The inverse along a product and its applications

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Abstract

In this paper, we study the recently defined notion of the inverse along an element. An existence criterion for the inverse along a product is given in a ring. As applications, we present the equivalent conditions for the existence and expressions of the inverse along a matrix.

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Von Neumann regularity, Inverse along an element, Green’s relations, Matrices over a ring
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1. Introduction

In this paper, $R$ is an associative ring with unity 1. An element $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such $x$, an inner inverse of $a$, is denoted by $a^-$. We call $b$ an outer inverse of $a$ provided that $bab = b$. If $b$ is both an inner and an outer inverse of $a$, then it is a reflexive inverse of $a$, and is denoted by $a^+$. Given a semigroup $S$, $S^1$ denotes the monoid generated by $S$. Following Green [1], Green’s preorders and relations in a semigroup are defined by

- $a \leq_L b \iff S^1a \subset S^1b \iff$ there exists $x \in S^1$ such that $a = xb$.
- $a \leq_R b \iff aS^1 \subset bS^1 \iff$ there exists $x \in S^1$ such that $a = bx$.
- $a \leq_H b \iff a \leq_L b$ and $a \leq_R b$.
- $a \leq_L b \iff S^1a = S^1b \iff$ there exist $x, y \in S^1$ such that $a = xb$ and $b = ya$.

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\[ aRb \iff aS^1 = bS^1 \iff \text{there exist } x, y \in S^1 \text{ such that } a = bx \text{ and } b = ay. \]
\[ aHb \iff aLb \text{ and } aRb. \]

Recently, Mary [4] introduced the notion of the inverse along an element that is based on Green’s relation in a semigroup \( S \). Given \( a, d \in S \), an element \( a \in S \) is invertible along \( d \) [4] if there exists \( b \) such that \( dab = d = bad \) and \( b \leq_H d \). The element \( b \) above is unique if it exists, and is denoted by \( a^\parallel d \). Recall that \( a^\parallel d \) exists implies that \( d \) is regular. Later, Mary and Patrício [5] proved that \( a \) is invertible along \( d \) if and only if \( dHdad \), which gave a new existence criterion for the inverse along an element. Further, given a regular element \( d \), they [5, 6] characterized the existence of \( a^\parallel d \) by means of a unit and \( d^- \) in a ring. Moreover, the representation of \( a^\parallel d \) is given. As applications, they [6] derived the equivalent conditions for the existence and the formula of the inverse along a regular lower triangular matrix. More results on the inverse along an element can be found in mathematical literature [3, 9].

Motivated by papers [5, 6], we investigate the inverse along a product \( pmq \) (\( m \) is regular) in a ring, extending the results in [5, 6]. As applications, the inverse along a regular matrix \[
\begin{bmatrix}
d_1 & d_3 \\
d_2 & d_4
\end{bmatrix}
\] is given under some conditions.

## 2. The inverse along a product \( pmq \)

In this section, we begin with some lemmas which play important roles in the sequel.

**Lemma 2.1.** Given \( a, b \in R \), then \( 1 + ab \) is invertible if and only if \( 1 + ba \) is invertible. Moreover, \( (1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a \).

Lemma 2.1 is known as the Jacobson’s Lemma (see e.g. [2]).

**Lemma 2.2.** ([8, Theorem 1]) Let \( R \) be a ring and \( e \) an idempotent in \( R \). Then \( exe + 1 - e \) is invertible in \( R \) if and only if \( exe \) is invertible in \( eRe \).

The next theorem, a main result of this paper, gives an existence criterion of the inverse along a product \( pmq \) in a ring.

**Theorem 2.3.** Let \( p, a, q, m \in R \) with \( m \) regular. If \( m \leq_L pm \) and \( m \leq_R mq \), then the following conditions are equivalent

- (i) \( a \) is invertible along \( pmq \).
- (ii) \( u = mqap + 1 - mm^- \) is invertible.
(iii) $v = qapm + 1 - m^-m$ is invertible.

In this case,

$$a^\parallel_{pmq} = pu^{-1}mq = pmv^{-1}q.$$ 

PROOF. It follows from Lemma 2.1 that (ii)$\Leftrightarrow$(iii). Next, it is sufficient to prove (i)$\Leftrightarrow$(ii).

