Multiplicity Lists for Symmetric Matrices whose Graphs Have Few Missing Edges

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Abstract

We characterize the possible lists of multiplicities occurring among the eigenvalues of real symmetric (or Hermitian) matrices whose graph is one of K_n, K_n less an edge, or both possibilities for K_n less two edges. The lists are quite different from those for trees. Some construction techniques are developed here and additional results with more missing edges are given, including the case of several independent edges.

Key Words and Phrases:
Few missing edges; Multiplicity lists; Real symmetric matrix; Undirected simple graph.
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1 Introduction

Let G be an (undirected) simple graph on n vertices and $\mathcal{S}(G)$ be the collection of all n-by-n real symmetric matrices, the graph of whose (nonzero)

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off-diagonal entries is G. No restriction is placed by G upon the diagonal entries of $A \in \mathcal{S}(G)$. We are interested in all possible lists of multiplicities for the eigenvalues of matrices in $\mathcal{S}(G)$. Let $\mathcal{L}(G)$ be the set of all such lists. Since the total of the multiplicities is n, view these as partitions of n.

It is natural to consider connected graphs G, and in the minimally connected case of trees, the possible lists $\mathcal{L}(G)$ have been heavily studied [7, 8, 9, 10, 11, 12] etc and have much special structure. However, a complete characterization is known only for some classes of trees.

We are interested here in the case in which G has few missing adges (the other extreme from trees), i.e., G is the complete graph K_n, or K_n with a few edges deleted. Of course, the maximum possible multiplicity, $\mathrm{M}(G)$, occurring in $\mathcal{L}(G)$ is an important constraint on these lists. Since, for symmetric matrices, algebraic multiplicity equals geometric multiplicity, $M(G) = n - mr(G)$, in which $mr(G)$ is the smallest rank occuring among matrices in $\mathcal{S}(G)$.

In general, $mr(G)$ is difficult to know, but, fortunately, when there are just a few edges missing from K_n, it is not hard to determine.

In the case of trees, there is a nice characterization of $mr(G)$ [7], but there are many additional constraints on $\mathcal{L}(G)$, such as at least two eigenvalues of multiplicity 1.

The case of high edge-density graphs seems to be in strong contrast to trees in several ways. Besides the $mr(G)$ constraint, there are often, but not always, no other constraints, and subject to the possible multiplicities, any eigenvalues are often possible. i.e., the inverse eigenvalue problem (IEP) is equivalent to the multiplicity list problem. It is an interesting question for which graphs 1) $\mathcal{L}(G)$ is all lists allowed by $mr(G)$ and 2) the IEP for G is equivalent to the $\mathcal{L}(G)$ problem for G. When this occurs for the graphs we study, we make note of it.

2 Useful Tools

We identify the edges missing from K_n by the graph that they, together with their vertices, form. So, for a graph H on no more than n vertices, by $K_n - H$ we mean that the edges (only) of H are deleted from K_n. Let S_k denote the star on k vertices, and P_k the path on k vertices. We give certain graphs special names based on what is missing: $G_0 = K_n$, $G_1 = K_n$ missing one edge, $G_2 = K_n - S_3$, $G_{1,1} = K_n$ less two independent edges. More generally,

$G_k = K_n - S_{k+1}$ and $G_{1,1,\ldots,1} = K_n$ less k independent edges if there are k $1's$. The matrices in $\mathcal{S}(G_k)$ have all their 0 entries in one row and column, and the matrices in $\mathcal{S}(G_{1,1,\ldots,1})$ look like

$$\begin{bmatrix} * & 0 & & & & & & & \\ 0 & * & & & & \Huge{*} & & & \\ & & * & 0 & & & & & \\ & & 0 & * & & & & & \\ & & & & * & 0 & & & \\ & & & & 0 & * & & & \\ & & & & & & \ddots & & \\ & & \Huge{*} & & & & & * & \\ & & & & & & & & * \end{bmatrix}.$$

We note that $*'s$ appearing off the diagonal in such a display must be nonzero (but otherwise are free), while $*'s$ appearing on the diagonal may be 0 or nonzero.

It is easy to check that

$$\begin{aligned} mr(G_0) &= 1 \\ mr(G_1) &= 2 \\ mr(G_k) &= 2, \qquad \text{for } k < n-1. \\ mr(G_{1,1,..,1}) &= 2 \end{aligned}$$

Therefore to attain $mr = 3$, at least 3 edges must be removed from the graph, and this occurs for $K_4 - P_4 = P_4$ as $mr(P4) = 3$.

