The Moore-Penrose inverse of differences and products of projectors in a ring with involution

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Abstract: In this paper, we study the Moore-Penrose inverses of differences and products of projectors in a ring with involution. Also, some necessary and sufficient conditions for the existence of the Moore-Penrose inverse are given. Moreover, the expressions of the Moore-Penrose inverses of differences and products of projectors are presented.

Keywords: Moore-Penrose inverses, normal elements, involutions, projectors

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1 Introduction

Throughout this paper, R is a unital $*$-ring, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfying that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$. Recall that an element $a \in R$ is said to have a Moore-Penrose inverse (abbr. MP-inverse) if there exists $b \in R$ such that the following equations hold [11]:

$$aba = a,\ bab = b,\ (ab)^* = ab,\ (ba)^* = ba.$$

Any b that satisfies the equations above is called a MP-inverse of a. The MP-inverse of $a \in R$ is unique if it exists and is denoted by $a^\dagger$. By $R^\dagger$ we denote the set of all MP-invertible elements in R.

MP-inverse of differences and products of projectors in various sets attracts wide attention from many scholars. For instance, Cheng and Tian [1] studied the MP-inverses

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of pq and $p-q$, where p, q are projectors in complex matrices. Li [10] investigated how to express MP-inverses of product pq and differences $p-q$ and $pq-qp$, for two given projectors p and q in a C^*-algebra. Later, Deng and Wei [3] derived some formulae for the MP-inverse of the differences and the products of projectors in a Hilbert space. Recently, Zhang et al. [12] obtained the equivalences for the existences of differences and products of projectors in a $*$-reducing ring. More results on MP-inverses can be found in [7, 8, 11].

Motivated by [9], we investigate the equivalences for the existences of the MP-inverse of differences and products of projectors in a ring with involution. Moreover, the expressions of the MP-inverse of differences and products of projectors are presented. Some well-known results in C^*-algebras are extended.

Note that neither dimensional analysis nor special decomposition in Hilbert spaces and C^*-algebras can be used in rings. The results in this paper are proved by a purely ring theoretical method.

2 Some lemmas

In 1992, Harte and Mbekhta [5] showed an excellent result in C^*-algebras, i.e., if a is MP-invertible, then $a^*c = ca^*$ and $ac = ca$ imply $a^\dagger c = ca^\dagger$. In 2013, Drazin [4] extended this result to a $*$-semigroup case in Lemma 2.1 below.

Lemma 2.1. [4, Corollary 2.7] *Let S be any $*$-semigroup, let $a_1, a_2, d \in S$, and suppose that a_1 and a_2 each have Moore-Penrose inverses $a_1^\dagger$, $a_2^\dagger$, respectively. Then, for any $d \in S$, $da_1 = a_2d$ and $da_1^* = a_2^*d$ together imply $a_2^\dagger d = da_1^\dagger$.*

The following result in C^*-algebras was considered by Koliha [6]. For the convenience of the reader, we give its proof in a ring.

Lemma 2.2. *Let $a, b \in R^\dagger$ with $ab = ba$ and $a^*b = ba^*$. Then $ab \in R^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$.*

Proof. It follows from Lemma 2.1 that $a^\dagger b = ba^\dagger$ and $b^\dagger a = ab^\dagger$. As $b^*a = ab^*$ and $b^*a^* = a^*b^*$, then $b^*a^\dagger = a^\dagger b^*$, which together with $ba^\dagger = a^\dagger b$ imply $a^\dagger b^\dagger = b^\dagger a^\dagger$. Note that $aa^\dagger$ commutes with b and $b^\dagger$. Also, $bb^\dagger$ commutes with a and $a^\dagger$. Hence, $b^\dagger a^\dagger$ satisfies four equations of Penrose. Indeed, we have

(i) $(abb^\dagger a^\dagger)^* = (aba^\dagger b^\dagger)^* = (aa^\dagger bb^\dagger)^* = bb^\dagger aa^\dagger = aa^\dagger bb^\dagger = aba^\dagger b^\dagger = abb^\dagger a^\dagger$.