(i)$\Rightarrow$(ii) Suppose that $a$ is invertible along $pmq$. From $m \leq_L pm$ and $m \leq_R mq$, then there exist $p'$ and $q'$ such that $p'pm = m = mqq'$. In view of [5, Theorem 2.2], we know that $a$ is invertible along $pmq$ if and only if $pmqHpmqapmq$. There are $x, y \in R$ such that

$$pmq = xpmqapmq = pmqapmqy. \quad (1)$$

Multiplying the above equation (1) by $p'$ on the left yields

$$mq = mqapmqy.$$ 

Multiplying the above equation (1) by $q'$ on the right yields

$$pm = xpmqapm.$$ 

Hence,

$$mqapmm^-mqyqm^-mm^- = mm^- = (mm^-p'xpmmm^-)mqapmm^-.$$ 

The equalities above show that $mqapmm^-$ is invertible in $mm^-Rmm^-$. By Lemma 2.2, $mqapmm^- + 1 - mm^-$ is invertible in $R$. Again, Lemma 2.1 ensures that $u = mqap + 1 - mm^-$ is invertible.

(ii)$\Rightarrow$(i) Suppose that $u$, therefore $v$ are invertible. From $um = mv = mqapm$, it follows that $pmq = pu^{-1}mqapmq = pmqapmv^{-1}q$ and $pu^{-1}mq = pmv^{-1}q$. Pose $b = pu^{-1}mq = pmv^{-1}q$, then $b \leq_R pmq$ since $pu^{-1}mq = pu^{-1}p'pmq = pmqq'v^{-1}q$.

Hence, $a$ is invertible along $pmq$. Moreover,

$$a^\parallel_{pmq} = pu^{-1}mq = pmv^{-1}q.$$ 

The proof is completed. $\Box$

If $p$ is left invertible and $q$ is right invertible, then $m\mathcal{L}pm$ and $m\mathcal{R}mq$. As a special result of Theorem 2.3, we have the following corollary.
Corollary 2.4. Let \( p, a, q, m \in R \) with \( m \) regular. If \( p \) is left invertible and \( q \) is right invertible, then the following conditions are equivalent

(i) \( a \) is invertible along \( pmq \).
(ii) \( u = mqap + 1 - mm^{-} \) is invertible.
(iii) \( v = qapm + 1 - m^{-}m \) is invertible.

In this case, \( a^\|pmq = pu^{-1}mq = pmv^{-1}q \).

Taking \( p = q = 1 \), we get

Corollary 2.5. ([5, Theorem 3.2] and [6, Theorem 1.3]) Let \( m \) be a regular element of a ring \( R \). Then the following are equivalent

(i) \( a \) is invertible along \( m \).
(ii) \( u = ma + 1 - mm^{-} \) is invertible.
(iii) \( v = am + 1 - m^{-}m \) is invertible.

In this case, \( a^\|m = u^{-1}m = mv^{-1} \).

3. Applications to the inverse along a matrix

Mary, Patrício [6] gave some equivalent conditions for the existence of the inverse along a regular lower triangular matrix \( \begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix} \) over a Dedekind-finite ring. It would be interesting to find the related existence criteria and formula of the inverse along a regular matrix \( D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \), in the general case.

By \( R_{2 \times 2} \) we denote the ring of \( 2 \times 2 \) matrices over \( R \). Let \( D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in R_{2 \times 2} \) and \( D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: PMQ \). Given a lower triangular matrix \( M = \begin{bmatrix} d_2 & 0 \\ d_1 & d_3 \end{bmatrix} \) with \( d_2 \) and \( d_3 \) regular, Patrício and Puystjens [7] proved that \( M \) is regular if and only if \( w = (1 - d_3d_3^{-})d_1(1 - d_2^{-}d_2) \) is regular. In this case,

\[
MM^{-} = \begin{bmatrix} d_2d_2^{-} & 0 \\ (1 - ww^{-})(1 - d_3d_3^{-})d_1d_2^{-} & d_3d_3^{-} + ww^{-}(1 - d_3d_3^{-}) \end{bmatrix}.
\]
Next, we consider the inverse along a regular matrix, whose $(2, 2)$ entry is zero.

**Theorem 3.1.** Let \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad D = \begin{bmatrix} d_1 & d_3 \\ d_2 & 0 \end{bmatrix} \in \mathbb{R}_{2 \times 2} \) with \( d_2 \) and \( d_3 \) regular. If \( c \| d_2 \) exists, then \( A^D \) exists if and only if \( \xi = \beta - \alpha c \| d_2 a \) is invertible.