Methodologically, we rely on the use of orthogonal similarities, beginning with a diagonal matrix. Specifically, we use 2-by-2 orthogonal similarities working on two rows and the same columns at a time, so that the actual similarity matrix looks like

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & \mathbf{0} & \\ & & * & 0 & \cdots & 0 & * & & \\ & & 0 & 1 & & & 0 & & \\ & & \vdots & & \ddots & & \vdots & & \\ & & 0 & & & 1 & 0 & & \\ & & * & 0 & \cdots & 0 & * & & \\ & \mathbf{0} & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}.$$

These are often called Givens transformations and our method might be called "reverse Givens" as we seek to make entries of the object matrix nonzero, rather than zero. And we call a matrix without zero entries "full matrix". The two simple observations we use are embodied in the following two lemmas and .

Lemma 2.1 *The matrix* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(\mathbb{R})$ *is transformed to a full symmetric matrix* $\begin{bmatrix} a' & c \\ c & b' \end{bmatrix}$ *with* a' *and* b' *any numbers properly between* a *and* b*, subject to* $a' + b' = a + b$*, by an orthogonal similarity via* U *if and only if* U *is full and* $a \neq b$*. If* $a = b$*, the similarity returns the same diagonal matrix.*

Lemma 2.2 *If* U *is a full 2-by-2 orthogonal matrix and* $x = \alpha e_1$ *or* βe_2*,* $\alpha, \beta \neq 0$*, then* Ux *is a full 2-vector. If* $x_1, ..., x_k$ *are full 2-vectors, then* U *may be chosen so that* $Ux_1, ..., Ux_k$ *are also full.*

We will often denote the 2-by-2 orthogonal similarity that we use by the rows and columns on which it operates. The transform ij would alter only rows and columns i and j. Usually a generic, full 2-by-2 similarity is sufficient. According to the lemma, if the diagonal entries are not equal and the ij entry was 0, the result will be that the ij entry becomes nonzero and that the diagonal entries are pushed together in a manner we may choose. In some cases, we choose a particular similarity so that one new diagonal entry is a particular number between a and b. Other entries in rows and columns i or j may be made nonzero. If the diagonal entries are the same, we may still use the ij transform, only to alter some off-diagonal entries other than i, j, which will be left 0.

We illustrate how our similarities may be used with an important case.

Theorem 2.3 *A diagonal matrix $D \in M_n(R)$ is orthogonally similar to a symmetric matrix with all off-diagonal entries nonzero, unless D is a multiple of I.*

Proof. Necessity is clear. For sufficiency, we may assume wlog that the last two diagonal entries of D are distinct. Thus, this 2-by-2 diagonal principal submatrix may be replaced by a full 2-by-2 symmetric matrix whose diagonal entries differ from the other original diagonal entries, using an $n-1, n$ transform.

Now, an $n-2, n-1$ transform leaves the lower right 3-by-3 principal submatrix full, as well as its diagonal entries different from the remaining original diagonal entries. Continuing in this manner, transforming rows and columns $n-3$ and $n-2$ next and so on, results in a full matrix. □

3 Main Results

We give our main results for graphs with few missing edges. The first deals with the complete graph K_n on $n \geq 2$ vertices, which is covered by theorem 2.3 of the last section. This fact has been known for some time, having been noticed by the author Johnson and mentioned in talks by him on the subject for many years. The same has also been noticed much later in [2]. Here we consider it as a starting point.

Theorem 3.1 *$\mathcal{L}(K_n)$ consists of all multiplicity lists with at least two distinct eigenvalues, or, equivalently, all lists in which every eigenvalue has multiplicity less than n. Moreover, subject to this condition, the eigenvalues are arbitrary; i.e, the inverse eigenvalue problem is equivalent to the multiplicity list problem for K_n.*

From the above theorem, we know that for the complete graph any multiplicity list with at least two distinct eigenvalues may occur. One may suspect that graphs that are complete, except for missing a few edges, would also host many multiplicity lists. This is so, and we will discuss a few natural cases here, e.g., the complete graph missing just one or two edges. The next case, the complete graph, less one edge was left as an open question in [2].

Since $mr(G_1) = mr(G_{1,1}) = 2$, the candidate multiplicity lists are the same, and, as will be seen, all of the lists, subject to $mr = 2$, do occur. The case of G_1 could be deduced from $G_{1,1}$ by using one transform, indexed by the vertices of one of the missing independent edges, unless the two diagonal entries are equal for each of the missing edges. This can be avoided, but it is perhaps simplest to do the cases of G_1 and $G_{1,1}$, separately.

