# About the von Neumann regularity of triangular block matrices

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**Abstract**

Necessary and sufficient conditions are given for the von Neumann regularity of triangular block matrices with von Neumann regular diagonal blocks over arbitrary rings. This leads to the characterization of the von Neumann regularity of a class of triangular Toeplitz matrices over arbitrary rings. Some special results and a new algorithm are derived for triangular Toeplitz matrices over commutative rings. Finally, the Drazin invertibility of some companion matrices over arbitrary rings is considered, as an application.

Keywords: Block matrices over rings, von Neumann regularity, triangular Toeplitz matrices.

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## 1 Introduction

Let $R$ be an arbitrary ring with unity 1, $Mat\,(n, R)$ the full ring of $n \times n$ matrices over $R$ and $Tri\,(n, R)$ the subring of lower triangular $n \times n$ matrices over $R$. The identity matrix with the appropriate size is denoted by $I$. An $m \times n$ matrix $A$ over $R$ is said to be *von Neumann regular* over $R$ if there exists an $n \times m$ matrix $A^{(1)}$ over $R$ such that $AA^{(1)}A = A$. Let

$$T_k = T\begin{pmatrix} a \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{bmatrix} a & 0 & \cdots & 0 & 0 \\ a_1 & a & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ a_{k-2} & \cdots & a_1 & a & 0 \\ a_{k-1} & a_{k-2} & \cdots & a_1 & a \end{bmatrix},$$

with components $a, a_1, ..., a_{k-1}$ in $R$, be a lower triangular Toeplitz matrix over $R$. It has to be remarked first that the von Neumann regularity of $T_k$ does *not* imply the von Neumann regularity of (nonzero) components or blocks of $T_k$. Indeed, consider $R = Mat\,(4, \mathbb{Z}_{12})$ with

$$a = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right], a_1 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right].$$

Then $a$ and $a_1$ are *not* von Neumann regular in $R = Mat\,(4, \mathbb{Z}_{12})$ but

$$\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$$

is von Neumann regular in $Mat\,(2, R)$. Secondly, there is a difference between the von Neumann regularity of $T_k$ in the ring $Tri\,(k, R)$ and in the full matrix ring $Mat\,(k, R)$. Indeed, if $a$ is of the form $a_r + a_j$, with $a_r$ von Neumann regular in $R$ and $a_j$ is in the Jacobson radical of $R$, then

$$T_k = T\begin{pmatrix} a_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + T\begin{pmatrix} a_j \\ a_1 \\ \vdots \\ a_{k-1} \end{pmatrix},$$

in which the first term is von Neumann regular and the second term is in the Jacobson radical of $Tri\,(k, R)$. This means that Theorem 1 of [2] can be applied to see if $T_k$ is von Neumann regular in $Tri\,(k,R)$. Clearly, if $T_k$ is von Neumann regular in $Tri\,(k,R)$ then $T_k$ is von Neumann regular in $Mat\,(k,R)$, but the following example shows that there are $T_k$ matrices not von Neumann regular in $Tri\,(k,R)$ and von Neumann regular in $Mat\,(k,R)$. Indeed, 2 is not von Neumann regular and 8 is von Neumann regular in $\mathbb{Z}_{12}$, and

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 1 & 6 \end{bmatrix}.$$

Theorem 1 of [2] shows that there is no lower triangular von Neumann inverse of $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, although $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is von Neumann regular in $Mat\,(2,\mathbb{Z}_{12})$ because it is equivalent with $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ which is von Neumann regular in $Mat\,(2,\mathbb{Z}_{12})$. We clearly state that we wanted to investigate the von Neumann regularity of lower triangular Toeplitz matrices in the full matrix ring $Mat\,(k,R)$. The only restriction we had to make in the general case is that the lower triangular Toeplitz matrices are of the form $\begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$, with $U$ and $W$ von Neumann regular over $R$. Therefore, we considered first the von Neumann regularity of block matrices of the form $\begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$ with $U$ and $W$ square or non-square, but von Neumann regular over arbitrary rings. As special cases, we consider lower triangular Toeplitz matrices with von Neumann regular components in a commutative ring and also lower triangular Toeplitz matrices with a von Neumann regular diagonal over a commutative ring. As an application, we consider the Drazin invertibility of some companion matrices over arbitrary rings, see also [5]. We refer to [1] for definitions and notations used in this paper.

## 2 The general case

In this section, all matrices are over an arbitrary ring.

