Characterizations and representations of core and dual core inverses in rings

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Abstract

In this paper, double commutativity and reverse order law for core inverse are considered. Then, new characterizations of Moore-Penrose inverse are given by one-sided invertibilities in a ring. Also, we characterize core inverse and dual core inverse of a regular element by units in a ring \( R \). Moreover, their expressions are shown.

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1. Introduction

In this paper, \( R \) means an associative ring with unity 1. We say that \( a \in R \) is (von Neumann) regular if there exists \( x \in R \) such that \( axa = a \). Such \( x \) is called an inner inverse of \( a \), and is denoted by \( a^{-} \). Let \( a\{1\} \) be the set of all inner inverses of \( a \). Recall that an element \( a \in R \) is said to be group invertible if there exists \( x \in R \) such that \( axa = a, xax = x \) and \( ax = xa \). The element \( x \) satisfying the conditions above is called a group inverse of \( a \). The group inverse of \( a \) is unique if it exists, and is denoted by \( a^{\#} \).

An involution in \( R \) is an anti-isomorphism of degree 2, which satisfies \( (a^{*})^{*} = a, (a + b)^{*} = a^{*} + b^{*} \) and \( (ab)^{*} = b^{*}a^{*} \) for all \( a, b \in R \). An element

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$a \in R$ is called Moore-Penrose invertible (see [8]) if there exists $x \in R$ satisfying the following equations

(i) $axa = a$  (ii) $xax = x$  (iii) $(ax)^* = ax$  (iv) $(xa)^* = xa$.

Any element $x$ satisfying the equations (i)-(iv) is called a Moore-Penrose inverse of $a$. If such $x$ exists, it is unique and is denoted by $a^\dagger$. If $x$ satisfies the conditions (i) and (iii), then $x$ is called a $\{1,3\}$-inverse of $a$, and is denoted by $a^{(1,3)}$. If $x$ satisfies the conditions (i) and (iv), then $x$ is called a $\{1,4\}$-inverse of $a$, and is denoted by $a^{(1,4)}$. The symbols $R^{-1}$, $R^\#$, $R^\dagger$, $R^{(1,3)}$ and $R^{(1,4)}$ denote the sets of all invertible, group invertible, Moore-Penrose invertible, $\{1,3\}$-invertible and $\{1,4\}$-invertible elements in $R$, respectively.

The concept of core inverse of a complex matrix was first introduced by Baksalary and Trenkler in [2]. Recently, Rakić et al. [10] gave an equivalent definition of core inverse in rings. An element $a \in R$ is core invertible (see [10, Definition 2.3]) if there exists $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$. It is known that the core inverse $x$ of $a$ is unique if it exists, and is denoted by $a_@$. The dual core inverse of $a$ when exists is defined as the unique $a_{@}$ such that $aa_{@}a = a$, $a_{@}R = a^*R$ and $Ra_{@} = Ra$. By $R^\odot$ and $R_{@}$ we denote the sets of all core invertible and dual core invertible elements in $R$, respectively.

In this paper, double commutativity and reverse order law for core inverse proposed in [1] are considered. Also, we characterize the Moore-Penrose inverse of a regular element by one-sided invertibilities in a ring $R$. Further, new existence criteria of core inverse and dual core inverse of a regular element are given by units. Moreover, their expressions are shown.

2. Main results

In what follows, $R$ always denotes an associative unital ring with involution. We first give the representation of (dual) core inverse of $a$ in $R$.

**Proposition 2.1.** Let $a \in R$. Then

(i) $a \in R^\odot$ if and only if $a \in R^\# \cap R^{(1,3)}$. In this case, $a_{@} = a^\#aa^{(1,3)}$.

(ii) $a \in R_{@}$ if and only if $a \in R^\# \cap R^{(1,4)}$. In this case, $a_{@} = a^{(1,4)}aa^\#$.