Theorem 3.2 *Suppose $n \geq 3$ and let $G_1 = K_n -$ an edge, the graph on n vertices with one edge missing from the complete graph. Then, $\mathcal{L}(G_1)$ consists of all multiplicity lists in which no eigenvalue has multiplicity more than $n-2$. Moreover, subject to this condition, the eigenvalues are arbitrary; i.e., the inverse eigenvalue problem for G_1 is equivalent to the multiplicity list problem for G_1.*

Proof. Since rank $A \geq 2$ for every $A \in S(G_1)$, the stated condition is clearly necessary.

For sufficiency of the condition, we consider two cases. (1) First, suppose that there are at least 3 distinct eigenvalues (which implies the condition) among the list a: $a_1, ..., a_n$ of real numbers and suppose that a_1 is neither the largest nor the smallest. Array them as a diagonal matrix with a_1 first, so that $a_2, ..., a_n$ include, at least, 2 distinct eigenvalues. Then, by theorem 2.3, the lower right $(n-1)$-by-$(n-1)$ principal submatrix is orthogonally similar to a symmetric matrix A_2 whose graph is K_{n-1}. Furthermore, the 1,1 entry of A_2 may be taken to be a_1, as any value in the convex hull of the eigenvalues may appear on the diagonal of a orthogonally similarity [5]. So $D = diag(a_1, a_2, ..., a_n)$ is unitarily similar to an Hermitian matrix A_1 of the form

$$A_1 = \left[\begin{array}{c|ccc} a_1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array}\right].$$

in which a_1 is also the 1, 1 entry of A_2 and $G(A_2) = K_{n-1}$. Now, apply a 1,2 transform, which will yield a matrix A of the form

$$A = \begin{bmatrix} a_1 & 0 & * & \dots & * \\ 0 & a_1 & * & \dots & * \\ * & * & & & \\ \vdots & \vdots & & \ast & \\ * & * & & & \end{bmatrix}$$

whose graph is G_1.

(2) In the remaining cases, there are just two distinct eigenvalues: a_1 with multiplicity k and a_2 with multiplicity l, satisfying $2 \leq k \leq l \leq n-2$ and $k+l=n$. When $n=3$, it is impossible. So, $n \geq 4$. We construct a matrix $A \in \mathcal{S}(G_1)$ with eigenvalues $a_1 = 1$ and $a_2 = 0$ and then the eigenvalues may be made arbitrary, distinct real numbers with a linear transformation applied to A. Such a linear transformation does not change the graph. Let V be a k-by-$(n-2)$ full matrix with orthonormal rows. Let $0 < s, t < 1$ and scale the first 2 rows of V by $\sqrt{1-s}$ and $\sqrt{1-t}$, respectively, to get $\widetilde{V}$ and note that V, s and t could be chosen so that no two columns of $\widetilde{V}$ are orthogonal. We assume this. Now,

$$W = \begin{bmatrix} \sqrt{s} & 0 & \\ 0 & \sqrt{t} & \\ 0 & 0 & \widetilde{V} \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

is k-by-n and has orthonormal rows. Then, $W^T W \in \mathcal{S}(G_1)$ and its spectrum consists of k $1's$ and l $0's$, which completes the proof. $\square$

We note that the second case of the proof importantly uses the theory of DM matrices described in [4]. The proof of the first case cannot be adapted to the second case.

Now, it turns out that if several independent edges are missing from K_n, the possible multiplicity lists are similar.

Theorem 3.3 *Let $n \geq 2k$, and let $G_{1,1,\ldots,1}$ (k subscripted 1's) be $K_n - k$ independent edges, the graph on n vertices with k non-adjacent edges missing from the complete graph. Then, $\mathcal{L}(G_{1,1,\ldots,1})$ consists of all multiplicity lists, in which no eigenvalue has multiplicity more than $n-2$. Moreover, subject to this condition, the eigenvalues are arbitrary; i.e., the inverse eigenvalue problem is equivalent to the multiplicity list problem for $G_{1,1,\ldots,1}$.*

Proof. It suffices to prove the claim for two arbitrary different eigenvalues $a, a, ..., a$ and b, b. Of course, if some of the eigenvalues marked "a" are actually different, or some marked "b" are different (as long as no a's coincide with b's), the same strategy works and the nonzeros are more obvious.

For convenience, we call the entries a_{ij}, with $j = i + k$, in a matrix A the *k-diagonal*. So the main diagonal has $k = 0$, the superdiagonal has $k = 1$, the diagonal above the superdiagonal has $k = 2$, etc.

