EXPLORING PATTERNS USING INQUIRY-BASED LESSONS

Joana Silva and Ema Mamede

CIEC – University of Minho

This study focuses on the exploration of patterns by 6th-graders in their mathematics class. It addresses two questions: 1) How do students embrace tasks involving patterns? 2) What difficulties do they face when solving these tasks? An intervention of three inquiry-based mathematics lessons comprising 3 tasks of patterns (3 to continue a sequence and 2 to investigate the sequence) was carried out. Qualitative methods were used to analyse students' solving the tasks. Results show that students succeeded in the majority of the tasks. The inquiry-based lessons gave them an opportunity to solve problems and stimulate mathematical reasoning and students' oral mathematics communication, creating a challenging environment in the classroom.

PATTERNS IN THE TEACHING OF MATHEMATICS

In agreement with Orton (2005), patterns are difficult to define as the word has several meanings. In the literature patterns relates to expressions such as regularities, sequences, order and structure. Vale, Palhares, Cabrita and Borralho (2006) argue that the concept of pattern is used when one refers to a particular disposition of shapes, colours or sounds that involve regularities. Devlin (2002) points out that patterns are the essence of mathematics, and what mathematicians do is to examine abstract patterns – numeric patterns, patterns of shape, patterns of movement, among others.

Concerning the types of patterns, Orton (2005) distinguishes geometric patterns, in which the regularity relates to some idea of symmetry, and numeric patterns, involving numeric sequences. Vale and Pimentel (2009) distinguish patterns of repetition, that has an identified motive that repeats cyclically indefinitely, and patterns of growth, in which each term changes in a predictable way from the previous one. There are linear and non-linear patterns of growth, which the related algebraic expression can be, or not, represented by a 1st-grade polynomial expression. Zazkis and Liljedahl (2012) distinguish numeric patterns, geometric patterns, computational procedures patterns, linear and quadratic patterns, and repeated patterns. But more important than discuss the diversity of classifications of type of patterns given in the literature is to focus the attention in the context in which the patterns can emerge.

In this study the numeric and figurative contexts are used to present patterns to the students. To know more about students reactions solving tasks of patterns, to focus attention on students reasoning during their resolutions gives relevant information about how teachers can approach the tasks in the classroom.

To explore patterns in the mathematics classroom is fundamental and pertinent as it appeals to the development of creativity, makes possible the establishment of several connections between distinct topics, it develops the ability to classify and order information as well as to understand the connection between mathematics and the world (Vale & Pimentel, 2009). Patterns allow students to discover connections, establish generalizations and make predictions. In the classroom, patterns can be an excellent context to develop mathematical concepts, beyond being an opportunity to develop the ability to solve problems, reasoning and communicate mathematically (Vale, Palhares, Cabrita & Borralho, 2006; Vale & Pimentel, 2009). The development of algebraic thinking requires an approach to Algebra through problem solving of patterns, as the exploration of patterns is a powerful strategy of problem solving; to solve non-routine problems can become a way for students to explore and formalize patterns, allowing them establish conjectures, verbalize relations between distinct elements of a pattern and generalize (Vale & Pimentel, 2009). Vale and Fonseca (2011) point out that patterns can give the students an opportunity to develop mathematical knowledge as they allow students to relate different concepts and contents in distinct contexts. Thus, patterns are a relevant topic concerning the development of students' ability in mathematical processes such as problem solving, mathematical reasoning and mathematical communication. These ideas are also expressed in Portuguese official curricula documents (see DGIDC, 2007), and also international documents (see NCTM, 2007).