**Theorem 1.** *Let* $M = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$ *be a* $2 \times 2$ *block matrix such that the square or non-square* $U$ *and* $W$ *are von Neumann regular with* $(1)$*-inverses* $U^{(1)}$ *and* $W^{(1)}$*, respectively. Then* $M$ *is von Neumann regular if and only if*

$$T = \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right)$$

*is von Neumann regular. Moreover, for any von Neumann inverse* $T^{(1)}$ *of* $T$*, the product*

$$\begin{bmatrix} I & 0 \\ -W^{(1)}V & I \end{bmatrix} \begin{bmatrix} U^{(1)}UU^{(1)} & \left(I - U^{(1)}U\right) T^{(1)} \left(I - WW^{(1)}\right) \\ 0 & W^{(1)}WW^{(1)} \end{bmatrix} \times$$
$$\times \begin{bmatrix} I & 0 \\ -\left(I - WW^{(1)}\right) VU^{(1)} & I \end{bmatrix}$$

*is a von Neumann inverse of* $M$.

*Proof.* Firstly, assume $\left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right)$ is von Neumann regular. Let

$$\begin{aligned} H &= I - U^{(1)}U, \\ G &= I - WW^{(1)}, \\ E &= \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}, \\ F &= \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

We remark that $UH = 0$ and so $U(I - H) = U$, and that $GW = 0$ and therefore $(I - G)W = W$. Moreover,

$$\begin{bmatrix} U & 0 \\ GVH & W \end{bmatrix} = E \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix} F + (I - E) \begin{bmatrix} U & 0 \\ 0 & W \end{bmatrix} (I - F).$$

Setting $X = E \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix} F$ and $Y = (I - E) \begin{bmatrix} U & 0 \\ 0 & W \end{bmatrix} (I - F)$ and since $GVH, U$ and $W$ are von Neumann regular, then $X$ and $Y$ are also von

Neumann regular. Let $X^{(1)}$ and $Y^{(1)}$ be von Neumann inverses of $X$ and $Y$, respectively. Furthermore, since

$$\begin{aligned} EX &= XF = X, \\ (I-E)Y &= Y(I-F) = Y, \end{aligned}$$

it follows that

$$\begin{aligned} & (EXF + (I-E)Y(I-F))\left(FX^{(1)}E + (I-F)Y^{(1)}(I-E)\right) \times \\ & \times (EXF + (I-E)Y(I-F)) \\ = \; & (EXF + (I-E)Y(I-F)) \end{aligned}$$

and $\begin{bmatrix} U & 0 \\ GVH & W \end{bmatrix}$ is von Neumann regular. To show that $\begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$ is von Neumann regular, we remark that

$$\begin{bmatrix} I & 0 \\ -GVU^{(1)} & I \end{bmatrix} \begin{bmatrix} U & 0 \\ V & W \end{bmatrix} \begin{bmatrix} I & 0 \\ -W^{(1)}V & I \end{bmatrix} = \begin{bmatrix} U & 0 \\ GVH & W \end{bmatrix}. \qquad (1)$$

Let us now compute a von Neumann inverse of $M$. Let

$$\begin{aligned} T &= \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right), \\ X &= \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}, \\ X^{(1)} &= \begin{bmatrix} 0 & T^{(1)} \\ 0 & 0 \end{bmatrix}, \\ Y &= \begin{bmatrix} U & 0 \\ 0 & W \end{bmatrix}, \\ Y^{(1)} &= \begin{bmatrix} U^{(1)} & 0 \\ 0 & W^{(1)} \end{bmatrix}. \end{aligned}$$

It was shown in the first part of this proof that $FX^{(1)}E + (I-F)Y^{(1)}(I-E)$ is a von Neumann inverse of $X+Y$, i.e.

$$\begin{bmatrix} U^{(1)}UU^{(1)} & \left(I - U^{(1)}U\right) T^{(1)} \left(I - WW^{(1)}\right) \\ 0 & W^{(1)}WW^{(1)} \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} U & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & W \end{bmatrix}$. Using (1),

$$\begin{bmatrix} 1 & 0 \\ -W^{(1)}V & 1 \end{bmatrix} \begin{bmatrix} U^{(1)}UU^{(1)} & \left(I - U^{(1)}U\right) T^{(1)} \left(I - WW^{(1)}\right) \\ 0 & W^{(1)}WW^{(1)} \end{bmatrix} \begin{bmatrix} I & 0 \\ -GVU^{(1)} & I \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$.