**Proof.** (i) “$\Rightarrow$” By [10, Theorem 2.14], we have $a \in R^{(1,3)}$. Also, $a = aa_{@}a = aa(a_{@})^2a = a^2(a_{@})^2a \in a^2R$, which combines with $a = a_{@}a^2 \in Ra^2$ yield $a \in R^\#$. Hence, $a \in R^\# \cap R^{(1,3)}$.
“⇐” Let \( x = a^\# aa^{(1,3)} \). We next show that \( x \) is the core inverse of \( a \).

1. It is direct to check that \( axa = a \).

2. We have \( xR = a^\# aa^{(1,3)} R = aa^\# a^{(1,3)} R \subseteq aR \) and \( aR = a^\# a^2 R = a^\# aa^{(1,3)} a^2 R \subseteq xR \).

3. From \( x = a^\# aa^{(1,3)} = a^\# (a^{(1,3)}) a^* \) and \( a^* = a^* aa^{(1,3)} = a^* ax \), it follows that \( Rx = Ra^* \).

Hence, \( a^\# = a^\# aa^{(1,3)} \).

(ii) By a similar proof of (i).

It is known that \( a \in R^\dagger \) if and only if \( a \in R^{(1,3)} \cap R^{(1,4)} \). By Proposition 2.1, we obtain \( a \in R^\# \cap R^\# \iff a \in R^{(1,3)} \cap R^{(1,4)} \iff a \in R^\# \cap R^\dagger \).

We next give a result regarding commutativity. Firstly, we show the following lemma.

**Lemma 2.2.** Let \( a, x \in R \) with \( xa = ax \) and \( xa^* = a^* x \). If \( a^{(1,3)} \) exists, then \( aa^{(1,3)} x = xaa^{(1,3)} \).

**Proof.** From \( xa = ax \), it follows that

\[
\begin{aligned}
xa^{(1,3)} &= ax^{(1,3)} = aa^{(1,3)} ax^{(1,3)} \\
&= aa^{(1,3)} xaa^{(1,3)}.
\end{aligned}
\]

The condition \( xa^* = a^* x \) implies that

\[
\begin{aligned}
aa^{(1,3)} x &= (a^{(1,3)})^* a^* x = (a^{(1,3)})^* xa^* \\
&= (a^{(1,3)})^* x(aa^{(1,3)} a)^* = (a^{(1,3)})^* xa^* aa^{(1,3)} \\
&= (a^{(1,3)})^* a^* xaa^{(1,3)} \\
&= aa^{(1,3)} xaa^{(1,3)}.
\end{aligned}
\]

Hence, \( aa^{(1,3)} x = xaa^{(1,3)} \). \( \square \)

Applying Lemma 2.2, we obtain the following result.

**Theorem 2.3.** Let \( a, x \in R \) with \( xa = ax \) and \( xa^* = a^* x \). If \( a^\# \) exists, then \( a^\# x = xa^\# \).

**Proof.** Since \( a^\# = a^\# aa^{(1,3)} \) and \( a^\# x = xa^\# \), it follows that

\[
a^\# x = a^\# aa^{(1,3)} x = a^\# xaa^{(1,3)} = xaa^\# aa^{(1,3)} = xa^\#.
\]
Hence, $a^@x = xa^@$.  

Baksalary and Trenkler [1] asked the following question: Given complex matrices $A$ and $B$, if $A^@$, $B^@$ and $(AB)^@$ exist, does it follow that $(AB)^@ = B^@A^@$. Later, Cohen, Herman and Jayaraman [3] presented several counterexamples for this problem.

Next, we show that the reverse order law for core inverse holds under certain conditions in a general ring case.

**Theorem 2.4.** Let $a, b \in R$ with $ab = ba$ and $ab^* = b^*a$. If $a^@$ and $b^@$ exist, then $(ab)^@$ exists and $(ab)^@ = b^@a^@ = a^@b^@$.

**Proof.** It follows from Theorem 2.3 that $b^@a = ab^@$ and $a^@b = ba^@$.

Also, the conditions $b^*a = ab^*$ and $a^*b^* = b^*a^*$ guarantee that $b^*a^@ = a^@b^*$, which together with $a^@b = ba^@$ imply $a^@b^@ = b^@a^@$ according to Theorem 2.3.