Frobisher and Threlfall (2005) referred that, in the first years of contact with patterns, students should develop abilities to describe, complete and create patterns, transform a written expression into a symbolic one, or vice-versa, extend a pattern to solve problems, explain a generalization of a pattern and also use patterns to establish relations. Also Garrick, Threlfall and Orton (2005) argue that to explore tasks involving regularities, in group or individually, challenge students to verbalize their ideas and present their knowledge, improving their ability to communicate mathematically. Regarding these issues, literature suggests that teachers should select tasks for their students that challenge them to find regularities, make conjectures, discuss ideas, improve their argumentation based on coherent and valid justifications, and also to improve their ability to reasoning and communicate mathematically (see Vale & Pimentel, 2009). Thus, it becomes essential to challenge students to describe patterns, by their own words, and justify how they can extend or create it.

Orton and Orton (2005) consider that the learning of patterns is influenced by the level of conceptual difficulty beyond students' motivation. For the authors, students' level of abstraction affects their understanding of mathematical concepts, as there are students that simply cannot make sense of essential aspects of mathematics. If the mathematics teaching do not comprise relevant experiences for students, it will be difficult for them to improve their level of abstraction. Thus, teachers assume a crucial role in promoting relevant learning for their students.

The tasks of patterns

Concerning the tasks of patterns, Vale, Palhares, Cabrita and Borralho (2006) underlie the idea that teachers should present their students with tasks relevant to students' algebraic thinking. These tasks should comprise the identification of patterns, the creation and continuation of patterns, and the exploration of different properties of relations. Vale and Pimentel (2009) point out that tasks of patterns should comprise the multiple representation of a pattern, the recognition of a pattern, the prediction of terms, the generalization and building of a pattern, and the oral and written description of a pattern.

Patterns can emerge from numeric, visual or mixt approaches (Orton, 2005). Since early school years, students should have the opportunity to observe patterns and represent them geometric and numerically, starting to establish connections between geometry and arithmetic. Vale (2012) argues that mathematics teaching should include challenging tasks that emphasize the understanding of a generalization, through numeric and figurative aspects, making the most of students' ability to think visually. To improve students' visual representations may comprise their use of different representations, such as describe patterns using tables and adequate numeric expressions (Vale, 2012). For Vale (2012) generalization involves higher order thinking such as reasoning, abstraction, or visualization. Thus, the selection of tasks is crucial if one intends to offer problem solving experiences that allow students to make generalizations.

This study tries to understand how sixth-graders explore patterns. It tries to address two questions: 1) How do students embrace tasks involving patterns of repetition on and growth in the mathematics class? 2) What difficulties do they face when solving these tasks?

Methods

Qualitative research methods were adopted, and an interpretative approach was used. Bogdan and Bicklen (1994) argue that qualitative methods occurs in a natural environment that provides a natural source of data in which the researcher is the main tool. This research is strongly descriptive, keeping the focus on the processes rather than the products.

The participants were 28 students of a sixth-grade from a state supported school form Braga, Portugal. There were no students with special education needs.

An intervention was conducted in three mathematics lessons focused on patterns that took place during two consecutive weeks. Each session last 90 minutes long. The intervention was implemented in the class by one of the authors of this paper.

Tasks of problem solving that require students' ability to explore patterns, establishing conjectures, verbalizing relations between the elements of the pattern and accomplish generalization were selected. The tasks presented to the students comprised on: continuation of sequences; identification of the rule (or law) of formation of a sequence; types of sequences; determination of terms of a given sequence; and exploration of sequences. Figures 1 and 2 present examples of some tasks to continue a sequence presented to students in the intervention sessions.

Observe and complete the sequences.

25, 22, 19, 16, ________________

Figure 1: Examples of tasks involving figurative and numeric pattern.

Observe the following sequence:

47, 49, 51, 53, 55, …

Indicate the next four terms of the sequence. ____________________

Indicate the rule to build the sequence. ____________________

____

Is 86 a possible term of this sequence? Why? ____________________

____

Figure 2: Example of a task to continue and determine the term of a sequence.

After each session, an analysis of the tasks implementation and of the students' difficulties solving them was carried out by the researcher, one of the authors of this paper.