(ii) $(b^\dagger a^\dagger ab)^* = (b^\dagger ba^\dagger a)^* = a^\dagger ab^\dagger b = b^\dagger a^\dagger ab$.

(iii) $abb^\dagger a^\dagger ab = aa^\dagger bb^\dagger ab = aa^\dagger bb^\dagger ba = aa^\dagger ba = aa^\dagger ab = ab$.

(iv) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger ba^\dagger ab^\dagger a^\dagger = b^\dagger ba^\dagger aa^\dagger b^\dagger = b^\dagger ba^\dagger b^\dagger = b^\dagger a^\dagger$.

Therefore, $ab \in R^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger = a^\dagger b^\dagger$. □

Penrose [11, p. 408] presented the MP-inverse of $A + B$, where A and B are complex matrices such that $A^*B = 0$ and $AB^* = 0$. His formula indeed holds in a ring with involution.

Lemma 2.3. *Let $a, b \in R^\dagger$ such that $a^*b = ab^* = 0$. Then $(a + b)^\dagger = a^\dagger + b^\dagger$.*

3 Main results

We say that an element p is a projector if $p^2 = p = p^*$. Throughout this paper, the elements p, q are projectors from the ring R.

Theorem 3.1. *Let $a, b \in R^\dagger$ with $a^*p = pa^*$ and $b^*p = pb^*$. Then $ap + b(1 - p) \in R^\dagger$ and $(ap + b(1 - p))^\dagger = a^\dagger p + b^\dagger(1 - p)$.*

Proof. As $a^*p = pa^*$, then $ap = pa$ since p is a projector. Similarly, $bp = pb$. We have $(ap)^*b(1 - p) = 0$. Indeed, $(ap)^*b(1 - p) = pa^*(1 - p)b = a^*p(1 - p)b = 0$. Also, $ap(b(1-p))^* = 0$. By Lemma 2.2, it follows that $(ap)^\dagger = a^\dagger p$ and $(b(1-p))^\dagger = b^\dagger(1-p)$. In view of Lemma 2.3, we obtain $ap+b(1-p) \in R^\dagger$ and $(ap+b(1-p))^\dagger = a^\dagger p+b^\dagger(1-p)$. □

Recall from [8] that an element $a \in R$ is $*$-cancellable if $a^*ax = 0$ implies $ax = 0$ and $xaa^* = 0$ implies $xa = 0$. A ring R is called $*$-reducing ring if all elements in R are $*$-cancellable. We get the following result, under the condition of $*$-cancellabilities of some elements, rather than $*$-reducing rings in [12].

Proposition 3.2. *Let $p(1-q)$ and $q(1-p)$ be $*$-cancellable. Then the following conditions are equivalent:*

(1) $1 - pq \in R^\dagger$, (2) $1 - pqp \in R^\dagger$, (3) $p - pqp \in R^\dagger$, (4) $p - pq \in R^\dagger$, (5) $p - qp \in R^\dagger$,

(6) $1 - qp \in R^\dagger$, (7) $1 - qpq \in R^\dagger$, (8) $q - qpq \in R^\dagger$, (9) $q - qp \in R^\dagger$, (10) $q - pq \in R^\dagger$.

Proof. (1) $\Leftrightarrow$ (6) Note that $a \in R^{\dagger}$ if and only if $a^* \in R^{\dagger}$. Hence, it is sufficient to prove that (1) – (5).

(1) $\Leftrightarrow$ (2) By [12, Theorem 4].

(2) $\Rightarrow$ (3) Noting $p-pqp = p(1-pqp) = (1-pqp)p$, it is an immediate result of Lemma 2.2.

(3) $\Rightarrow$ (2) Since $1 - pqp = p(p - pqp) + 1 - p$ and $(p - pqp)^* = p - pqp$, it follows from Theorem 3.1 that $1 - pqp \in R^{\dagger}$.

(3) $\Leftrightarrow$ (4) Note that $a \in R^{\dagger} \Leftrightarrow aa^* \in R^{\dagger}$ and a is $*$-cancellable by [8, Theorem 5.4]. As $p(1 - q)(p(1 - q))^* = p - pqp \in R^{\dagger}$ and $p - pq$ is $*$-cancellable, the result follows.