For the matrix $aI_{n-2} \oplus bI_2$, we begin to create nonzeros from the upper right corner, then move southwest. First, we only transform the even labeled diagonals. For example, when n is even, perform 1,n-1 and 2,n transforms to make the $(n-2)^{th}$-diagonal nonzero, and then 1,n-3; 2,n-2; 3,n-1 and 4,n transforms make the $(n-4)^{th}$-diagonal nonzero, and go on with this procedure, until $k = 2$, to make the matrix looks like a chess board,

$$\begin{bmatrix} a_1 & 0 & * & \ddots & * & 0 & * & 0 & * & 0 \\ 0 & a_2 & 0 & * & \ddots & * & 0 & * & 0 & * \\ * & 0 & a_3 & 0 & * & \ddots & * & 0 & * & 0 \\ \ddots & * & 0 & a_4 & 0 & \ddots & \ddots & * & 0 & * \\ * & \ddots & * & 0 & \ddots & \ddots & \ddots & \ddots & * & 0 \\ 0 & * & \ddots & \ddots & \ddots & \ddots & \ddots & * & \ddots & * \\ * & 0 & * & \ddots & \ddots & \ddots & \ddots & 0 & * & \ddots \\ 0 & * & 0 & * & \ddots & * & 0 & \ddots & 0 & * \\ * & 0 & * & 0 & * & \ddots & * & 0 & a_{n-1} & 0 \\ 0 & * & 0 & * & 0 & * & \ddots & * & 0 & a_n \end{bmatrix}.$$

While we perform this process, we must keep in mind that our objective is to make just one zero in certain rows and columns. For example, if we make $a_1 = a_2$, by lemma 2.1, then a 1,2 transform will make the first and second rows and columns nonzero except the (1,2) and (2,1) entries. If we make $a_3 = a_4$, then a 3,4 transform will make the third and fourth rows and columns nonzero except the (3,4) and (4,3) entries, etc. At the end, we get a matrix of the form

$$\begin{bmatrix} * & 0 & * & & \cdots & & & & & * \\ 0 & * & * & * & & \cdots & & & & * \\ * & * & * & 0 & * & & & \cdots & & * \\ * & * & 0 & * & * & \ddots & & & & \\ & * & * & * & * & 0 & & & & \\ \vdots & & & & 0 & * & & \ddots & & \\ & \vdots & & & & & \ddots & \ddots & & \\ & \vdots & & & & & \ddots & \ddots & & \\ & & \vdots & & & & & & * & * \\ * & & & & & & & & * & * \end{bmatrix}.$$

When n is odd, begin with a 1,n transform, which makes the 1,n and n,1 entries nonzero. Following this with a series of transforms, as in the even case, will give the desired pattern. □

We illustrate the idea with a 6-by-6 example. WLOG, let the eigenvalues of A be $1, 1, 1, 1, 0, 0$ and suppose we want to end up with 3 independent zeros.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{1,5 transform}} \begin{bmatrix} a & 0 & 0 & 0 & * & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ * & 0 & 0 & 0 & 1-a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{then, 2,6 transform}]{\text{where a} \in \text{(0,1)}}$$

$$\begin{bmatrix} a & 0 & 0 & 0 & * & 0 \\ 0 & a & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ * & 0 & 0 & 0 & 1-a & 0 \\ 0 & * & 0 & 0 & 0 & 1-a \end{bmatrix} \xrightarrow{\text{1,3 transform}} \begin{bmatrix} a' & 0 & * & 0 & * & 0 \\ 0 & a & 0 & 0 & 0 & * \\ * & 0 & 1+a-a' & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ * & 0 & * & 0 & 1-a & 0 \\ 0 & * & 0 & 0 & 0 & 1-a \end{bmatrix}$$

$$\xrightarrow[\text{2,4 transform}]{\text{where } a' \in \text{(a,1)}} \begin{bmatrix} a' & 0 & * & 0 & * & 0 \\ 0 & a' & 0 & * & 0 & * \\ * & 0 & 1+a-a' & 0 & * & 0 \\ 0 & * & 0 & 1+a-a' & 0 & * \\ * & 0 & * & 0 & 1-a & 0 \\ 0 & * & 0 & * & 0 & 1-a \end{bmatrix}.$$

Then perform 1,2 and 3,4 transforms, to get the desired pattern

$$\begin{bmatrix} * & 0 & * & * & * & * \\ 0 & * & * & * & * & * \\ * & * & * & 0 & * & * \\ * & * & 0 & * & * & * \\ * & * & * & * & * & 0 \\ * & * & * & * & 0 & * \end{bmatrix}.$$

If the matrix were larger, we would also need a 5,6 transform.