Conversely, let us now assume the von Neumann regularity of $\begin{bmatrix} U & 0 \\ V & W \end{bmatrix}$. We remark the following equalities:

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & 0 \end{bmatrix} \\
= &\begin{bmatrix} U & 0 \\ V & W \end{bmatrix} \begin{bmatrix} I - U^{(1)}U & 0 \\ -W^{(1)}V\left(I - U^{(1)}U\right) & I - W^{(1)}W \end{bmatrix} \\
= &\begin{bmatrix} I - UU^{(1)} & 0 \\ -\left(I - WW^{(1)}\right) VU^{(1)} & I - WW^{(1)} \end{bmatrix} \begin{bmatrix} U & 0 \\ V & W \end{bmatrix} \\
= &\begin{bmatrix} 0 & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & 0 \end{bmatrix} \begin{bmatrix} I - U^{(1)}U & 0 \\ -W^{(1)}V\left(I - U^{(1)}U\right) & I - W^{(1)}W \end{bmatrix}.
\end{aligned}$$

As there exists a matrix $Z$ such that

$$\begin{bmatrix} U & 0 \\ V & W \end{bmatrix} Z \begin{bmatrix} U & 0 \\ V & W \end{bmatrix} = \begin{bmatrix} U & 0 \\ V & W \end{bmatrix},$$

then multiplying on the left by

$$\begin{bmatrix} I - UU^{(1)} & 0 \\ -\left(I - WW^{(1)}\right) VU^{(1)} & I - WW^{(1)} \end{bmatrix}$$

and on the right by

$$\begin{bmatrix} I - U^{(1)}U & 0 \\ -W^{(1)}V\left(I - U^{(1)}U\right) & I - W^{(1)}W \end{bmatrix},$$

we obtain

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & 0 \end{bmatrix} Z \begin{bmatrix} 0 & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & 0 \end{bmatrix} \\
= &\begin{bmatrix} 0 & 0 \\ \left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right) & 0 \end{bmatrix},
\end{aligned}$$

from which follows that $\left(I - WW^{(1)}\right) V \left(I - U^{(1)}U\right)$ is von Neumann regular. □

**Lemma 2.** *Let $U$ and $V$ be two matrices with the same number of rows. Then $\begin{bmatrix} U & V \end{bmatrix}$ is von Neumann regular iff $\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}$ is von Neumann regular.*

*Proof.* Assume that $\begin{bmatrix} U' \\ V' \end{bmatrix}$ is a von Neumann inverse of $\begin{bmatrix} U & V \end{bmatrix}$, where the blocks $U'$ and $V'$ are such that the products $UU'$ and $VV'$ exist. As

$$\begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} U' \\ V' \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix}$$

then $(UU' + VV')\,U = U$ and $(UU' + VV')\,V = V$. Simple calculations show that $\begin{bmatrix} U' & 0 \\ V' & 0 \end{bmatrix}$ is a von Neumann inverse of $\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}$.

Conversely, if $\begin{bmatrix} U' & * \\ V' & * \end{bmatrix}$ is a von Neumann inverse of $\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}$, then the equalities $(UU' + VV')\,U = U$ and $(UU' + VV')\,V = V$ hold. Thus,

$$\begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} U' \\ V' \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix}$$

and $\begin{bmatrix} U & V \end{bmatrix}$ is von Neumann regular. □

The next result generalizes Theorem 3.1 in [4].

**Theorem 3.** *Let $M$ be a matrix partitioned column-wise as*

$$M = \begin{bmatrix} U & V \end{bmatrix},$$

*where $U$ has a von Neumann inverse $U^{(1)}$. Then $M$ is von Neumann regular iff $S = \left(I - UU^{(1)}\right) V$ is von Neumann regular. In this case,*

$$\begin{bmatrix} U^{(1)}UU^{(1)} - U^{(1)}VS^{(1)}\left(I - UU^{(1)}\right) \\ S^{(1)}\left(I - UU^{(1)}\right) \end{bmatrix},$$

*is a von Neumann inverse of $M$, where $S^{(1)}$ is any von Neumann inverse of $S$.*

*Proof.* If $\begin{bmatrix} U & V \end{bmatrix}$ is von Neumann regular, the previous lemma implies that $\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}$ is von Neumann regular. Then $\begin{bmatrix} 0 & 0 \\ V & U \end{bmatrix}$ is von Neumann regular since

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ V & U \end{bmatrix}.$$

As $U$ is von Neumann regular, and using Theorem 1, it follows that $\begin{bmatrix} U & V \end{bmatrix}$ is von Neumann regular iff $S = \left(I - UU^{(1)}\right) V$ is von Neumann regular. If $S^{(1)}$ is a von Neumann inverse of $S$, then

$$\begin{bmatrix} I & 0 \\ -U^{(1)}V & I \end{bmatrix} \begin{bmatrix} 0 & S^{(1)}\left(I - UU^{(1)}\right) \\ 0 & U^{(1)}UU^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & S^{(1)}\left(I - UU^{(1)}\right) \\ 0 & \begin{array}{c} -U^{(1)}VS^{(1)}\left(I - UU^{(1)}\right) \\ +U^{(1)}UU^{(1)} \end{array} \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} 0 & 0 \\ V & U \end{bmatrix}$. Multiplying on the left and on the right by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, we have that

$$\begin{bmatrix} -U^{(1)}VS^{(1)}\left(I - UU^{(1)}\right) + U^{(1)}UU^{(1)} & 0 \\ S^{(1)}\left(I - UU^{(1)}\right) & 0 \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} U & V \\ 0 & 0 \end{bmatrix}$. By the proof of the lemma,

$$\begin{bmatrix} -U^{(1)}VS^{(1)}\left(I - UU^{(1)}\right) + U^{(1)}UU^{(1)} \\ S^{(1)}\left(I - UU^{(1)}\right) \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} U & V \end{bmatrix}$. □