Once given the above conditions, it is straightforward to check that

(1) By Lemma 2.2, we have $abb^{(1,3)} = bb^{(1,3)}a$. Hence, $abb^@a^@ab = abb^{(1,3)}a^@b = bb^{(1,3)}a^@b = b^@a^@ab$.

(2) Since $abb^{(1,3)} = bb^{(1,3)}a$, it follows that $b^@a^@ = b^@bb^{(1,3)}a^@a^{(1,3)} = b^@bb^{(1,3)}a^@a^{(1,3)} = b^@abb^{(1,3)}a^@a^{(1,3)} = a^@b^@a^@a^{(1,3)}a$ and $ab = b^@b^2a = b^@bb^{(1,3)}b^2a = b^@ab^2 = b^@a^@a^{(1,3)}a^2b^2 = b^@a^@a^2b^2$.

Hence, $abR = b^@a^@R$.

(3) If $x$ in Lemma 2.2 is group invertible, then $aa^{(1,3)}x^@ = x^@aa^{(1,3)}$. We have

$b^@a^@ = b^@bb^{(1,3)}a^@a^{(1,3)} = b^@a^@bb^{(1,3)}a^{(1,3)} = b^@a^@(aa^{(1,3)}bb^{(1,3)})^* = b^@a^@(b^@a^@)(b^{(1,3)})^* (ab)^*$ and

$(ab)^* = b^*a^@a^{(1,3)} = a^*b^*a^{(1,3)} = a^*b^*bb^{(1,3)}a^{(1,3)} = b^*a^@a^{(1,3)}a^@bb^{(1,3)}a^{(1,3)} = b^*a^@bb^{(1,3)}a^@a^{(1,3)} = b^*a^@a^@a^{(1,3)}$.

So,

$Rb^@a^@ = R(ab)^*$.

Thus, $(ab)^@ = b^@a^@ = a^@b^@$.  

Herein, we first state several lemmas which play an important role in the sequel.

**Lemma 2.5.** Let $a, b \in R$. Then

(i) If $(1 + ab)x = 1$, then $(1 + ba)(1 - bxa) = 1$.

(ii) If $y(1 + ab) = 1$, then $(1 - bya)(1 + ba) = 1$. 
Lemma 2.6. [12, Theorems 2.16, 2.19 and 2.20] Let $S$ be a $\ast$-semigroup and $a \in S$. Then the following conditions are equivalent:

(i) $a \in S^\dagger$.
(ii) $a = aa^*ax$ for some $x \in S$.
(iii) $a = yaax^*a$ for some $y \in S$.

In this case, $a^\dagger = a^*ax^*a^* = a^*a^2a^* = a^*y^2a^*$.

Lemma 2.7. (see e.g. [5, Lemma 5.1]) Let $a \in R$. Then $a \in R^\dagger$ if and only if there exist $x, y \in R$ such that $axa = a = aya$, $(ax)^* = ax$ and $(ya)^* = ya$.

In this case, $a^\dagger = yax$.

In the following theorem, new characterizations of the Moore-Penrose inverse are given by one-sided invertibilities.

Theorem 2.8. Let $a \in R$ be regular with inner inverse $a^\ast$. Then the following conditions are equivalent:

(i) $a \in R^\dagger$.
(ii) $aa^* + 1 - a^\ast a$ is right invertible.
(iii) $a^\ast a + 1 - a^\ast a$ is right invertible.
(iv) $aa^\ast a^\ast + 1 - aa^\ast$ is right invertible.
(v) $a^\ast aa^\ast + 1 - a^\ast a$ is right invertible.
(vi) $aa^* + 1 - aa^\ast$ is left invertible.
(vii) $a^\ast a + 1 - a^\ast a$ is left invertible.
(viii) $aa^\ast a^\ast + 1 - aa^\ast$ is left invertible.
(ix) $a^\ast aa^\ast + 1 - a^\ast a$ is left invertible.

