Mathematical Properties of the Elasticity Difference Tensor

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Abstract. A tetrad, adapted to the principal directions of the unstrained reference tensor, is chosen and the elasticity difference tensor, as introduced in [1], is decomposed along those directions. The second order tensors obtained are studied and an example is presented.

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INTRODUCTION

Here we will consider a continuous medium possessing elastic properties, the collection of all its idealized particles being the 3-dimensional space $X$ - the material space. $(M, g)$ represents the space-time manifold, i.e. $M$ is a four-dimensional, connected, Hausdorff manifold and $g$ is a Lorentzian metric with signature $(-+++)$ such that $g = -u \otimes u + h$, where: (i) $h = \pi^* k$, $\pi^*$ being the usual canonical projection onto $X$ and $k$ being a metric in $X$; (ii) $\pi^{-1}(p \in X)$ defines a timelike curve in $M$ having $u$ as unit tangent vector field and represents the flowline of $p$; (iii) $\pi : U \subset M \longrightarrow X$ describes a state of matter.

Following [1], for an unrelaxed state of matter the unstrained reference tensor [2] can be written as $k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b$, the scalar fields $n_1, n_2, n_3$ being related to the eigenvalues along the principal directions of $k_{ab}$. An orthonormal tetrad \{u, x, y, z\}, with $u$ timelike and $x, y, z$ unit spacelike vector fields aligned with the eigenvectors of $k_{ab}$, will be used. On a local coordinate system, the metric $g$ can be written as

$$g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b. \tag{1}$$

In order to study elasticity properties of the space-time, the authors in [1] define the elasticity difference tensor:

$$S^a_{bc} = \frac{1}{2} k^{am} (D_b k_{mc} + D_c k_{mb} - D_m k_{bc}),$$

where $D$ denotes projected covariant derivative associated to $g$. A classification of $S$ will certainly be interesting for the characterization of the elasticity properties of the space-time. In order to do so, we decompose $S^a_{bc}$ along the principal directions of $k^a_{b}$:

$$S^a_{bc} = M_{bc} x^a + N_{bc} y^a + P_{bc} z^a.$$

The second order symmetric tensors $M, N, P$ are now investigated.
The following results for $M_{bc}$ were obtained, the proofs being in [4].

**Theorem 1** The general form of $M_{bc}$ is given by

\[
M_{bc} = u^m(x_{m,(b)u_c} + u_{(bc)x_c};m) + x_{(bc)} - x^m(x_{m,b};m) + \gamma_{011}u_{(b)c} - \gamma_{010}u_{b,c} + \frac{1}{n_2^2}\left[2n_{1,1}u_{(b)c} + 2n_{1,m}u_{(b)c} + 2n_{1,m}x_{m,b,c}\right] + \frac{1}{n_2^2}\left[-x^m(z_bz_{m,3},m) + y_{b,3}(m,n_{2,2},m) + \frac{1}{2}(\gamma_{021} - \gamma_{20})u_{(b)c} + x^m(y_{m,(b)c} - y_{(b)c}m)\right] + \frac{1}{n_2^2}\left[\gamma_{031} - \gamma_{302}(y_{b,c} + x^m(z_{m,c};c) - z_{(b)c}m)\right],
\]

where $\gamma_{abc}$ are the rotation coefficients and a comma represents a partial derivative.

**Theorem 2** $x$ is an eigenvector of $M_{bc}$ iff $n_1$ remains invariant along the directions of $y$ and $z$, i.e. $\Delta_y(\log n_1) = \Delta_z(\log n_2) = 0$, where $\Delta_y$ represents the intrinsic derivative along $y$. The corresponding eigenvalue is $\lambda = \Delta_z(\log n_1)$.

**Theorem 3** $y$ is an eigenvector of $M_{bc}$ iff $n_1$ remains invariant along the direction of $y$, i.e. $\Delta_y(\log n_1) = 0$, and $\frac{1}{2}\gamma_{232}(1 - (n_2^2/n_1^2)) + \frac{1}{2}\gamma_{232}(1 - (n_2^2/n_2^2)) + \frac{1}{2}\gamma_{231}(n_2^2/n_2^2) - (n_2^2/n_1^2) = 0$. The corresponding eigenvalue is $\lambda = -(n_2^2/n_1^2)\Delta_y n_2 + \gamma_{222}[-(n_2^2/n_1^2) + 1]$.

**Theorem 4** $z$ is an eigenvector of $M_{bc}$ iff $n_1$ remains invariant along the direction of $z$, i.e. $\Delta_z(\log n_1) = 0$, and $\frac{1}{2}\gamma_{232}(1 - (n_2^2/n_1^2)) + \frac{1}{2}\gamma_{232}(1 - (n_2^2/n_1^2)) + \frac{1}{2}\gamma_{231}(n_2^2/n_2^2) - (n_2^2/n_1^2) = 0$. The corresponding eigenvalue is $\lambda = -(n_2^2/n_1^2)\Delta_z n_3 - \gamma_{332}(n_2^2/n_1^2) - 1$.

Similar results have been obtained by the authors for $N$ and $P$ [4].

The following example illustrates the results above. We consider a spherically symmetric metric $g$ written in local coordinates $t, r, \theta, \phi$ as $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$ (see [3], p.186). If a radial deformation is considered such that $ds^2 = -dt^2 + n^2(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$, the only non-zero components of the elasticity difference tensor are $S'_{rr} = S'_{\theta\theta} = \frac{1}{n(r)} \frac{dn(r)}{dr}$ and $S'_{\phi\phi} = \frac{r^2 \sin^2(\theta) \frac{dn(r)}{dr}}{n(r)} = \sin^2(\theta) S'_{\theta\theta}$. Then $M_{bc} = \lambda_1(x_{b,xc} - y_{b,yc} - z_{b,zc})$, $N_{bc} = 2\lambda_2(x_{b,yc} + x_{c,yb})$, and $P_{bc} = 2\lambda_3(x_{b,zc} + x_{c,zb})$, where $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{n(r)} \frac{dn(r)}{dr}$. Therefore, the eigenvalue associated with the eigenvector $u$ vanishes identically. The remaining eigenvectors are: (i) $\{x, y, z\}$ for $M_{bc}$, $\frac{1}{n(r)} \frac{dn(r)}{dr}$, $-\frac{1}{n(r)} \frac{dn(r)}{dr}$, $-\frac{1}{n(r)} \frac{dn(r)}{dr}$ being the corresponding eigenvalues, so that the Segre type is $\{1, 1, 1\}$; (ii) $\{x+y, x-y, z\}$ for $N_{bc}$ with eigenvalues $\frac{1}{n(r)} \frac{dn(r)}{dr}$, $-\frac{1}{n(r)} \frac{dn(r)}{dr}$, and zero, respectively, the Segre type being $\{1, 1, 0\}$; (iii) $\{x+z, x-z, y\}$ for $P_{bc}$ with eigenvalues $\frac{1}{n(r)} \frac{dn(r)}{dr}$, $-\frac{1}{n(r)} \frac{dn(r)}{dr}$, and zero, respectively, the Segre type being then $\{1, 1, 1\}$.

**REFERENCES**