**Theorem 4.** *Let $M$ be a matrix partitioned row-wise as*

$$M = \begin{bmatrix} U \\ V \end{bmatrix},$$

*where $U$ has a von Neumann inverse $U^{(1)}$. Then $M$ is von Neumann regular iff $S = V\left(I - U^{(1)}U\right)$ is von Neumann regular. In this case,*

$$\begin{bmatrix} U^{(1)}UU^{(1)} - \left(I - U^{(1)}U\right) S^{(1)}VU^{(1)} & \left(I - U^{(1)}U\right) S^{(1)} \end{bmatrix}$$

*is a von Neumann inverse of $M$, where $S^{(1)}$ is any von Neumann inverse of $S$.*

*Proof.* Similar to the proof of Theorem 3. □

We can apply now the results of the preceding theorems to the characterization of the von Neumann regularity of triangular Toeplitz matrices.

**Proposition 5.** *Let $a$ be von Neumann regular. Then*

$$\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} \text{ is von Neumann regular}$$

*iff* $\lambda = \left(1 - aa^{(1)}\right) a_1 \left(1 - a^{(1)}a\right)$ *is von Neumann regular.*

*Moreover, for any von Neumann inverse $\lambda^{(1)}$ of $\lambda$, the product*

$$\begin{bmatrix} 1 & 0 \\ -a^{(1)}a_1 & 1 \end{bmatrix} \begin{bmatrix} a^{(1)}aa^{(1)} & \left(1 - a^{(1)}a\right) \lambda^{(1)} \left(1 - aa^{(1)}\right) \\ 0 & a^{(1)}aa^{(1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\left(1 - aa^{(1)}\right) a_1 a^{(1)} & 1 \end{bmatrix}$$

*is a von Neumann inverse of* $\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$.

*Proof.* Clear from Theorem 1. □

**Proposition 6.** *Let $a$ be von Neumann regular. If* $\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$ *is von Neumann regular, which is equivalent with the von Neumann regularity of* $\lambda = \left(1 - aa^{(1)}\right) a_1 \left(1 - a^{(1)}a\right)$*, then the following conditions are equivalent:*

1. $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$ *is von Neumann regular.*
2. $\left(1 - aa^{(1)}\right) \begin{bmatrix} a_2 & a_1 \end{bmatrix} \left(I - \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{(1)} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}\right)$ *is von Neumann regular.*
3. $\left(I - \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{(1)}\right) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \left(1 - a^{(1)}a\right)$ *is von Neumann regular.*
4. $\left(1 - \lambda\lambda^{(1)}\right) \kappa \left(1 - \lambda^{(1)}\lambda\right)$ *is von Neumann regular, with* $\kappa = \left(1 - aa^{(1)}\right) \left(a_2 - a_1 a^{(1)} a_1\right) \left(1 - a^{(1)}a\right)$.

*Proof.* (1) ⇔ (2) ⇔ (3) follow from Theorem 1.

(1) ⇔ (4) Firstly, we assume the von Neumann regularity of $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$, i.e., there exists a matrix $T$ such that

$$\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} T \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}. \tag{2}$$

Let $\kappa = \left(1 - aa^{(1)}\right)\left(a_2 - a_1 a^{(1)} a_1\right)\left(1 - a^{(1)} a\right)$ and $\lambda = \left(1 - aa^{(1)}\right) a_1 \left(1 - a^{(1)} a\right)$. We remark the following equalities:

$$\begin{bmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \kappa & \lambda & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a^{(1)}a_1 & 1 & 0 \\ \left(a^{(1)}a_1\right)^2 - a^{(1)}a_2 & -a^{(1)}a_1 & 1 \end{bmatrix} \left(1 - a^{(1)}a\right)$$

$$= \left(1 - aa^{(1)}\right) \begin{bmatrix} 1 & 0 & 0 \\ -a_1 a^{(1)} & 1 & 0 \\ \left(a_1 a^{(1)}\right)^2 - a_2 a^{(1)} & -a_1 a^{(1)} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$$

$$= \left(1 - aa^{(1)}\right) \begin{bmatrix} 1 & 0 & 0 \\ -a_1 a^{(1)} & 1 & 0 \\ \left(a_1 a^{(1)}\right)^2 - a_2 a^{(1)} & -a_1 a^{(1)} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ -a^{(1)}a_1 & 1 & 0 \\ \left(a^{(1)}a_1\right)^2 - a^{(1)}a_2 & -a^{(1)}a_1 & 1 \end{bmatrix} \left(1 - a^{(1)}a\right).$$