(4) $\Leftrightarrow$ (5) As $(p - pq)^* = p - qp$ and $a \in R^{\dagger} \Leftrightarrow a^* \in R^{\dagger}$, then $p - pq \in R^{\dagger} \Leftrightarrow p - qp \in R^{\dagger}$. □

Recall that an element $a \in R$ is normal if $aa^* = a^*a$. Further, if a normal element a is MP-invertible, then $aa^{\dagger} = a^{\dagger}a$ by Lemma 2.2.

In 2004, Koliha, Rakočević and Straškraba [9] showed that $p - q$ is nonsingular if and only if $1 - pq$ and $p + q - pq$ are both nonsingular, for projectors p, q in complex matrices. It is natural to consider whether the same property can be inherited to the MP-inverse in a ring with involution. The following result illustrates its possibility.

Theorem 3.3. *Let $p - q$, $p(1 - q)$ and $q(1 - p)$ be $*$-cancellable. Then the following conditions are equivalent*:

(1) $p - q \in R^{\dagger}$,

(2) $1 - pq \in R^{\dagger}$,

(3) $p + q - pq \in R^{\dagger}$.

Proof. (1) $\Rightarrow$ (2) Note that $p - q$ is normal. It follows from Lemma 2.2 that $((p - q)^2)^{\dagger} = ((p - q)^{\dagger})^2$. As $p(p - q)^2 = (p - q)^2 p = p - pqp$, then $1 - pqp = (p - q)^2 p + 1 - p$ and hence $1 - pqp \in R^{\dagger}$ according to Theorem 3.1. So, $1 - pq \in R^{\dagger}$ by [12, Theorem 4].

(2) $\Rightarrow$ (1) By [12, Theorem 4], we know that $1 - pq \in R^{\dagger}$ implies $1 - pqp \in R^{\dagger}$. Let $\overline{p} = 1 - p$ and $\overline{q} = 1 - q$. Note that $p(1 - q)$ is $*$-cancellable. We have $1 - pq \in R^{\dagger} \Rightarrow p - pq = \overline{q} - \overline{p}\,\overline{q} \in R^{\dagger}$ by (1) $\Rightarrow$ (4) in Proposition 3.2. Also, as $\overline{q}(1 - \overline{p}) = p(1 - q)$ is $*$-cancellable, then $\overline{q} - \overline{p}\,\overline{q} \in R^{\dagger}$ implies $1 - \overline{q}\,\overline{p} \in R^{\dagger}$ by (10) $\Rightarrow$ (6) in Proposition 3.2,

which means $1-\overline{p}\,\overline{q}\in R^{\dagger}$ since $a\in R^{\dagger}\Leftrightarrow a^{*}\in R^{\dagger}$. Again, applying [12, Theorem 4], it follows that $1-\overline{p}\,\overline{q}\,\overline{p}\in R^{\dagger}$.

Setting $a=1-pqp$ and $b=1-\overline{p}\,\overline{q}\,\overline{p}$, then $a^{*}p=pa^{*}$ and $b^{*}p=pb^{*}$. Since $(p-q)^{2}=ap+b(1-p)$, we obtain $(p-q)^{2}=(p-q)(p-q)^{*}\in R^{\dagger}$ by Theorem 3.1 and hence $p-q\in R^{\dagger}$ from [8, Theorem 5.4].

(1) $\Leftrightarrow$ (3) In (1) $\Leftrightarrow$ (2), replacing p, q by $1-p$, $1-q$, respectively. □

Next, we mainly consider the representations of the MP-inverse by aforementioned results.

Theorem 3.4. *Let $p-q\in R^{\dagger}$. Define F, G and H as*

$$F=p(p-q)^{\dagger},\ G=(p-q)^{\dagger}p,\ H=(p-q)(p-q)^{\dagger}.$$

Then, we have

(1) $F^{2}=F=(p-q)^{\dagger}(1-q)$,

(2) $G^{2}=G=(1-q)(p-q)^{\dagger}$,

(3) $H^{2}=H=H^{*}$.

Proof. (1) We first prove $F=(p-q)^{\dagger}(1-q)$.