Proof. (ii) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv), (iii) $\Leftrightarrow$ (v), (vi) $\Leftrightarrow$ (vii), (vi) $\Leftrightarrow$ (viii) and (vii) $\Leftrightarrow$ (ix) are followed from Lemma 2.5.

(i) $\Rightarrow$ (ii) If $a \in R^\dagger$, then there exists $x \in R$ such that $a = aa^*ax$ from Lemma 2.6. As $(aa^*a^\ast + 1 - aa^\ast)(axa^\ast - 1 - aa^\ast) = 1$, then $aa^*a^\ast + 1 - aa^\ast$ is right invertible. Hence, $aa^* + 1 - aa^\ast$ is right invertible by Lemma 2.5.

(ii) $\Rightarrow$ (i) As $aa^* + 1 - aa^\ast$ is right invertible, then $a^\ast a + 1 - a^\ast a$ is also right invertible by Lemma 2.5. Hence, there is $s \in R$ such that $(a^\ast a + 1 - a^\ast a)s = 1$. We have $a = a(a^*a + 1 - a^\ast a)s = aa^*as \in aa^*aR$. So $a \in R^\dagger$ by Lemma 2.6.

(i) $\Rightarrow$ (vi) It is similar to the proof of (i) $\Rightarrow$ (ii).

(vi) $\Rightarrow$ (i) As $aa^* + 1 - aa^\ast$ is left invertible, then $t(aa^* + 1 - aa^\ast) = 1$ for some $t \in R$. Also, $a = 1 - a = t(aa^* + 1 - aa^\ast)a = taa^*a \in Raa^*a$, which ensures $a \in R^\dagger$ according to Lemma 2.6.
We get the following result from Theorem 2.8.

**Corollary 2.9.** [7, Theorem 1.2] Let \( a \in R \) be regular with inner inverse \( a^- \).

Then the following conditions are equivalent:

(i) \( a \in R^\dagger \).
(ii) \( aa^* + 1 - aa^- \) is invertible.
(iii) \( a^*a + 1 - a^-a \) is invertible.
(iv) \( aa^*aa^- + 1 - aa^- \) is invertible.
(v) \( a^-aa^*a + 1 - a^-a \) is invertible.

**Theorem 2.10.** Let \( a \in R \) be regular with inner inverse \( a^- \). Then the following conditions are equivalent:

(i) \( a \in R^\dagger \) and \( aR = a^2R \).
(ii) \( u = aa^*a + 1 - aa^- \) is right invertible.
(iii) \( v = a^*a^2 + 1 - a^-a \) is right invertible.

**Proof.** (i) \( \Rightarrow \) (ii) As \( aR = a^2R \), then \( a + 1 - aa^- \) is right invertible by [9, Theorem 1]. Also, \( a \in R^\dagger \) can conclude \( aa^*aa^- + 1 - aa^- \) is invertible by Corollary 2.9. Hence, \( u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-) \) is right invertible.

(ii) \( \Leftrightarrow \) (iii) Follows from Lemma 2.5.

(iii) \( \Rightarrow \) (i) Since \( v \) is right invertible, there exists \( v_1 \in R \) such that \( vv_1 = 1 \). Then \( a = avv_1 = a(a^*a^2 + 1 - a^-a)v_1 = aa^*a^2v_1 \in aa^*aR \) and hence \( a \in R^\dagger \) by Lemma 2.6. It follows from Corollary 2.9 that \( a \in R^\dagger \) implies that \( w = a^*a + 1 - a^-a \in R^{-1} \). As \( v = (a^*a + 1 - a^-a)(a^-a^2 + 1 - a^-a) \) is right invertible, then \( a^-a^2 + 1 - a^-a = w^{-1}v \) is right invertible, and hence \( a + 1 - a^-a \) is also right invertible. So, \( aR = a^2R \) by [9, Theorem 1].

