L. (2012), "Experimental intervention studies on word problem solving and math disabilities: a selective analysis of the literature", *Learning Disability Quarterly*, Vol. 36 No. 2, pp. 97-111, doi: 10.1177/0731948712444277.

Further reading

Fuson, K.C., Carroll, W.M. and Drueck, J.V. (2000), "Achievement results for second and third graders using the standards-based curriculum everyday mathematics", *Journal for Research in Mathematics Education*, Vol. 31 No. 3, pp. 277-295, available at: www.jstor.org/stable/t49808

Geary, D.C. (2007), "An evolutionary perspective on learning disabilities in mathematics", *Developmental Neuropsychology*, Vol. 32 No. 1, pp. 471-519.

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