As $(p-q)^{*}=p-q$ and $p-q\in R^{\dagger}$, then $(p-q)^{2}\in R^{\dagger}$ by Lemma 2.2. Moreover, $((p-q)^{2})^{\dagger}=((p-q)^{\dagger})^{2}$. Also, $(p-q)(p-q)^{\dagger}=(p-q)^{\dagger}(p-q)$. From $p(p-q)^{2}=(p-q)^{2}p$ and $p((p-q)^{2})^{*}=((p-q)^{2})^{*}p$, we have $p((p-q)^{\dagger})^{2}=((p-q)^{\dagger})^{2}p$ using Lemma 2.1.

Hence,

$$\begin{aligned}(p-q)^{\dagger}(1-q) &= ((p-q)^{\dagger})^{2}(p-q)(1-q)=((p-q)^{\dagger})^{2}p(1-q)\\ &= ((p-q)^{\dagger})^{2}p(p-q)=p((p-q)^{\dagger})^{2}(p-q)\\ &= p(p-q)^{\dagger}\\ &= F.\end{aligned}$$

We now show $F^{2}=F$. Since $p(p-q)^{\dagger}=(p-q)^{\dagger}(1-q)$, one can get

$$\begin{aligned}F^{2} &= (p-q)^{\dagger}(1-q)p(p-q)^{\dagger}\\ &= (p-q)^{\dagger}(1-q)(p-q)(p-q)^{\dagger}\end{aligned}$$

$$
\begin{aligned}
&= p(p-q)^\dagger(p-q)(p-q)^\dagger \\
&= p(p-q)^\dagger \\
&= F.
\end{aligned}
$$

(2) By $F^* = G$.

(3) It is trivial. □

Under the same symbol in Theorem 3.4, more relations among F, G and H are given in the following result.

Corollary 3.5. *Let $p - q \in R^\dagger$. Then*

(1) $q(p-q)^\dagger = (p-q)^\dagger(1-p)$,

(2) $qH = Hq$,

(3) $G(1-q) = (1-q)F$.

Proof. (1) can be obtained by a similar proof of Theorem 3.4(1).

(2) Taking involution on (1), it follows that $(1-p)(p-q)^\dagger = (p-q)^\dagger q$ and hence

$$
\begin{aligned}
qH &= q(p-q)(p-q)^\dagger = q(p-1)(p-q)^\dagger \\
&= -q(p-q)^\dagger q = -(p-q)^\dagger(1-p)q \\
&= -(p-q)^\dagger(q-p)q \\
&= Hq.
\end{aligned}
$$

(3) We have

$$
\begin{aligned}
G(1-q) &= (p-q)^\dagger(p-q)(1-q) = (p-q)^\dagger p(p-q) \\
&= (1-q)(p-q)^\dagger(p-q) \\
&= (1-q)F.
\end{aligned}
$$

□

Keeping in mind the relations in Theorem 3.4 and Corollary 3.5, we give the following equalities, where $\overline{a}$ denotes $1-a$.

Corollary 3.6. *Let $p-q \in R^{\dagger}$. Then*

(1) $Fp = pG = pH = Hp$,

(2) $qHq = qH = Hq = HqH$,

(3) $\overline{q}\overline{F} = \overline{G}\overline{q} = \overline{q}\overline{F}\overline{q}$,

(4) $(p-q)^{\dagger} = F + G - H$.

In general, $p-q \in R^{\dagger}$ can not imply $p+q \in R^{\dagger}$. Such as, take $R = \mathbb{Z}$ and $1 = p = q \in R$, then $p - q = 0 \in R^{\dagger}$, but $p + q = 2 \notin R^{\dagger}$ since 2 is not invertible.

The next theorem presents the necessary and sufficient conditions for the existence of $(p+q)^{\dagger}$.

Theorem 3.7. *Let 2 be invertible in R. Then the following conditions are equivalent*:

(1) $pH = p$,

(2) $(p+q)H = (p+q)$,

(3) $p+q \in R^{\dagger}$ and $(p+q)^{\dagger} = (p-q)^{\dagger}(p+q)(p-q)^{\dagger}$.