Corollary 3.4 *Let $n \geq 4$ and $G_{1,1} = K_n -$two independent edges, the graph on n vertices with two non adjacent edges missing from the complete graph. Then, $\mathcal{L}(G_{1,1})$ consists of all multiplicity lists in which no eigenvalue has multiplicity more than $n-2$. Moreover, subject to this condition, the eigenvalues are arbitrary; i.e., the inverse eigenvalue problem is equivalent to the multiplicity list problem for $G_{1,1}$.*

Interestingly, the case of G_2 is rather different from G_1 or $G_{1,1}$, though the minimum rank is the same.

Theorem 3.5 *Let $G_2 = K_n - S_3$, the graph on n vertices with two adjacent edges missing from K_n. Then, $\mathcal{L}(G_2)$ consists of all multiplicity lists in which no eigenvalue has multiplicity more than $n-2$, except for the list $2,2$ when $n=4$. Moreover, subject to this condition, the eigenvalues are arbitrary; i.e., the inverse eigenvalue problem is equivalent to the multiplicity list problem for $G_2 = K_n - S_3$.*

Proof. The graph G_2, for $n=4$ does not permit the multiplicity list $2,2$. The pattern for a matrix in $\mathcal{S}(G_2)$ may be displayed as

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}.$$

Suppose the eigenvalues are a,a,b,b. Then $rank(A-aI)=2$, which implies rows 1 and 3 and rows 1 and 4 each form a linearly independent set, and that rows 3 and 4 are a dependent set. The same is true for $rank(A-bI)=2$. But rows 3 and 4 cannot be dependent in both $(A - aI)$ and $(A - bI)$. (Another more general result can be found in [3]).

Now, consider the multiplicity list to be $2, 1, 1$ (or $1, 1, 1, 1$), with $n = 4$. Begin with

$$\begin{bmatrix} a & & & \\ & b & & \\ & & a & \\ & & & c \end{bmatrix}.$$

Suppose that $b \in (a, c)$. Perform a 3, 4 transform, transforming a to b

$$\begin{bmatrix} a & & & \\ & b & & \\ & & b & * \\ & & * & c' \end{bmatrix}.$$

followed by a 2,3 and then a 1,2 transform to arrive at

$$\begin{bmatrix} a' & * & 0 & * \\ * & b' & 0 & * \\ 0 & 0 & b & * \\ * & * & * & c' \end{bmatrix},$$

which is permutation similar to a matrix in $\mathcal{S}(G_2)$. The argument for $1, 1, 1, 1$ is essentially the same. In fact, this pattern is realizable if and only if there is at most one equality in the four eigenvalues, see Theorem 5.1, third bullet in [1]. For $n = 5$, begin with

$$\begin{bmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & b_1 & \\ & & & & b_2 \end{bmatrix}.$$

Here $a_1 = a_2 = a_3$ and $b_1 = b_2$ is allowed, but each a_i is distinct from each b_j.

Perform transforms 1,4; 3,4; 2,5; and then a 1,2 transform, to get

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & 0 & 0 & * \end{bmatrix}.$$

Again this is permutation similar to something in $\mathcal{S}(G_2)$.

For $n > 5$, we may begin with

$$\left[\begin{array}{ccc|ccccc} a_{n-2} & & & & & & & \\ & \ddots & & & & & & \\ & & a_4 & & & & & \\ \hline & & & a_1 & & & & \\ & & & & a_2 & & & \\ & & & & & a_3 & & \\ & & & & & & b_1 & \\ & & & & & & & b_2 \end{array}\right],$$

then process the lower right 5×5 as above to make the lower right 5-by-5 principal submatrix full, and then perform n-5,n-4; n-6,n-5;...; and 1,2 transforms to get the desired pattern. We require that each transform make its diagonal entries different from the remaining original diagonal entries. □

Despite the fact that the list $(2,2) \notin \mathcal{L}(G_2)$ when $n = 4$, it should be noted that the list $(3,2) \in \mathcal{L}(G_{1,2})$ when $n = 5$. It is also straightforward to show that all other lists with largest multiplicity 3 also lie in $\mathcal{L}(G_{1,2})$, when $n > 4$.