Multiplying (2) on the right by $\begin{bmatrix} 1 & 0 & 0 \\ -a^{(1)}a_1 & 1 & 0 \\ \left(a^{(1)}a_1\right)^2 - a^{(1)}a_2 & -a^{(1)}a_1 & 1 \end{bmatrix} \left(1 - a^{(1)}a\right)$ and on the left by $\left(1 - aa^{(1)}\right) \begin{bmatrix} 1 & 0 & 0 \\ -a_1 a^{(1)} & 1 & 0 \\ \left(a_1 a^{(1)}\right)^2 - a_2 a^{(1)} & -a_1 a^{(1)} & 1 \end{bmatrix}$, we obtain

$$\begin{bmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \kappa & \lambda & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \kappa & \lambda & 0 \end{bmatrix} T \begin{bmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \kappa & \lambda & 0 \end{bmatrix},$$

and therefore $\begin{bmatrix} \lambda & 0 \\ \kappa & \lambda \end{bmatrix}$ is von Neumann regular. As $\lambda$ is von Neumann regular, this is equivalent to $\left(1 - \lambda\lambda^{(1)}\right) \kappa \left(1 - \lambda^{(1)}\lambda\right)$ be von Neumann regular. Let us now assume that $\left(1 - \lambda\lambda^{(1)}\right) \kappa \left(1 - \lambda^{(1)}\lambda\right)$ is von Neumann regular with $\kappa = \left(1 - aa^{(1)}\right)\left(a_2 - a_1 a^{(1)} a_1\right)\left(1 - a^{(1)}a\right)$. Setting

$$Y = \begin{bmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \kappa & \lambda & 0 \end{bmatrix},$$

$$\begin{aligned}
X &= \begin{bmatrix} a & 0 & 0 \\ -aa^{(1)}a_1a^{(1)}a & a & 0 \\ -aa^{(1)}\left(a_2 - a_1a^{(1)}a_1\right)a^{(1)}a & -aa^{(1)}a_1a^{(1)}a & a \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -aa^{(1)}a_1a^{(1)} & 1 & 0 \\ -aa^{(1)}\left(a_2 - a_1a^{(1)}a_1\right)a^{(1)} & -aa^{(1)}a_1a^{(1)} & 1 \end{bmatrix} a,
\end{aligned}$$

then both $X$ and $Y$ are von Neumann regular. Let $e = aa^{(1)}$ and $f = a^{(1)}a$. Then

$$\begin{aligned}
Xf &= X = eX, \\
Y(1-f) &= Y = (1-e)Y.
\end{aligned}$$

Therefore, for any $X^{(1)}, Y^{(1)}$,

$$S = fX^{(1)}e + (1-f)Y^{(1)}(1-e),$$

is a von Neumann inverse of $X + Y$ since

$$\begin{aligned}
(eXf + (1-e)Y(1-f))S(eXf + (1-e)Y(1-f)) &= eXf + (1-e)Y(1-f) \\
&= X + Y
\end{aligned}$$

which equals

$$\begin{aligned}
&\begin{bmatrix} a & 0 & 0 \\ a_1 - aa^{(1)}a_1 - a_1aa^{(1)} & a & 0 \\ \left(a_2 - a_1a^{(1)}a_1\right) - aa^{(1)}\left(a_2 - a_1a^{(1)}a_1\right) - \\ - \left(a_2 - a_1a^{(1)}a_1\right)a^{(1)}a & a_1 - aa^{(1)}a_1 - a_1aa^{(1)} & a \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -a_1a^{(1)} & 1 & 0 \\ \left(a_1a^{(1)}\right)^2 - a_2a^{(1)} & -a_1a^{(1)} & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \times \qquad (3) \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 \\ -a^{(1)}a_1 & 1 & 0 \\ \left(a^{(1)}a_1\right)^2 - a^{(1)}a_2 & -a^{(1)}a_1 & 1 \end{bmatrix}
\end{aligned}$$

from which follows that $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$ is von Neumann regular.