**Remark 2.11.** In general, \( a \in R^\dagger \) and \( aR = a^2R \) can not imply \( a \in R^\# \). Such as, let \( R \) be the ring of all bi-finite infinite complex matrices with transpose as involution, where an infinite matrix is said to be bi-finite if it is both row-finite and column-finite. Let \( a = \sum_{i=1}^{\infty}e_{i,i+1} \in R \), where \( e_{i,j} \) denotes the infinite matrix whose \((i,j)\)-entry is 1 and other entries are zero. Then \( aa^* = 1 \) and \( a^*a = \sum_{i=2}^{\infty}e_{i,i} \). So, \( a^\dagger = a^* \) and \( aR = a^2R \). But \( a \notin R^\# \). In fact, if \( a \in R^\# \), then \( a^\dagger a = aa^\# = aa^#aa^* = aa^* = 1 \), which implies \( a \) is invertible. Contradiction.

Dually, we have the following result.
Theorem 2.12. Let $a \in R$ be regular with inner inverse $a^-$. Then the following conditions are equivalent:

(i) $a \in R^\dagger$ and $Ra = Ra^2$.
(ii) $u = aa^*a + 1 - a^-a$ is left invertible.
(iii) $v = a^2a^* + 1 - aa^-$ is left invertible.

Lemma 2.13. ([6, Proposition 2.1] and [9, Corollary 2]) Let $a \in R$ be regular with inner inverse $a^-$. Then the following conditions are equivalent:

(i) $a^\#$ exists.
(ii) $a + 1 - aa^-$ is invertible.
(iii) $a + 1 - a^-a$ is invertible.
(iv) $a^2 + 1 - aa^-$ is invertible.

We next give existence criteria and representations of core inverse and dual core inverse by units in a ring.

Theorem 2.14. Let $a \in R$ be regular with inner inverse $a^-$. Then the following conditions are equivalent:

(i) $a \in R^\# \cap R^\dagger$.
(ii) $a \in R^\# \cap R_\otimes$.
(iii) $u = aa^*a + 1 - aa^-$ is invertible.
(iv) $v = aa^*a + 1 - a^-a$ is invertible.
(v) $s = a^*a^2 + 1 - a^-a$ is invertible.
(vi) $t = a^2a^* + 1 - aa^-$ is invertible.

In this case,

$$a^\otimes = u^{-1}a a^*, \quad a_\otimes = a^*av^{-1},$$

$$a^\dagger = (t^{-1}a^2)^* = (a^2s^{-1})^* \quad \text{and}$$

$$a^\# = (aa^*t^{-1})^2a = a(s^{-1}a^*a)^2.$$  

**Proof.** (i) $\iff$ (ii) By Proposition 2.1.

(iii) $\iff$ (v) and (iv) $\iff$ (vi) are obtained by Lemma 2.5.

(i) $\Rightarrow$ (iii) In virtue of Lemma 2.13 and Corollary 2.9, $a \in R^\# \cap R^\dagger$ implies that $a + 1 - aa^-$ and $aa^*aa^- + 1 - aa^-$ are both invertible. Hence, $u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible. Hence, $u = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible.

(iii) $\Rightarrow$ (i) Suppose that $u = aa^*a + 1 - aa^-$ is invertible. Then $a \in R^\dagger$ from Theorem 2.10 and hence $aa^*aa^- + 1 - aa^-$ is invertible by Corollary 2.9. As $u = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible, then $a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)^{-1}u$ is invertible, i.e., $a \in R^\#$ by Lemma 2.13.
(i) ⇔ (iv) can be obtained by a similar proof of (i) ⇔ (iii).

Next, we give representations of \( a^\# \), \( a_\# \), \( a^\dagger \) and \( a^\# \), respectively. Herein, we recall in [4, Proposition 7] and [11, Corollary 5] that \( a \in R^\# \) and only if \( a = a^2x \) and \( a = ya^2 \) for some \( x, y \in R \). In this case, \( a^\# = yax = y^2a = ax^2 \).

Since \( ua = aa^*a^2 \), \( a = (u^{-1}aa^*)a^2 \). As \( a^\# \) exists, then \( a^\# = (u^{-1}aa^*)^2a \).