Theorem 3.6 *Let $n = 5$ and $G_{1,2} = K_5$ less an edge and an independent S_3, so that $A \in \mathcal{S}(G_{1,2})$ has a single 0 in one row and column and two $0'$s in a different row column pair. Then, $\mathcal{L}(G_{1,2})$ consists of all multiplicity lists in which no eigenvalue has multiplicity more than 3. Moreover, the inverse eigenvalue problem is equivalent to the multiplicity list problem for $G_{1,2}$.*

Proof. We list all the possible eigenvalues of this case:

1. $a_1 > a_2 > a_3 > a_4 > a_5$;
2. $a_1 > a_2 > a_3 > a_4 = a_5$;
3. $a_1 > a_2 > a_3 = a_4 > a_5$;
4. $a_1 > a_2 = a_3 > a_4 > a_5$;
5. $a_1 = a_2 > a_3 > a_4 > a_5$;
6. $a_1 > a_2 > a_3 = a_4 = a_5$;
7. $a_1 > a_2 = a_3 = a_4 > a_5$;
8. $a_1 = a_2 = a_3 > a_4 > a_5$;
9. $a_1 > a_2 = a_3 > a_4 = a_5$;
10. $a_1 = a_2 > a_3 = a_4 > a_5$;
11. $a_1 = a_2 > a_3 > a_4 = a_5$;
12. $a_1 = a_2 = a_3 > a_4 = a_5$;

13. $a_1 = a_2 > a_3 = a_4 = a_5$;

We pick up one case to show our strategy, for example, when the list is (3; 2) and the two different eigenvalues are arbitrary. Begin with the diagonal matrix,

$$\begin{bmatrix} a & & & & \\ & a & & & \\ & & b & & \\ & & & b & \\ & & & & a \end{bmatrix},$$

and perform a 4,5 transform followed by a 2,5 transform to arrive at

$$\begin{bmatrix} a & & & & \\ & a'' & & * & * \\ & & b & & \\ & * & & b' & * \\ & * & & * & a' \end{bmatrix},$$

Now, perform a special 1,3 transform to arrive at

$$\begin{bmatrix} a'' & & * & & \\ & a'' & & * & * \\ * & & b'' & & \\ & * & & b' & * \\ & * & & * & a' \end{bmatrix},$$

As the first and second diagonal entries are equal (which can be arranged by between-ness), a 1,2 transform produces the desired result

$$\begin{bmatrix} a'' & & * & * & * \\ & a'' & * & * & * \\ * & * & b'' & & \\ * & * & & b' & * \\ * & * & & * & a' \end{bmatrix},$$

In fact, we need list the eigenvalues in such a way that the $(4, 4)$ entry is different from $(5, 5)$ entry and the $(2, 2)$ entry is in between the $(1, 1)$ and $(3, 3)$ entries or after 2,5 transform the $(2, 2)$ entry is in between the $(1, 1)$ and $(3, 3)$ entries. All cases can be done by the above mentioned process except case (7), which we do in a different way.

Suppose the eigenvalues of case (7) are $a > b = b = b > c$. By lemma 4.1 in Appendix, we have a matrix like

$$\begin{bmatrix} b & & & & \\ & b & & & \\ & & * & x & \\ & & x & b & y \\ & & & y & * \end{bmatrix}, xy \neq 0, b \neq *$$

the lower right 3-by-3 principal submatrix is with a, b and c as eigenvalues.

Perform a 2,3 transform, we get

$$\begin{bmatrix} b & & & & \\ & * & * & * & \\ & * & * & * & \\ & * & * & b & y \\ & & & y & * \end{bmatrix},$$

then a 1,4 transform give the required form

$$\begin{bmatrix} b & * & * & & * \\ * & * & * & * & \\ * & * & * & * & \\ & * & * & b & * \\ * & & & * & * \end{bmatrix}.$$

□

Remark 3.7 *In spite of the fact that the list* $(3, 2) \in \mathcal{L}(G_{1,2})$*, for* $n = 5$*, we do not know if the list* $(4, 2) \in \mathcal{L}(G_{1,2})$*, for* $n = 6$*.*

4 Appendix

Lemma 4.1 *Given* $\alpha_1 > \alpha_2 > \alpha_3$*, there is a matrix* $A = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$, $xy \neq 0$*, such that* $\alpha_1, \alpha_2, \alpha_3$ *are eigenvalues of* A *with* $b \in (\alpha_1, \alpha_3)$ *prescribed.*

Proof. We need only consider the facts that

$$trA : a + b + c = \alpha_1 + \alpha_2 + \alpha_3 \tag{1}$$

$$ab - x^2 + ac + bc - y^2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \tag{2}$$

$$detA : abc - cx^2 - ay^2 = \alpha_1\alpha_2\alpha_3. \tag{3}$$

Now
(3) $- c \times$ (2) gives

$$y^2 = \frac{(c - \alpha_1)(c - \alpha_2)(c - \alpha_3)}{a - c}$$

(3) $- a \times$ (2) gives

$$x^2 = \frac{-(a - \alpha_1)(a - \alpha_2)(a - \alpha_3)}{a - c}$$

Prescribe the b in (α_1, α_3) and choose $a \in (\alpha_1, \alpha_2), c \in (\alpha_2, \alpha_3)$ (or $c \in (\alpha_1, \alpha_2), a \in (\alpha_2, \alpha_3)$) to make x^2 and y^2 positive. This way we get the desired matrix.