Proof. (1) $\Rightarrow$ (2) If $pH = p$, then $qH = q$ by the symmetry of p and q. Hence $(p+q)H = (p+q)$.

(2) $\Rightarrow$ (1) Note that $H = (p-q)(p-q)^{\dagger}$ and $p-q$ is normal. We have $(p-q)H = p-q$ and $p+q = (p+q)H = (q-p)H + 2pH = -(p-q) + 2pH$, which implies $2pH = 2p$. Hence, $pH = p$ since 2 is invertible.

(2) $\Rightarrow$ (3) Let $x = (p-q)^{\dagger}(p+q)(p-q)^{\dagger}$. We prove that x is the MP-inverse of $p+q$ by checking four equations of Penrose.

(i) $((p+q)x)^* = (p+q)x$. Indeed,

$$\begin{aligned}(p+q)x &= (p+q)(p-q)^{\dagger}(p+q)(p-q)^{\dagger} \\ &= (p-q)^{\dagger}(1-q+1-p)(p+q)(p-q)^{\dagger} \\ &= (p-q)^{\dagger}(p-q)^2(p-q)^{\dagger} \\ &= (p-q)(p-q)^{\dagger}.\end{aligned}$$

(ii) $(x(p+q))^* = x(p+q)$. By similar proof of (i), we have $x(p+q) = (p-q)^{\dagger}(p-q)$.

(iii) Note that the relations $pH = Hp$ and $qH = Hq$ in Corollary 3.6. Then

$$(p+q)x(p+q) = (p-q)(p-q)^{\dagger}(p+q)$$

$$
\begin{aligned}
&= H(p+q) = (p+q)H \\
&= p+q.
\end{aligned}
$$

(iv) It follows that $x(p+q)x = (p-q)^\dagger(p+q)(p-q)^\dagger(p-q)(p-q)^\dagger = x$.

(3) $\Rightarrow$ (2) As $p+q \in R^\dagger$ with $(p+q)^\dagger = (p-q)^\dagger(p+q)(p-q)^\dagger$, then

$$
\begin{aligned}
p+q &= (p+q)(p+q)^\dagger(p+q) = (p+q)(p-q)^\dagger(p+q)(p-q)^\dagger(p+q) \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger(1-q+1-p)(p+q) \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger[(1-q)p+(1-p)q] \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger[(p-q)p+(q-p)q] \\
&= (p+q)(p-q)^\dagger(p-q)^\dagger(p-q)p-(p+q)(p-q)^\dagger(p-q)^\dagger(p-q)q \\
&= (p+q)(p-q)^\dagger(p-q)(p-q)^\dagger p-(p+q)(p-q)^\dagger(p-q)(p-q)^\dagger q \\
&= (p+q)(p-q)^\dagger p-(p+q)(p-q)^\dagger q \\
&= (p+q)(p-q)^\dagger(p-q) \\
&= (p+q)H.
\end{aligned}
$$

□

Next, we give a new necessary and sufficient condition of the existence of $(p+q)^\dagger$.

Theorem 3.8. *Let $p, q \in R$ with $pq = qp$. Then $p+q \in R^\dagger$ if and only if $1+pq \in R^\dagger$.*

In this case, $(p+q)^\dagger = (1+pq)^\dagger p + q(1-p)$ and $(1+pq)^\dagger = (p+q)^\dagger p + 1 - p$.

Proof. Suppose $p+q \in R^\dagger$. As $1+pq = p(p+q)+1-p$, then $(1+pq)^\dagger = (p+q)^\dagger p + 1 - p$ by Theorem 3.1.

Conversely, let $x = (1+pq)^\dagger p + q(1-p)$. We next show that x is the MP-inverse of $p+q$.

(i) $[(p+q)x]^* = (p+q)x$. We have

$$
\begin{aligned}
(p+q)x &= (p+q)[(1+pq)^\dagger p + q(1-p)] \\
&= (1+pq)^\dagger p + (1+pq)^\dagger pq + q(1-p) \\
&= (1+pq)^\dagger(1+pq)p + q(1-p).
\end{aligned}
$$

Hence, $[(p+q)x]^* = (p+q)x$.