By Proposition 2.1, we have

\[
a^\# = a^\#aa^{(1,3)} = u^{-1}aa^*u^{-1}aa^*a^2a^{(1,3)} = u^{-1}aa^*a(1,3)s = u^{-1}aa^*.
\]

Similarly, it follows that \( a^\# = a(a^*av^{-1})^2 \) and \( a_\# = a^*av^{-1} \).

As \( a^s = aa^*a^2 \) and \( ta = a^2a^*a \), then we have \( a = aa^*(a^2s^{-1}) = (t^{-1}a^2)a^*a \).

It follows from Lemma 2.7 that \( a \in R^\dagger \) and

\[
a^\dagger = (a^2s^{-1})^*(a(t^{-1}a^2)^*(s^{-1})(a^2)^*a(a^2)^*(t^{-1})^* = (s^{-1}a^2)^*(a^2)^*(s^{-1}a^2)^*(a^2)^*(t^{-1})^* = (a^2)^2(t^{-1})^* = (t^{-1}a^2)^*.
\]

Similarly, \( a^\dagger = (a^2s^{-1})^* \).

Noting \( sa^{-1}a = a^*a^2 \), we have \( a^{-1}a = s^{-1}a^*a^2 \) and \( a = aa^{-1} = (as^{-1}a^*)^2a \).

Hence, it follows that \( a^\# = (as^{-1}a^*)^2a = a(s^{-1}a^*a)^2 \) since \( a \in R^\# \).

We can also get \( a^\# = (aa^*t^{-1})^2a \) by a similar way. \( \square \)

**Theorem 2.15.** Let \( a \in R \) be regular. Then the following conditions are equivalent:

(i) \( a^\# \) exists.

(ii) \( a + 1 - aa^- \) and \( a^* + 1 - aa^- \) are invertible for some \( a^- , a^\in a\{1\} \).

(iii) \( a + 1 - aa^- \) is invertible and \( a^* + 1 - aa^- \) is left invertible for some \( a^- , a^\in a\{1\} \).

(iv) \( a^*a + 1 - aa^- \) and \( (a^*)^2 + 1 - aa^- \) are invertible for some \( a^- , a^\in a\{1\} \).

(v) \( a^*a + 1 - aa^- \) and \( (a^*)^2 + 1 - aa^- \) are left invertible for some \( a^- , a^\in a\{1\} \).

In this case, \( a^\# = (a^*a + 1 - aa^-)^{-1}a^* = a[(a^*)^2 + 1 - aa^-]^{-1}]^* \).

**Proof.** (i) ⇒ (ii) Since \( a \in R^\#, \ a \in R^{(1,3)} \) by Proposition 2.1. Let \( a^- , a^\in a\{1,3\} \). Then \( a + 1 - aa^- \) and \( a + 1 - aa^- \) are invertible by Lemma 2.13 and hence \( a^* + 1 - aa^- = (a + 1 - aa^-)^* \) is invertible.
(ii) ⇒ (iii) It is clear.

(iii) ⇒ (i) As \( a^* + 1 - aa^- \) is left invertible, then there exists \( s \in R \) such that \( s(a^* + 1 - aa^-) = 1 \). Hence, \( a = s(a^* + 1 - aa^-)a = sa^*a \in Ra^*a \), i.e., \( a^{(1,3)} \) exists by [13, Lemma 2.2]. Also, \( a^* + 1 - aa^- \in R^{-1} \) concludes that \( a \in R^\# \) by Lemma 2.13. So, \( a \in R^\# \) by Proposition 2.1.

(i) ⇒ (iv) Let \( a^- \in a\{1, 3\} \). Then \( a + 1 - aa^- \) and \( a^* + 1 - aa^- \) are invertible. Hence, \( a^*a + 1 - aa^- = (a^* + 1 - aa^-)(a + 1 - aa^-) \) is invertible. Also, it follows from Lemma 2.13 that \( a^2 + 1 - aa^- \in R^{-1} \) since \( a \in R^\# \). So, \( (a^*)^2 + 1 - aa^- = (a^2 + 1 - aa^-)^* \in R^{-1} \).

(iv) ⇒ (v) Clearly.