In order to give the expression of a von Neumann inverse of $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$, we set $A = \begin{bmatrix} \lambda & 0 \\ \kappa & \lambda \end{bmatrix}$ which is von Neumann invertible. Then

$$Y^{(1)} = \begin{bmatrix} 0 & A^{(1)} \\ 0 & 0 \end{bmatrix}$$

is a von Neumann inverse of $Y$ and

$$\begin{aligned} X^{(1)} &= a^{(1)} \begin{bmatrix} 1 & 0 & 0 \\ -aa^{(1)}a_1a^{(1)} & 1 & 0 \\ -aa^{(1)}\left(a_2 - a_1a^{(1)}a_1\right)a^{(1)} & -aa^{(1)}a_1a^{(1)} & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} a^{(1)} & 0 & 0 \\ a^{(1)}aa^{(1)}a_1a^{(1)} & a^{(1)} & 0 \\ a^{(1)}aa^{(1)}a_2a^{(1)} & a^{(1)}aa^{(1)}a_1a^{(1)} & a^{(1)} \end{bmatrix} \end{aligned}$$

is a von Neumann inverse of $X$. We recall that

$$S = fX^{(1)}e + (1-f)\,Y^{(1)}\,(1-e)$$

which equals

$$\begin{bmatrix} \tilde{a} & 0 & 0 \\ \tilde{a}a_1\tilde{a} & \tilde{a} & 0 \\ \tilde{a}a_2\tilde{a} & \tilde{a}a_1\tilde{a} & \tilde{a} \end{bmatrix} + \begin{bmatrix} 0 & \left(1-a^{(1)}a\right)A^{(1)}\left(1-aa^{(1)}\right) \\ 0 & 0 \end{bmatrix}$$

is a von Neumann inverse of $X+Y$, where $\tilde{a} = a^{(1)}aa^{(1)}$, and therefore, using (3),

$$\begin{bmatrix} 1 & 0 & 0 \\ -a^{(1)}a_1 & 1 & 0 \\ \left(a^{(1)}a_1\right)^2 - a^{(1)}a_2 & -a^{(1)}a_1 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 0 & 0 \\ -a_1a^{(1)} & 1 & 0 \\ \left(a_1a^{(1)}\right)^2 - a_2a^{(1)} & -a_1a^{(1)} & 1 \end{bmatrix}$$

is a von Neumann inverse of $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$. $\square$

**Remark.**

Concerning 2, some calculations establish that the matrix

$$\left(1-aa^{(1)}\right)\begin{bmatrix} a_2 & a_1 \end{bmatrix}\left(I - \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{(1)} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}\right)$$

is of the form $\left[\begin{array}{cc} * & \left(1-aa^{(1)}\right) a_1 \left(1-a^{(1)}a\right) \end{array}\right]$, and the regularity of this class of matrices can be characterized using Theorem 3.

**Corollary 7.** *If* $\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$ *is von Neumann regular then*

$$\begin{bmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & a_1 & a & 0 \\ a_3 & a_2 & a_1 & a \end{bmatrix} \text{ is von Neumann regular}$$

*iff*

$$\left(I - \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{(1)}\right) \begin{bmatrix} a_2 & a_1 \\ a_3 & a_2 \end{bmatrix} \left(I - \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{(1)} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}\right)$$

*is von Neumann regular.*

*Proof.* Clear from Proposition 5. □

**Corollary 8.** *If* $a$ *and* $\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}$ *are von Neumann regular, then the following conditions are equivalent:*

1. $\begin{bmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & a_1 & a & 0 \\ a_3 & a_2 & a_1 & a \end{bmatrix}$ *is von Neumann regular.*

2. $\left(I - \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}^{(1)}\right) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \left(1 - a^{(1)}a\right)$ *is von Neumann regular.*

3. $\left(1 - aa^{(1)}\right) \left[\begin{array}{ccc} a_3 & a_2 & a_1 \end{array}\right] \left(I - \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}^{(1)} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}\right)$ *is von Neumann regular.*

*Proof.* Clear from Theorem 1. □

## 3 The commutative case

In this section, the matrices considered are over a commutative ring. For any matrix over a commutative ring $R$, there is the following theorem by Bapat-Prasad-Rao, see [3], that reduces the problem of the von Neumann regularity of the matrix to a problem in the commutative base ring $R$.

**Theorem 9.** *Let $A$ be an $m \times n$ matrix of rank $r$ over $R$. Then $A$ is regular if and only if there are Rao-regular matrices $A_k$ each of which is either a zero matrix or of rank $k$ with orthogonal Rao-idempotents, for $k = 1, 2, \ldots, r$, such that*

$$A = A_r + A_{r-1} + \cdots + A_1.$$

*Such a decomposition is unique.*

This [B-P-R] result should be used for triangular Toeplitz matrices over commutative rings if there is no additional information on the components. However, here we want to show that if there is more information on the components of the triangular Toeplitz matrix we can derive some other results.

**Proposition 10.** *Let $R$ be a commutative ring and $T_k$ a $k \times k$ lower triangular Toeplitz matrix over $R$. If all components of $T_k$ are von Neumann regular in $R$ then $T_k$ is von Neumann regular in $Mat(k, R)$. Moreover, there is a simple reduction formula to compute a special von Neumann inverse $T_k^{\wedge}$ of $T_k$.*

*Proof.* We first remark that if $e = e^2$ in $R$ and $D$ and $N$ are von Neumann regular in $Mat(k, R)$ then $De + N(1-e)$ is von Neumann in $Mat(k, R)$ and $D^{(1)}e + N^{(1)}(1-e)$ are all von Neumann inverses of $De + N(1-e)$.