The first session comprised three tasks; the first and the second tasks had four questions each, the third had three questions. The session involved repetition and growth patterns (numeric, geometric and figurative). The repetition patterns comprised two, three and four repetition terms; the students had to observe and continue the sequences, identify the rule of formation, and verify the existence of particular terms. The second session comprise two tasks, each with four questions. Students had to draw the following terms of a given sequence, find out and explain the rule of formation of the sequences, find the terms of particular positions in the sequence, analyse the possible existence of particular terms and their positions in the sequence. Both tasks comprise patterns of growth, an illustrative and a geometric one. The third session comprised three tasks, with three, one and four question each. Students were supposed to continue patterns, explain their reasoning, find out a particular term and its position in the sequence, find out the possible existence of others, and explore the growth of a sequence to establish a generalization. The three tasks involved patterns of growth, of numeric and geometric types.

An inquiry-based approach was adopted in the sessions during the implementation of the tasks in which students centred activities were developed. The students were challenged explore the patterns and the questions presented to them prompt their work. They were encouraged to regulate their own activity while exploring the questions. Nevertheless, this was the first time students contacted with tasks of sequences, and very seldom they were challenge to discuss the problems in small groups. Naturally they felt more comfortable in solving the tasks individually, and then share the resolutions in the whiteboard with the whole class.

Data collection was accomplished using videorecorder and photos, researcher field notes and also students written work. All the students names used in this work are fictitious.

Results

Lesson 1

The lesson started with a whole class discussion about regularities and patterns. Students knew expressions from their informal settings, but struggle with the types of sequences. Soon students become familiar with the terminology and types of patterns. When analysing one of the example given to students, they were asked about the existence of a particular term, a debate emerged and the solution was found. Immediately after, one of the students asked about the rule of formation of a particular sequence. The teacher sent the question to the whole class and a debate of different arguments took place to reach the solution. At this moment, the motivation was high.

The first task was presented to the students; two illustrative sequences involving repetitive patterns were given and students were asked to extend them. Then students were challenged to complete two patterns of growth (see Figure 1). They start to develop work individually and then shared their resolutions with their pair, waiting for the whole group discussion. Only in the following sessions students were able to work together to reach a solution to the tasks. The first two questions, concerning patterns of repetition, were correctly solved by the students with no major difficulties; in the illustrative pattern of Figure 1 most of the students attended only to the numbers and, due to miscomputation, failed to extend the sequence. They extended the sequence only considering that the difference between consecutive terms was 3, presenting the sequence 1, 3, 6, 9, 12 e 15 as result, drawing the correspondent geometric picture, as given in Figure 3. When discussing their solving processes in whole group, they realize that should had attended to both, the numeric and the geometric sequence. The numeric sequence was easily accomplished by the students.

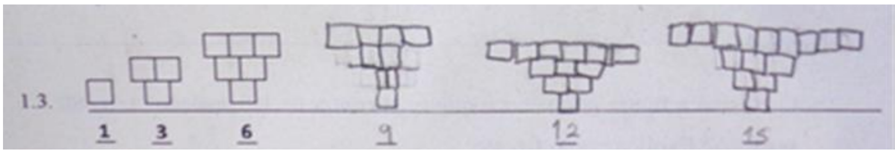

Figura 3: Incorrect resolution made by a student.

The second task (see Figure 2) was successful solve by students, presenting different justifications for the last question, arguing that 86 cannot belong to the sequence "because it ends in 6" or "because the sequence only has odd numbers and 86 is an even number". In the whole class discussion students start to discuss other numbers, such as 733, 200 or 2015. Students were enthusiastic about the tasks and when any resolution was not correct, they argue about it and correct it by their own.

The last task (see Figure 4) comprised three questions. The first two concern the extension of the sequence and was easily solved by students, in spite of some discussion about the rigour of the drawings of geometric shapes that emerged between them when sharing their answers.