□

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I am not sure if I should include the following theorem in this paper, or we should prepare another paper on $K_n - star_k$, I have some thought on this issue, I don't know if they are solved.

Theorem 5.1 *Let $G_{1,2} = K_n$ less an edge and an independent S_3, so that $A \in \mathcal{S}(G_{1,2})$ has a single 0 in one row and column and two $0's$ in a different row column pair. If there are at least 3 distinct eigenvalues and at least another eigenvalue is in between the largest and smallest eigenvalues, then, there is a $A \in \mathcal{S}(G_{1,2})$ with prescribed eigenvalues.*

Proof. We begin with $n = 6$ to show our strategy. Suppose that the eigenvalues are

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \tag{4}$$

We first consider that there are at most two equalities in (4).

Let A be

$$A = \begin{bmatrix} u_1 & & & & & \\ & u_2 & & & & \\ & & u_3 & & & \\ & & & a & x & 0 \\ & & & x & b & y \\ & & & 0 & y & c \end{bmatrix}, \quad xy \neq 0,$$

where the lower right 3-by-3 principal submatrix is with a_1, a_i (the first different from a_1) and a_6 as eigenvalues and $a_6 < b < a_1$, $u_i s$ are the rest eigenvalues. We choose $a \neq u_3$ and u_1 is in between u_2 and b.

Now perform (3,4) transform, we get

$$A = \begin{bmatrix} u_1 & & & & & \\ & u_2 & & & & \\ & & u_3' & * & * & 0 \\ & & * & a' & x' & 0 \\ & & * & x' & b & y \\ & & 0 & 0 & y & c \end{bmatrix},$$

as u_1 is in between u_2 and b, perform (2, 5) transform to transfer b to u_1, we get

$$A = \begin{bmatrix} u_1 & & & & & \\ & u_2' & * & * & * & * \\ & * & u_3' & * & * & 0 \\ & * & * & a' & * & 0 \\ & * & * & * & u_1 & * \\ & * & 0 & 0 & * & c \end{bmatrix},$$

At the end, a (1, 5) transform gives us the desired form.

When there are 3 equalities in (4). They are only five possible cases (the other cases can not guarantee that besides the three distinct eigenvalues, another is in between the largest and smallest eigenvalues):

1. $a_1 = a_2 = a_3 > a_4 = a_5 > a_6$;
2. $a_1 = a_2 > a_3 = a_4 = a_5 > a_6$;
3. $a_1 = a_2 > a_3 = a_4 > a_5 = a_6$;
4. $a_1 > a_2 = a_3 = a_4 > a_5 = a_6$;
5. $a_1 > a_2 = a_3 = a_4 = a_5 > a_6$.

Case (1) to case (4) can be done by using essentianlly the same procedure as aboved. Here we consider only case (5). Let A be

$$A = \begin{bmatrix} a_2 & & & & & \\ & a_2 & & & & \\ & & a_2 & & & \\ & & & a & x & 0 \\ & & & x & b & y \\ & & & 0 & y & c \end{bmatrix}, \quad xy \neq 0,$$

where the lower right 3-by-3 principal submatrix is with a_1, a_2 and a_6 as eigenvalues, we put $a_1 > a > a_2$, $a_2 > b > a_6$. Now perform (3, 4) transform, we get

$$A = \begin{bmatrix} a_2 & & & & & \\ & a_2 & & & & \\ & & a_2 + k & * & * & 0 \\ & & * & a - k & * & 0 \\ & & * & * & b & y \\ & & 0 & 0 & y & c \end{bmatrix}.$$

Then a (2, 3) transform

$$A = \begin{bmatrix} a_2 & & & & & \\ & a_2 + \frac{k}{2} & * & * & * & 0 \\ & * & a_2 + \frac{k}{2} & * & * & 0 \\ & * & * & a-k & * & 0 \\ & * & * & * & b & y \\ & 0 & 0 & 0 & y & c \end{bmatrix},$$

Now the (1,1) entry is in between (2,2) and (5,5) entries, a (2, 5) transform makes (1,1) entry and (2,2) entry the same,

$$A = \begin{bmatrix} a_2 & & & & & \\ & a_2 & * & * & * & * \\ & * & a_2 + \frac{k}{2} & * & * & 0 \\ & * & * & a-k & * & 0 \\ & * & * & * & * & * \\ & * & 0 & 0 & * & c \end{bmatrix},$$

then a (1, 2) transform gives us the desired form.