Conversely, if $De + N(1-e)$ is von Neumann regular in $Mat(k, R)$ then $De$ and $N(1-e)$ are von Neumann regular in $Mat(k, R)$. Indeed, it follows from

$$[De + N(1-e)]\, X\, [De + N(1-e)] = De + N(1-e)$$

that $DeXDe = De$ and $N(1-e)XN(1-e) = N(1-e)$ after multiplying with $e$ and $1-e$, respectively.

Now, if

$$P_0 = P \begin{pmatrix} 1 \\ a_1 a^{\#} \\ a_2 a^{\#} \\ \vdots \\ a_{k-1} a^{\#} \end{pmatrix} \text{ and } P_i = P \begin{pmatrix} 1 \\ a_{i+1} a_i^{\#} \\ \vdots \\ a_{k-1} a_i^{\#} \end{pmatrix}, i = 1, \ldots, k-2.$$

are lower triangular and invertible Toeplitz matrices then the given lower triangular Toeplitz matrix

$$T\begin{pmatrix} a \\ a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix} = P\begin{pmatrix} 1 \\ a_1 a^{\#} \\ a_2 a^{\#} \\ \vdots \\ a_{k-1} a^{\#} \end{pmatrix} \cdot T\begin{pmatrix} a \\ a_1\left(1-aa^{\#}\right) \\ a_2\left(1-aa^{\#}\right) \\ \vdots \\ a_{k-1}\left(1-aa^{\#}\right) \end{pmatrix}$$

$$= \quad P\begin{pmatrix} 1 \\ a_1 a^{\#} \\ a_2 a^{\#} \\ \vdots \\ a_{k-1} a^{\#} \end{pmatrix} \left( \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} aa^{\#} + \left[\begin{array}{ccc|c} 0 & & 0 & 0 \\ \hline a_1 & & & 0 \\ \vdots & \ddots & & \vdots \\ a_{k-1} & \cdots & a_1 & 0 \end{array}\right] \left(1-aa^{\#}\right) \right)$$

and the reduction formula is

$$T\begin{pmatrix} a \\ a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix}^{\wedge} = a^{\#}P_0^{-1} + \left(1-aa^{\#}\right) \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & T\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix}^{\wedge} \\ \hline 0 & 0 \ \cdots \ 0 \end{array}\right] P_0^{-1}$$

$$= \quad a^{\#}P_0^{-1} +$$

$$\left(1-aa^{\#}\right) \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & P_1^{-1}a_1^{\#} + \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & T\begin{pmatrix} a_2 \\ \vdots \\ a_{k-1} \end{pmatrix}^{\wedge} \\ \hline 0 & 0 \ \cdots \ 0 \end{array}\right] P_1^{-1}\left(1-a_1a_1^{\#}\right) \\ \hline 0 & 0 \ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \ 0 \end{array}\right] P_0^{-1}. \ \square$$

**Corollary 11.** *Under the conditions of Proposition 10,*

*1.*

$$\begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}^{\wedge} = \begin{bmatrix} a^{\#} & a_1^{\#}\left(1-aa^{\#}\right) \\ -a_1\left(a^{\#}\right)^2 & a^{\#} \end{bmatrix}$$

*2.*

$$\begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix}^{\wedge} =$$

$$\begin{bmatrix} a^{\#} & a_1^{\#}\left(1-aa^{\#}\right) & a_2^{\#}\left(1-a_1a_1^{\#}\right)\left(1-aa^{\#}\right) \\ -a_1\left(a^{\#}\right)^2 & a^{\#} & a_1^{\#}\left(1-aa^{\#}\right) \\ a_1^2\left(a^{\#}\right)^3-a_2\left(a^{\#}\right)^2 & -a_1\left(a^{\#}\right)^2 & a^{\#} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_2\left(a_1^{\#}\right)^2\left(1-aa^{\#}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

*3.*

$$\begin{bmatrix} a & 0 & 0 & 0 \\ a_1 & a & 0 & 0 \\ a_2 & a_1 & a & 0 \\ a_3 & a_2 & a_1 & a \end{bmatrix}^{\wedge} =$$

$$\begin{bmatrix} a^{\#} & a_1^{\#}e & a_2^{\#}e_1e & a_3^{\#}e_2e_1e \\ -a_1a^{\#\,2} & a^{\#} & a_1^{\#}e & a_2^{\#}e_1e \\ a_1^2a^{\#\,3}-a_2a^{\#\,2} & -a_1a^{\#\,2} & a^{\#} & a_1^{\#}e \\ 2a_1a_2a^{\#\,3}-a_3a^{\#\,2}-a_1^3a^{\#\,4} & a_1^2a^{\#\,3}-a_2a^{\#\,2} & -a_1a^{\#\,2} & a^{\#} \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -a_2\left(a_1^{\#}\right)^2e & -a_3a_2^{\#\,2}e_1e & 0 \\ 0 & \left(a_2^2a_1^{\#\,3}-a_3a_1^{\#\,2}\right)e-a_2a^{\#\,2} & -a_2\left(a_1^{\#}\right)^2e & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