In this case, \( A^D = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c \| d_2) & \xi^{-1}d_3 \\ c \| d_2 [1 - a \xi^{-1}(d_1 - \alpha c \| d_2)] & -c \| d_2 a \xi^{-1}d_3 \end{bmatrix}, \)

\[
\begin{align*}
\alpha &= d_1c + d_3d - (1 - w w^-)(1 - d_3d_2^+)d_1d_2^+,
\beta &= d_1a + d_3b + (1 - w w^-)(1 - d_3d_2^+),
\xi &= \beta - \alpha c \| d_2 a.
\end{align*}
\]

**Proof.** We have \( MAP = \begin{bmatrix} d_2c & d_2a \\ d_1c + d_3d & d_1a + d_3b \end{bmatrix}. \) Hence,

\[
U = MAP + I - MM^- = \begin{bmatrix} u & d_2a \\ \alpha & \beta \end{bmatrix}, \quad \text{where}
\]

\[
\begin{align*}
u &= d_2c + 1 - d_2d_2^+,
\alpha &= d_1c + d_3d - (1 - w w^-)(1 - d_3d_2^+)d_1d_2^+,
\beta &= d_1a + d_3b + (1 - w w^-)(1 - d_3d_2^+).
\end{align*}
\]

Since \( c \| d_2 \) exists, it follows that \( u = d_2c + 1 - d_2d_2^+ \) is invertible and \( c \| d_2 = u^{-1}d_2 \). Using Schur complements we get the factorization

\[
U = \begin{bmatrix} 1 & 0 \\ \alpha^{-1}u^{-1} & 1 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1 & c \| d_2 a \\ \alpha^{-1}u^{-1} & 1 \end{bmatrix},
\]

where \( \xi = \beta - \alpha c \| d_2 a \). Hence, \( U \) is invertible if and only if \( \xi \) is invertible.

Note that \( U^{-1} = \begin{bmatrix} 1 & -c \| d_2 a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha^{-1}u^{-1} & 1 \end{bmatrix}. \) Then

\[
A^D = PU^{-1}M = \begin{bmatrix} \xi^{-1}(d_1 - \alpha c \| d_2) & \xi^{-1}d_3 \\ c \| d_2 [1 - a \xi^{-1}(d_1 - \alpha c \| d_2)] & -c \| d_2 a \xi^{-1}d_3 \end{bmatrix}.
\]

The proof is completed. \( \square \)
Remark 3.2. In the above Theorem, if $c$ is not invertible along $d_2$, $A^{||D}$ may exist. Next, we give an example to illustrate it.

Take $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A$ to be the $2 \times 2$ identity matrix over any field. Since $0$ is not invertible along $1$ then the $(1,2)$ entry of $A$ is not invertible the $(2,1)$ entry of $D$, and yet $A$ is invertible along $D$ since they are both invertible.

Now, suppose that $d_4$ in the matrix $D$ is regular and set $e = 1 - d_4d_4^+$, $f = 1 - d_4^+d_4$ and $s = d_1 - d_3d_4^+d_2$. We have the following decomposition

$$D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3d_4^+ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & d_3f \\ ed_2 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^+d_2 & 1 \end{bmatrix} =: PMQ.$$  

We next discuss the inverse of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ along a regular matrix $D$, under certain conditions.

Theorem 3.3. Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times2}$ with $d_4$ regular. With the notations above, if $d_3f = 0$ and $a^{||s}$ exists, then $A^{||D}$ exists if and only if\[\xi = \beta - \alpha a^{||s}(ad_3d_4^+ + c)\]
is invertible.

In this case, $A^{||D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^{||s}) & \xi^{-1}d_4 \end{bmatrix}$, where

\[
\begin{align*}
u &= sa + 1 - ss^+,
\alpha &= d_2a + d_4b - (1 - tt^+)ed_2s^+,
\beta &= (d_2a + d_4b)d_3d_4^+ + d_2c + d_4d + (1 - tt^-)e,
\xi &= \beta - \alpha a^{||s}(ad_3d_4^+ + c),
x_1 &= [(1 - a^{||s})d_3d_4^+ - a^{||s}c]\xi^{-1},
x_2 &= u^{-1} - x_1\alpha u^{-1}.
\end{align*}
\]

Proof. If $d_3f = 0$, then $M = \begin{bmatrix} s & 0 \\ ed_2 & d_4 \end{bmatrix}$. Note that the regularity of $D$ is equivalent to the regularity of $M$. Hence, it follows from [7, Theorem 1] that

$$I - MM^- = \begin{bmatrix} 1 - ss^+ & 0 \\ -(1 - tt^-)ed_2s^+ & (1 - tt^-)e \end{bmatrix},$$

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where $t = ed_2(1 - s^+s)$.

Note that $MQAP = \begin{bmatrix} sa & s(ad_3d_4^++c) \\ d_2a + d_4b & (d_2a + d_4b)d_3d_4^++d_2c + d_4d \end{bmatrix}$. We have

$$U = MQAP + I - MM = \begin{bmatrix} u & s(ad_3d_4^++c) \\ \alpha & \beta \end{bmatrix},$$

where

$$u = sa + 1 - ss^+, \quad \alpha = d_2a + d_4b - (1 - tt^-)ed_2s^+, \quad \beta = (d_2a + d_4b)d_3d_4^++d_2c + d_4d + (1 - tt^-)e.$$

In this case,

$$U^{-1} = \begin{bmatrix} 1 & -a \parallel s(ad_3d_4^++c) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha u^{-1} & 1 \end{bmatrix},$$

where $\xi = \beta - \alpha a \parallel s(ad_3d_4^++c)$.