Figure 4: Sequence of the last task presented in the first session.

In the second question, students were asked to identify the geometric shape of the fourteenth term. More than half of the student referred that have had used counting as a strategy to answer the question. Immediately the researcher asked "What about the 765th position?" provoking a general laugh in the class. Shortly Bill explained "We could use the multiplication table by 3 as the repetition group has 3 distinct pictures, and look for the number that multiplied by 3 is 765… or close to it.". In the last question, the students were asked "Which geometric figure is in the 19th position? And in the 88th position? Explain how do you know it?". This was the most difficult question of the task.

Students discussed a lot between them when solving this problem, presenting their arguments, trying to reach a solution. When sharing their resolutions with the whole class, the students presented their justifications. One student argued that "as the repetition group comprises 3 terms and the first is a triangle, I counted 1, 4, 7, 10, 13 and realised that these positions matched always a triangle, thus the picture of the 19th position must be a triangle.". Other argued that "thought of the multiplication table by 3 and realised that 3 times 6 is 18, and was a circle. As I wanted the 19th, had to consider one term more in the sequence, which is a triangle".

When finding the picture in the 88th position, another student shared his resolution with the whole class, and argued "the sequence has 3 terms and I know that 3 times 10 is 30 and that the circle was the last one. Thus, 3 times 20 is 60, 3 times 30 is 90 and, as I want the 88, I thought 3 times 29 that is 87. Then was just to add 1 term to get 88, which was a triangle.”.

Lesson 2

Two tasks were presented to students in this lesson. In the first task, the illustrative pattern of growth of Figure 5 was presented to students. The prompt question was “How many dots would have the 100th picture? How do you know it?”. Immediately students started to discuss ideas, sharing strategies, having to listen and understand each other ideas and argue about it.

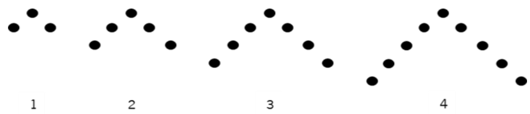

Figure 5: Sequence given to students in lesson 2.

Students realised that they need to find the law of formation to discover the 100th term, and several strategies emerged to achieve it. Explanations such as “the number of term is also the number of dots of the right side of the picture, ignoring the top one. The left side has 1 more dot than the right one.”, or “I multiplied the number of the term of the picture by two, added 1 unit and got the total number of dots of the picture. Thus, for 100 is 100 times 2 makes 200, and 200 plus 1 is 201.”, or even that “I saw that the number of a term plus the number of the next term equals the number of dots of that term. For instance, the number of dots of term 2 equals the number of that term, that is 2, plus the number of the next term, that is 3. That means that 2 + 3 = 5, which is the number of dots of the 2nd term. Thus, the number of dots of the 100th term is 100 + 101 = 201.”. One student with poor achievement in Maths explained also that “the 1st term has 2 dots on the left side, the 2nd has 3, the 3rd has 4, thus the 100th has 101; on the right side, the 1st has 1, the 2nd has 2, the 3rd has 3, so the 100 will have 100. All together will be 201”. Another student, based on the idea of symmetry of the pictures of each term, also explained “I realised that the term is the number of dots in each side of the picture ignoring the top one. So, the 100th term will have 100 dots in each side (100 + 100) plus the top one, making a total of 201 dots”. Figure 6 shows different students’ resolutions presented in the whiteboard to the whole class.