We $n > 6$, suppose the eigenvalues are $a_1 \geq a_2 \cdots \geq a_n$ (except $a_1 > a_2 = \cdots = a_{n-1} > a_n$, which we perform separately), and let A be

$$A = \begin{bmatrix} u_1 & & & & & & \\ & \ddots & & & & & \\ & & u_{n-4} & & & & \\ & & & u_{n-3} & & & \\ & & & & a & x & 0 \\ & & & & x & b & y \\ & & & & 0 & y & c \end{bmatrix}, \quad xy \neq 0,$$

where the lower right 3-by-3 principal submatrix is with a_1, a_i (first differ from a_1) and a_n as eigenvalues, u_1 is in between u_{n-4} and b. Then a (n-3,n-2) transform followed by a (n-4,n-1) transform makes $b = u_1$,

$$A = \begin{bmatrix} u_1 & & & & & & \\ & \ddots & & & & & \\ & & * & * & * & * & * \\ & & * & * & * & * & 0 \\ & & * & * & * & * & 0 \\ & & * & * & * & u_1 & * \\ & & * & 0 & 0 & * & c \end{bmatrix},$$

then perform n-5,n-4; n-6,n-5;...; and 2,3 transforms to get

$$A = \begin{bmatrix} u_1 & & & & & & & \\ & * & * & * & * & * & * & * \\ & * & \ddots & & & & & \\ & * & & * & * & * & * & * \\ & * & & * & * & * & * & 0 \\ & * & & * & * & * & * & 0 \\ & * & & * & * & * & u_1 & * \\ & * & & * & 0 & 0 & * & c \end{bmatrix},$$

We require that each transform make its diagonal entries different from the remaining original diagonal entries. At the last, a (1, n-1) transform gives us the desired form.

When $a_1 > a_2 = \cdots = a_{n-1} > a_n$. We begin with

$$A = \begin{bmatrix} a_2 & & & & & & \\ & \ddots & & & & & \\ & & a_2 & & & & \\ & & & a_2 & & & \\ & & & & a & x & 0 \\ & & & & x & b & y \\ & & & & 0 & y & c \end{bmatrix}, \quad xy \neq 0,$$

where the lower right 3-by-3 principal submatrix is with a_1, a_2 and a_n as eigenvalues, as in the case $n = 6$, we choose $a_1 > a > a_2$, $a_2 > b > c$, then use the same procedure to get

$$A = \begin{bmatrix} a_2 & & & & & & & \\ & \ddots & & & & & & \\ & & a_2 & & & & & \\ & & & a_2 + \frac{k}{2} & * & * & * & 0 \\ & & & * & a_2 + \frac{k}{2} & * & * & 0 \\ & & & * & * & a - k & * & 0 \\ & & & * & * & * & b & y \\ & & & 0 & 0 & 0 & y & c \end{bmatrix},$$

Now perform (n-4, n-1) transform, we get

$$A = \begin{bmatrix} a_2 & & & & & & & \\ & \ddots & & & & & & \\ & & a_2 & & & & & \\ & & & a_2 + k' & * & * & * & * \\ & & & * & a_2 + \frac{k}{2} & * & * & 0 \\ & & & * & * & a - k & * & 0 \\ & & & * & * & * & b' & * \\ & & & * & 0 & 0 & * & c \end{bmatrix},$$

keep $b' < a_2$, then go ahead with n-5,n-4; n-6,n-5;...; and 2,3 transforms to get

$$A = \begin{bmatrix} a_2 & & & & & & & \\ & a_2 + \delta & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & 0 \\ & * & * & * & * & * & * & 0 \\ & * & * & * & * & * & b & * \\ & * & * & * & 0 & 0 & * & c \end{bmatrix},$$

since a_2 is in between b and $a_2 + \delta$, perform (2, n-1) transform to make (2,2) entry a_2, i.e.,

$$A = \begin{bmatrix} a_2 & & & & & & & \\ & a_2 & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & 0 \\ & * & * & * & * & * & * & 0 \\ & * & * & * & * & * & * & * \\ & * & * & * & 0 & 0 & * & * \end{bmatrix}.$$

Then a (1,2) transform gives us the desired form.

□