By calculations, $A \parallel D = PU^{-1}MQ = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a \parallel s) & \xi^{-1}d_4 \end{bmatrix}$, where

$$x_1 = [(1 - a \parallel s a)d_3d_4^++a \parallel s c]\xi^{-1}, \quad x_2 = u^{-1} - x_1\alpha u^{-1}.$$

The proof is completed. \(\square\)

**Remark 3.4.** In Theorem 3.3, $A \parallel D$ may exist and yet $d_3f \neq 0$ and $a \parallel s$ exists.

Indeed, suppose that $R = \mathbb{Z}/6\mathbb{Z}$ and let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in R_{2 \times 2}$. Note that $A$ is invertible along $D$, using Corollary 2.5. From $d_4^+=2$, we have $f = 1 - d_4^+d_4 = 3$ and $d_3f = 3 \neq 0$. Note that $sa + 1 - ss^- = 5$ is invertible, from which $a$ is invertible along $s$.

In Theorem 3.3, if $d_4$ is invertible, then $e = f = 0$. Hence, we have the following corollary.

**Corollary 3.5.** Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2 \times 2}$ with $d_4$ invertible. If $a \parallel s$ exists, then $A \parallel D$ exists if and only if $\xi = \beta - \alpha a \parallel s(ad_3d_4^++c)$ is invertible.
In this case, \( A^D = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a^s) & \xi^{-1}d_4 \end{bmatrix} \), where

\[
\begin{align*}
    s &= d_1 - d_3d_4^{-1}d_2, \\
    u &= sa + 1 - ss^+, \\
    \alpha &= d_2a + d_4b, \\
    \beta &= \alpha d_3d_4^{-1} + d_2c + d_4d, \\
    \xi &= \beta - \alpha a^s(ad_3d_4^{-1} + c), \\
    x_1 &= [(1 - a^s a)d_3d_4^{-1} - a^s c]\xi^{-1}, \\
    x_2 &= u^{-1} - x_1\alpha^{-1}.
\end{align*}
\]

In Theorem 3.3, take \( d_4 = 0 \), then \( s = d_1 \). We can get the formula and equivalence for the existence of the inverse along a regular lower triangular matrix obtained in [6].

**Corollary 3.6.** ([6, Theorem 3.1]) Let \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \ D = \begin{bmatrix} d_1 & 0 \\ d_2 & d_4 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) with \( d_4 \) regular. With the notations above, if \( a^d_1 \) exists, then \( A^D \) exists if and only if \( \xi = \beta - \alpha a^s d_1 c \) is invertible.

In this case, \( A^D = \begin{bmatrix} a^d_1 & \xi^{-1}d_4 \\ \xi^{-1}(d_2 - \alpha d_1^s) & \xi^{-1}d_4 \end{bmatrix} \), where

\[
\begin{align*}
    u &= d_1a + 1 - d_1d_1^+, \\
    t &= ed_2(1 - d_1^+d_1), \\
    \alpha &= d_2a + d_4b - (1 - tt^+)ed_2d_1^+, \\
    \beta &= d_2c + d_4d + (1 - tt^+)e, \\
    \xi &= \beta - \alpha a^s d_1 c.
\end{align*}
\]

By taking \( ed_2 = 0 \) in Theorem 3.3, we get the following corollary.

**Corollary 3.7.** Let \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \ D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \) with \( d_4 \) regular. With the notations above, if \( ed_2 = d_3f = 0 \) and \( a^s \) exists, then \( A^D \) exists if and only if \( \xi = \beta - \alpha a^s(ad_3d_4^+ + c) \) is invertible.
In this case, \( A^{\parallel D} = \begin{bmatrix} x_1d_2 + x_2s & x_1d_4 \\ \xi^{-1}(d_2 - \alpha a\|s) & \xi^{-1}d_4 \end{bmatrix} \), where

\[
\begin{align*}
  u & = sa + 1 - ss^+, \\
  \alpha & = d_2a + d_4b, \\
  \beta & = \alpha d_3d_4^+ + d_2c + d_4d + e, \\
  \xi & = \beta - \alpha a\|s(\alpha d_3d_4^+ + c), \\
  x_1 & = [(1 - a\|s)a)d_3d_4^+ - a\|s]c]\xi^{-1}, \\
  x_2 & = u^{-1} - x_1\alpha u^{-1}.
\end{align*}
\]

Question 3.8. Given a regular matrix \( D \), can we give further equivalent conditions such that \( A^{\parallel D} \) exists without additional conditions?

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