(v) ⇒ (i) Since \( a^* + 1 - aa^- \) and \( (a^*)^2 + 1 - aa^- \) are both left invertible, there exist \( m, n \in R \) such that \( m(a^*a + 1 - aa^-) = 1 = n((a^*)^2 + 1 - aa^-) \). As \( a = m(a^*a + 1 - aa^-)a = ma^*a^2 \) and \( a = n((a^*)^2 + 1 - aa^-)a = n(a^*)^2a \), then \( ma^* = m(n(a^*)^2a)^* = (ma^*a^2)n^* = an^* \).

Let \( x = ma^* = an^* \). Then \( x \) is the core inverse of \( a \). Indeed, we have

1. \( (ax)^* = ax \) since \( ax = n(a^*)^2a(an^*) = (a^2n^*)^*a^2n^* \).
2. \( axa = (ax)^*a = (a^*ax)^* = (a^*a^2n^*)^* = n(a^*)^2a = a \).
3. \( xax = (ma^*)a(an^*) = (ma^*a^2)n^* = an^* = x \).
4. \( xax^2 = ma^*a^2 = a \).
5. \( ax^2 = ax(an^*) = (axa)n^* = an^* = x \).

It follows from [10, Theorem 2.14] that \( x = a^\circ \).

We next give the formulae of \( a^\circ \). In process of \( (v) \Rightarrow (i) \), \( a^*a + 1 - aa^- \) and \( (a^*)^2 + 1 - aa^- \) are both invertible from \( (iv) \Leftrightarrow (v) \). Hence, \( m = (a^*a + 1 - aa^-)^{-1} \) and \( n = ((a^*)^2 + 1 - aa^-)^{-1} \).

We obtain

\[
a^\circ = ma^* = (a^*a + 1 - aa^-)^{-1}a^*
= an^* = a[((a^*)^2 + 1 - aa^-)^{-1}]^*.
\]

The proof is completed. \( \square \)

**Proposition 2.16.** Let \( a \in R \) be regular. If \( a^* + 1 - aa^- \) is invertible for any \( a^- \in a\{1\} \), then \( a^\circ \) exists.

**Proof.** If \( u = a^* + 1 - aa^- \) is invertible, then \( a = u^{-1}a^*a \in Ra^*a \), hence \( a \) is \( \{1, 3\} \)-invertible by [13, Lemma 2.2].

As \( a + 1 - aa^{(1,3)} = (a^* + 1 - aa^{(1,3)})^* \) is invertible for \( a^{(1,3)} \in a\{1\} \), then \( a \in R^\# \) by Lemma 2.13. So, \( a^\circ \) exists from Proposition 2.1. \( \square \)
Proposition 2.17. Let $a \in R$ be regular. If $(a^*)^2 + 1 - aa^-$ is invertible for any $a^- \in a\{1\}$, then $a^\#$ exists.

Proof. Let $u = (a^*)^2 + 1 - aa^-$. Then $ua = (a^*)^2a$, it follows $a = u^{-1}(a^*)^2a \in Ra^*a$. So, $a$ is $\{1, 3\}$-invertible by [13, Lemma 2.2].

Also, $a^2 + 1 - aa^{(1, 3)} = ((a^*)^2 + 1 - aa^{(1, 3)})^* \in R^{-1}$ guarantees that $a \in R^\#$ from Lemma 2.13. Hence, it follows from Proposition 2.1 that $a^\#$ exists. □

The converse statements of Propositions 2.16 and 2.17 may not be true. In following Example 2.18, we find that $a$ is core invertible, but there exist some $a^- \in a\{1\}$ such that $(a^* + 1 - aa^-), (a^*)^2 + 1 - aa^-$ and $a^* a + 1 - aa^-$ are all not invertible.