*where* $e=1-aa^{\#}, e_1=1-a_1a_1^{\#}, e_2=1-a_2a_2^{\#}$.

Remark that in these three cases the special von Neumann inverse $T_k^{\wedge}$ of $T_k$ is the sum of a Toeplitz matrix and a zero bordered Toeplitz matrix.

**(II)**

If $A$ is a square matrix of full rank then $A$ is von Neumann regular iff $\det A$ is von Neumann regular and the matrix $\left(1-\det A\det^{(1)}A\right)A$ is von

Neumann regular. Indeed, $\det A \det^{(1)} A$ is a Rao-idempotent, $\left(\det A \det^{(1)} A\right) A$ is Rao-regular and there is the Pierce decomposition

$$A = \left(\det A \det^{(1)} A\right) A \oplus \left(1 - \det A \det^{(1)} A\right) A.$$

Also, $\det T_k = a^k$ can be von Neumann regular without $a$ to be von Neumann regular and in this situation the Pierce decomposition is not very helpful.

However, if we suppose $a$ to be von Neumann regular then we have the following:

(a) $n = 2$ : since $a^{(1)}$ exists, $a^2 \neq 0$,

$$\begin{aligned} T_2 &= \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} \\ &= a^2 \left(a^2\right)^{(1)} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} + \left(1 - a^2 \left(a^2\right)^{(1)}\right) \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} \\ &= aa^{(1)} \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix} + \left(1 - aa^{(1)}\right) \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix}. \end{aligned}$$

Therefore $T_2$ is von Neumann regular iff $s = \left(1 - aa^{(1)}\right) a_1$ is von Neumann regular. Clearly, the orthogonal idempotents are here $aa^{(1)}$ and $ss^{(1)}$.

(b) $n = 3$ : since $a^{(1)}$ exists, $a^3 \neq 0$.

$$\begin{aligned} T_3 &= \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \\ &= a^3 a^{3\,(1)} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} + \left(1 - a^3 a^{3\,(1)}\right) \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} \\ &= aa^{(1)} \begin{bmatrix} a & 0 & 0 \\ a_1 & a & 0 \\ a_2 & a_1 & a \end{bmatrix} + \left(1 - aa^{(1)}\right) \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_1 & 0 \end{bmatrix}. \end{aligned}$$

Therefore $T_3$ is von Neumann regular iff the $2 \times 2$ lower triangular Toeplitz matrix

$$T \begin{pmatrix} \left(1 - aa^{(1)}\right) a_1 \\ \left(1 - aa^{(1)}\right) a_2 \end{pmatrix}$$

is von Neumann regular which can be handled if $a_1$ is von Neumann regular.

(c) $n = 4, 5, \ldots$, in a similar way.

# 4 Application

Consider the lower companion matrix

$$L = \begin{bmatrix} 0 & a \\ I_{n-1} & \underline{k} \end{bmatrix}$$

with $\underline{k}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}$ over an arbitrary ring. In [6], the group inverse, i.e. the Drazin index 1 case, of $L$ has been considered. It follows from [5] and the results obtained here that

**Proposition 12.** *The following are equivalent:*

1. *$L$ has Drazin index 2,*
2. *$L$ does not have Drazin index 1, $T_2 = \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$ is von Neumann regular and*
$$U = L^4 \left(L^2\right)^{(1)} + I - L^2 \left(L^2\right)^{(1)}$$
*is invertible,*
3. *$L$ does not have Drazin index 1, $T_2 = \begin{bmatrix} a & 0 \\ a_1 & a \end{bmatrix}$ is von Neumann regular and*
$$V = \left(L^2\right)^{(1)} L^4 + I - \left(L^2\right)^{(1)} L^2$$
*is invertible,*

*and, if $a$ is von Neumann regular and $L$ does not have Drazin index 1, also equivalent with*

4. *$\left(1 - aa^{(1)}\right) a_1 \left(1 - a^{(1)}a\right)$ is von Neumann regular and $U$ is invertible,*
5. *$\left(1 - aa^{(1)}\right) a_1 \left(1 - a^{(1)}a\right)$ is von Neumann regular and $V$ is invertible.*

Indexes of higher order of $L$ can be also handled in a similar way.

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