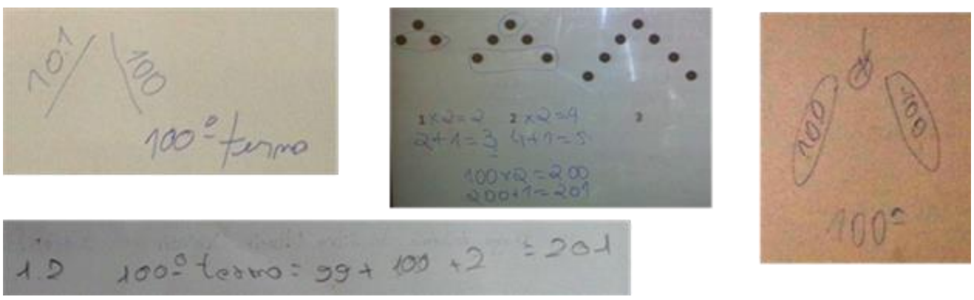

Figure 6: Different resolutions presented by students.

This task gave students an opportunity to realise that there was several correct strategies to solve the problem. Then the researcher asked students: “Is it possible to have a picture with 125 dots in this sequence? Where it will be in the sequence?”. After a short discussion, students realised that was possible because “all the pictures of the sequence have an odd number of dots”. Trying to find the position of the term, one student argued that “if we take 1 dot of the top, and add the two diagonals, is 124 dots. If we divide the 124 by 2, we will have get the number of dots of each diagonal”, using the word ‘diagonal’ referring to the laterals of the inverted V shaped made by the terms of the sequence.

In the second task, a geometric growth pattern was given (see Figure 7), and some prompt questions were presented “How many dark grey squares are there in the 30rd picture? And light-grey squares? How do you know that?

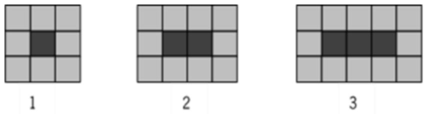

Figure 7: Geometric growth sequence presented to the students.

This was a more difficult task to students. They realised that was relevant to find out the law of formation, but found the job hard, and some difficulties concerning the mathematics communication become evident. The problem was solved by some students, but discussed only in lesson 3.

Lesson 3

This lesson started with last task of lesson 2. A debate on strategies used to find an answer emerged. One of the students explains that “In the first picture, there are 2 dark squares and above them there are a light square plus two more on the sides. Thus, if we imagine the 30rd picture, we would have 30 dark squares and above them we would add 2 more from the sides; that would make 32. We would do 32 times 2 because there are 32 squares on top and bottom, making 64. Then we would add 2 more for the sides, making a total of 66 squares.”. Students’ ability to communicate their ideas and arguments when solving the tasks improved since the first lesson, becoming more rigorous.

In another task, students were challenged to write the 15th term of the numeric sequence “1, 18, 3, 16, 5, 14, 7, …”, and also to find out if the 18th term was an odd or even number. The students realised that they would need to find the law of formation of this sequence, and soon some resolutions were presented by them to the whole class (see Figure 8).

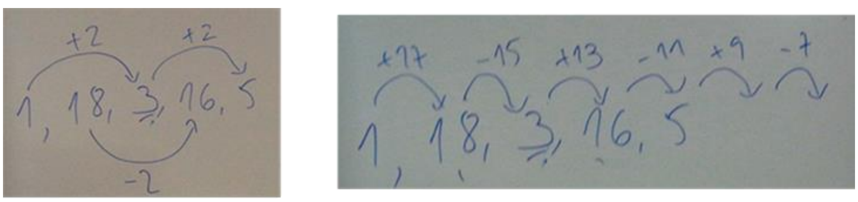

Figure 8: Students’ resolutions discussed with the whole class.

In the next task students were given a star with 5 vertices numbered from 1 to 5 following by the problem: “If we keep numbering the vertices using the same order, the number 6 will be in vertex 1, number 7 in vertex 2, number 8 in 3, and so on till 2007. In which vertex is the last number? Why do you think so?”. After some discussion between them, some answers were given and arguments such as “I realised that 1 + 5 = 6, 2 + 5 = 7 and so on, and I found out that all numbers ended in 2 or 7 were in vertex 2. As 2007 end in 7, it must be also in vertex 2.”. The last task of the lesson was the geometric growth pattern with equilateral triangles solved with no major difficulties by students, having presented their explanations.