(ii) It follows that $[x(p+q)]^* = x(p+q)$ since p and q commute.

(iii) $(p+q)x(p+q) = p+q$. Indeed,

$$\begin{aligned}(p+q)x(p+q) &= (p+q)[(1+pq)^\dagger(1+pq)p + q(1-p)] \\ &= (1+pq)^\dagger(1+pq)p + (1+pq)^\dagger(1+pq)pq + q(1-p) \\ &= (1+pq)^\dagger(1+pq)p(1+pq) + q(1-pq) \\ &= p(1+pq) + q(1-pq) \\ &= p+q.\end{aligned}$$

(iv) By a similar way of (3), we get $x(p+q)x = x$.

Thus, $(p+q)^\dagger = (1+pq)^\dagger p + q(1-p)$. □

The next theorem, a main result of this paper, admits proficient skills on F, G and H, expressing the formulae of the MP-inverse of difference of projectors.

Theorem 3.9. *Let $p-q \in R^\dagger$. Then*

(1) $(1-pqp)^\dagger = p((p-q)^\dagger)^2 + (1-p)$,

(2) $(1-pq)^\dagger = p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p$,

(3) $(p-pqp)^\dagger = p((p-q)^\dagger)^2$,

(4) *If $p-pq$ is $*$-cancellable, then* $(p-pq)^\dagger = (p-q)^\dagger p$,

(5) *If $p-pq$ is $*$-cancellable, then* $(p-qp)^\dagger = p(p-q)^\dagger$.

Proof. (1) As $1-pqp = p(p-q)^2 + 1 - p$, then $(1-pqp)^\dagger = p((p-q)^\dagger)^2 + 1 - p$ according to Theorem 3.1.

(2) It follows from Theorem 3.3 that $p-q \in R^\dagger$ implies $1-pq \in R^\dagger$. Let $x = p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p$. We next show that x is the MP-inverse of $1-pq$.

(i) We have

$$\begin{aligned}(1-pq)x &= (1-pq)[p((p-q)^\dagger)^2 - pq(p-q)^\dagger + 1 - p] \\ &= (p-pqp)((p-q)^\dagger)^2 - (1-pq)pq(p-q)^\dagger + (1-pq)(1-p) \\ &= p(p-q)^2((p-q)^\dagger)^2 - (p-pqp)(p-q)^\dagger(1-p) + (1-pq)(1-p) \\ &= p(p-q)(p-q)^\dagger - p(p-q)^2(p-q)^\dagger(1-p) + (1-pq)(1-p)\end{aligned}$$

$$
\begin{aligned}
&= p(p-q)(p-q)^\dagger - p(p-q)(1-p) + (1-pq)(1-p) \\
&= p(p-q)(p-q)^\dagger + 1 - p \\
&= pH + 1 - p.
\end{aligned}
$$

Hence, $((1-pq)x)^* = (1-pq)x$ since $pH = Hp$ and $H^* = H$.

(ii) We get $x(1-pq) = p(p-q)^\dagger p + 1 - p$. Hence, $(x(1-pq))^* = x(1-pq)$.

(iii) $(1-pq)x(1-pq) = 1-pq$. Indeed,

$$
\begin{aligned}
(1-pq)x(1-pq) &= (pH+1-p)(1-pq) = Hp(1-pq) + (1-p)(1-pq) \\
&= Hp(p-pq) + 1 - p = pH(p-pq) + 1 - p \\
&= pHp(p-q) + 1 - p = pH(p-q) + 1 - p \\
&= p(p-q) + 1 - p \\
&= 1 - pq.
\end{aligned}
$$

(iv) $x(1-pq)x = 1-pq$. Actually, we can obtain this result by a similar proof of (iii).

(3) Since $p - pqp = p(p-q)^2 = (p-q)^2 p$, we get $(p-pqp)^\dagger = p((p-q)^\dagger)^2$ by Lemma 2.2.

(4) Keeping in mind that $a^\dagger = a^*(aa^*)^\dagger = (a^*a)^\dagger a^*$, we have $(p-pq)^\dagger = (p-qp)p((p-q)^\dagger)^2 = (p-q)((p-q)^\dagger)^2 p = (p-q)^\dagger p$.