Example 2.18. Let $M_2(\mathbb{C})$ be the ring of 2 by 2 complex matrices and let involution $*$ be the conjugate transpose. Given $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \in M_2(\mathbb{C})$, then $A^2 = -A$ and hence $A^\#$ exists. So, $A^\#$ exists. Taking $A^* = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$, then $A^* + I - AA^- = \frac{1}{3} \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$, $(A^*)^2 + I - AA^- = \frac{1}{3} \begin{bmatrix} -2 & -4 \\ 4 & 8 \end{bmatrix}$ and $A^* A + I - AA^- = \frac{1}{3} \begin{bmatrix} 7 & -13 \\ -14 & 26 \end{bmatrix}$ are not invertible.

Remark 2.19. Even $a^* a + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, $a$ may not be core invertible. Let $R$ be a ring which is the same as the infinite matrix ring in Remark 2.11 and let $a = \sum_{i=1}^{\infty} e_{i+1,i}$. Then $a^* a = 1$, $a^* = \sum_{i=2}^{\infty} e_{i,i}$ and $a^t = a^*$. It is easy to know that $a^- = \sum_{i=1}^{\infty} e_{i, i+1} + \sum_{i=1}^{n} a_i e_{i, 1}$ for some $n$ and $a_i \in \mathbb{C}$. So, $a^* a + 1 - aa^- = 2 - aa^- = 2e_{1,1} - \sum_{i=1}^{\infty} a_i e_{i+1, 1} + \sum_{i=2}^{\infty} e_{i,i}$ and $(a^* a + 1 - aa^-)^{-1} = \frac{1}{2} e_{1,1} + \sum_{i=1}^{n} a_i e_{i+1, 1} + \sum_{i=2}^{\infty} e_{i,i}$. But $a \notin R^\#$, hence $a \notin R^\#$.

Proposition 2.20. Let $a \in R^\#$. Then $a \in R^1$ if and only if $a^* a + 1 - aa^\# \in R^{-1}$.

Proof. “⇒” Note that $a \in R^1$ implies $a^* a + 1 - aa^\# a \in R^{-1}$ by Corollary 2.9. As $a \in R^\#$, then $a + 1 - aa^\# \in R^{-1}$ from Lemma 2.13. Since $a^* a + 1 - aa^\# = (a^* + 1 - aa^\#)(a + 1 - aa^\#) \in R^{-1}$, it follows that $a^* a + 1 - aa^\# \in R^{-1}$.

“⇐” Let $u = a^* + 1 - aa^\#$ be invertible. Then $ua = a^* a$ and $au = aa^*$. Hence, $a = u^{-1} a^* a = aa^* u^{-1} = a(u^{-1} a^* u)^{-1} = aa^* a(u^{-1} a^* u)^{-1} \in aa^* aR$. So, $a \in R^1$ by Lemma 2.6. □
Recall that a ring $R$ is called Dedekind-finite ring if $ab = 1$ implies $ba = 1$, for all $a, b \in R$. We next give characterizations of core inverse in such a ring.

**Proposition 2.21.** Let $R$ be a Dedekind-finite ring. Then the following conditions are equivalent:

(i) $a^\circ$ exists.

(ii) $a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for any $a^{(1,3)}$.

(iii) $a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for some $a^{(1,3)}$.

In this case, $a^\circ = (a^*a + 1 - aa^{(1,3)})^{-1}a^*$.

**Proof.** (i) $\Rightarrow$ (ii) By Theorem 2.15 (i) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (iii) Clearly.

(iii) $\Rightarrow$ (i) Let $u = a + 1 - aa^{(1,3)}$. Then $u^*u = a^*a + 1 - aa^{(1,3)} \in R^{-1}$. As $R$ is a Dedekind-finite ring, then $u \in R^{-1}$, which guarantees $a \in R^\#$ by Lemma 2.13. Hence, $a \in R^\# \cap R^{(1,3)}$ is core invertible from Proposition 2.1. Now, $a^\circ = (a^*a + 1 - aa^{(1,3)})^{-1}a^*$ by Theorem 2.15.

**Corollary 2.22.** Let $R$ be a Dedekind-finite ring. If $a \in R^\dagger$, then $a \in R^\circ$ if and only if $a^*a + 1 - aa^{\dagger} \in R^{-1}$. In this case, $a^\circ = (a^*a + 1 - aa^{\dagger})^{-1}a^*$.

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