Final remarks

Approaching patterns with freedom to explore relations and properties was possible due to the inquiry-based lessons implemented that allowed students to reach their solutions. The tasks revealed to be adequate to students knowledge and required the investigation of resolution processes and strategies. In the solving process, students had the opportunity to develop their ability to solve problems, their ability to reasoning and to communicate mathematically. Students established mathematical relations when dealing with the sequences, made generalizations, and presented and justified their resolutions. The inquiry-based lessons carried out gave them the opportunity to assume an investigative attitude, preserving the motivation, discussing their ideas among them, in small groups or whole class.

Regarding their difficulties, to have to search for terms of higher order in the sequences revealed to be a challenge for this students as they were not familiar with these tasks. Students also revealed difficulties concerning the mathematical communication, as they were not used to explain their reasoning or procedures because they were used to solve problems with only one resolution method and with only one possible solution.

Another remarkable point concerns students’ attitudes when sharing their solutions. Initially, it was difficult for them to listen their colleagues and discuss their ideas; but by the end of the intervention, students easily embraced the tasks exploring patterns, trying to use different strategies, feeling proud in sharing them with the whole class.

This intervention allowed students to contact with relevant problems for the development of their algebraic thinking. Being able to explore pattern gave them the opportunity to make conjectures, formulate generalizations, establish mathematical relations and develop their ability to argue relying on valid justifications. This work offers the students the possibility to develop abilities related to their algebraic thinking that support their mathematical thinking, allowing them to go beyond their ability to compute.

References

Bogdan, R. & Biklen, S. (1994). *Investigação Qualitativa em Educação. Uma Introdução à Teoria e aos Métodos*. Porto: Porto Editora.

Devlin, K. (2002). *Matemática: A Ciência dos Padrões*. Porto: Porto Editora.

DGIDC (2007). *Programa de Matemática do Ensino Básico*. Lisboa: DGIDC-ME.

Frobisher, L. & Threlfall, J. (2005). Teaching and Assessing Patterns in Number in the Primary Years. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 84-103). London: Continuum.

Garrick, R., Threlfall, J. & Orton, A. (2005). Pattern in the Nursery. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 1-17). London: Continuum.

National Council Teachers of Mathematics (2007). Princípios e Normas para a Matemática Escolar. Lisboa: Associação de Professores de Matemática.

Orton, A. (2005). Children's Perception of Patterns in Relation to Shape. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp.149-167). London: Cassell.

Orton, A. & Orton, J. (2005). Pattern and the Approach to Algebra. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 104-120). London: Continuum.

Vale, I. (2012). As tarefas de padrões na aula de matemática: Um desafio para professores e alunos. *Interações, 20,* 181-207.

Vale, I. & Fonseca, L. (2011). Patterns tasks with geometric transformation in elementary teacher's training: some examples. *Journal of the European Teacher Education Network, 6*, 76-86.

Vale, I. & Pimentel, T. (2009). *Padrões no Ensino e Aprendizagem da Matemática – Propostas Curriculares para o Ensino Básico*. Viana do Castelo: ESE – IPVL.

Vale, I., Palhares, P., Cabrita, I. & Borralho, A. (2006). Os Padrões no Ensino e Aprendizagem da Álgebra. In I. Vale, T. Pimentel, A. Barbosa, L. Fonseca, L. Santos & P. Canavarro (Orgs.), *Números e Álgebra na aprendizagem da matemática e na formação de professores*, (pp. 193-211). Lisboa: SEM – SPCE.

Vieira, L. & Ferreira, D. (2009). Pensamento Algébrico no 1.º Ciclo. In E. Mamede (Coord.), *Matemática – Tarefas para o Novo Programa – 1.º Ciclo*, (pp. 57-104). Braga: AEME.

Zazkis, R. & Liljedahl, P. (2012). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49, 379-402.