(5) Note that a is $*$-cancellable if and only if a^* is $*$-cancellable. It follows from $(a^*)^\dagger = (a^\dagger)^*$ that $(p-qp)^\dagger = p(p-q)^\dagger$. □

Corollary 3.10. *Let $p - pq$ be $*$-cancellable and let $1 - pq \in R^\dagger$. Then $p - q \in R^\dagger$ and*

$$(p-q)^\dagger = (1-pq)^\dagger(p-pq) + (p+q-pq)^\dagger(pq-q).$$

Proof. From Theorem 3.3, we have $p - q \in R^\dagger \Leftrightarrow 1 - pq \in R^\dagger$.

By Theorem 3.9 (2), we have $(p+q-pq)^\dagger = (1-p)((p-q)^\dagger)^2 + (1-p)(1-q)(p-q)^\dagger + p$. It is straight to check that $(1-pq)^\dagger(p-pq) + (p+q-pq)^\dagger(pq-q)$ satisfies four equations of Penrose. □

The following result is motivated by [2], therein, Deng considered the Drazin inverses of difference of idempotent bounded operators on Hilbert spaces.

Theorem 3.11. *Let $pq - qp$ be $*$-cancellable. Then*

(1) $(p-q)^\dagger = p - q$ *if and only if* $pq = qp$,

(2) *If* 6 *is invertible in* R, *then* $(p+q)^\dagger = p + q$ *if and only if* $pq = 0$.

Proof. (1) If $pq = qp$, it is straightforward to check $(p-q)^\dagger = p - q$.

Conversely, $(p-q)^\dagger = p - q$ implies $(p-q)^3 = p - q$, we get $pqp = qpq$ and hence $(pq - qp)^*(pq - qp) = 0$. It follows that $pq = qp$ since $pq - qp$ is $*$-cancellable.

(2) Suppose $pq = 0$. Then $p^*q = pq^* = 0$ since p, q are projectors. Then $(p+q)^\dagger = p+q$ by Lemma 2.3.

Conversely, $(p+q)^\dagger = p+q$ concludes $(p+q)^3 = p+q$. By direct calculations, it follows that $2pq + 2qp + pqp + qpq = 0$. (3.1)

Multiplying the equality (3.1) by p on the left yields $2pq + 3pqp + pqpq = 0$. (3.2)

Multiplying the equality (3.1) by q on the right gives $2pq + 3qpq + pqpq = 0$. (3.3)

Combining the equalities (3.2) and (3.3), it follows that $pqp = qpq$ since 3 is invertible. As $pq - qp$ is $*$-cancellable, then $pqp = qpq$ implies $pq = qp$. Hence, equality (3.1) can be reduced to $6pq = 0$.

Thus, $pq = 0$. □

Theorem 3.12. *Let* $1 - p - q \in R^\dagger$. *Then*

(1) $pqp \in R^\dagger$ *and* $(pqp)^\dagger = p((1-p-q)^\dagger)^2 = ((1-p-q)^\dagger)^2 p$,

(2) *If* pq *is* $*$-*cancellable, then* $pq \in R^\dagger$ *and* $(pq)^\dagger = qp((1-p-q)^\dagger)^2$.

Proof. (1) Since $(1-p-q)^* = 1-p-q$, we have $((1-p-q)^2)^\dagger = ((1-p-q)^\dagger)^2$ by Lemma 2.2. As $pqp = p(1-p-q)^2 = (1-p-q)^2 p$, then $pqp \in R^\dagger$ from Lemma 2.2 and hence $(pqp)^\dagger = p((1-p-q)^\dagger)^2 = ((1-p-q)^\dagger)^2 p$.

(2) Note that $1 - p - q \in R^\dagger$ implies $pqp \in R^\dagger$. As $pqp = pq(pq)^*$ and pq is $*$-cancellable, then $pq \in R^\dagger$ by [8, Theorem 5.4]. The formula $a^\dagger = a^*(aa^*)^\dagger$ guarantees that $(pq)^\dagger = qp((1-p-q)^\dagger)^2$. □

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