THE TEACHING OF THE CONCEPT OF TANGENT LINE USING ORIGINAL SOURCES

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This paper reports a mathematics experiment conducted in a Portuguese secondary school, with 11^th^ and 12^th^ grades pupils (pre-university). We have used original historical sources for teaching the mathematical concept of tangent line to a curve as well as its connection with the concept of derivative. Emphasis will be given to the conception of the designed module, to the analysis of the pupils' answers and to the pupils' final outcomes. We will infer both benefits and disadvantages of using such activities into the teaching of mathematics.

Keywords: Tangent line; derivative; historical sources; history of mathematics; mathematics education.

INTRODUCTION

The use of history of mathematics in mathematics teaching is increasingly advocated by experts in mathematics, history of mathematics and mathematics education. Many reasons have been presented to justify why it is advantageous to use history of mathematics in mathematics teaching, within those we emphasise the humanization of mathematics, the motivational tool, the dynamical vision of mathematics and its evolution and/or the support for development of an internal image of mathematical concepts (Fauvel&Maanen, 2000; Jankvist, 2009).

Nevertheless some disadvantages have also been identified (Siu, 2006, p. 268-269) and on these we may highlight the time consuming problem, the pupils' concerns about the benefits of this use and the lack of teacher training.

In Portugal we may speak, at least theoretically, of a tradition in the use of history of mathematics in mathematics teaching, which dates back to 1772 when a Faculty of Mathematics was created at the newly reformed University of Coimbra (Mota & Ralha, 2011).

According to this long lasting tradition we have developed an experiment, as part of our PhD research on the historical evolution of the concept of tangent line and its teaching, which aimed at examining the use of original historical sources in teaching mathematical concepts.

THE TEACHING OF THE CONCEPT OF TANGENT LINE

The concept of tangent line to a curve is a concept that becomes familiar to most pupils during their compulsory education but, as Vinner showed (1991, p. 65-81), the pupils' concept definition and concept image of tangent line are not usually the same and this does not allow learners to correctly work with the concept. To avoid this situation, Tall (1990, p. 58) defended that students should face enriching learning experiences to help them construct concept images that are as close as possible to the concept definition.

As Tzanakis & Thomaidis (2011, p.1654-1655) showed history of mathematics can be used in mathematics teaching to, among others, uncover/unveil concepts, as a bridge between mathematics and other disciplines, to get insights into concepts by looking from a different point of view and to compare old and modern. Within our experiment we have tried to achieve such goals by using original sources. Our guidelines were: the works of Euclid, Archimedes and Apollonius, the work of Roberval, Descartes, Fermat, Newton and Leibniz, as well as the notion presented by the Portuguese mathematician José Anastácio da Cunha in the 18th century.

The concept of tangent line in the Portuguese curriculum and textbooks

In Portugal, on one hand, the curriculum for mathematics advocates explicitly (in every grade) the use of the history of mathematics. On the other hand, for our concept of tangent line to a curve we find it approached in the relationship between the derivative of a function at a given point and the slope of the tangent line to the graph of the function at that point. We may, for example, read [2]:

> The use of historical examples or references to the evolution of mathematical concepts will help pupils to appreciate the contribution of mathematics to Humanity problem understanding and problem solving. Some suggested situations: polynomials according to Pedro Nunes, history of differential calculus, history of complex numbers. (Departamento Ensino Secundário, 2001, p. 20)

Curriculum directives are, as expected, vague but, in Portugal, most textbooks and teachers follow only the so called “illumination approaches” (Jankvist, 2009, p. 245-246). In our opinion, Portuguese textbooks [3] aim at informative purposes more than formative ones, illustrated with the use of mathematicians pictures and very scarce references (sometimes even misleading) to the work of these mathematicians. Therefore such historical facts may often become more distracting than informative.

Methodology and data collection

Our experiment took place in two different moments with different approaches and was conditioned by the school executive board directives: we were asked to use the fewest possible classes and we were not given permission to the collection of images or for pupils’ interviews. Thus, data collection was carried out exclusively from the pupils’ written answers as well as our field notes but we believe that our results are still valid. In fact, having developed the experience in usual classroom environment we believe that we have gotten more genuine reviews and reactions from pupils.

In a first phase, involving an 11th grade class of twenty-one pupils, we have used “the modules approach” (Jankvist, 2009, p. 246) with a “historical package” that had the duration of three-class periods of ninety minutes each. This choice was due to the fact that we wanted to focus on a small topic, with strong ties to the curriculum, and we had limited class periods available.

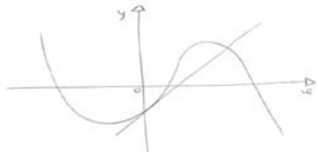
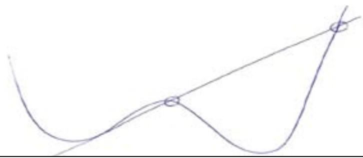
Table 1: Structure of the historical package.

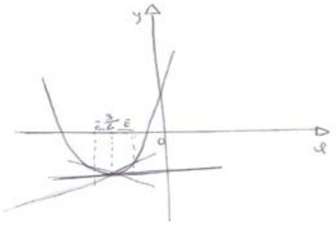
Author	Original Source used [4]	Question/task presented to pupils
Euclid	*Elements III*, Definition 2 (Heath, 1956, p. 1) and *Elements III*, 16 (corollary) (Heath, 1956, p. 39)	**1.** Is this definition suitable to all curves? Justify. **2.** Present a method to draw the tangent line to a circle at a given point.
Archimedes	*On Spirals*, Definition 1 (Heath, 2002, p. 165)	Is Euclid's definition suitable for Archimedes' spiral? In case of a negative answer present an alternative definition for a tangent line to a curve.
Roberval	*Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes curves* (Roberval, 1736, p. 22-23) *Traité des indivisibles* (Roberval, 1736, p. 209-212)	**1.** Use Roberval's method to draw the tangent line to a cycloid at a given point. Make a small report about the taken steps and justify it with transcriptions from the original text. **2.** Using the concepts presented by Roberval and your knowledge in the theory of movement establish connections between the concepts of movement, velocity and tangent line.
Fermat	*Methodus ad Disquirendam Maximam et Minimam* (Fermat, 1891, p. 121)	**1.** Using Fermat's method, determine the extremes of a 2^{nd} degree polynomial function. **2.** What is, geometrically, the interpretation of Fermat's method?
Leibniz	*Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, qua nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro illi Calculi Genus* (Leibniz, 1684/1983, p. 5-7)	**1.** Using an image of the differential triangle, establish a relationship between the slope of a tangent line to a curve at a given point and the function derivative at that point. **2.** Establish a relationship between function monotony and the sign of the derivative. **3.** Using the established relationship present a method to determine the extreme of a function.

Both authors and tasks were chosen to represent different approaches to the concept of tangent line as well as its relation with other concepts, namely the Greek "touch without cut" approach, the relationship with physics and, in particular, with movement, the limit position of the secant approach and the relationship with the concept of derivative.

Our pupils worked in five small heterogeneous groups but their answers, in the different groups, were quite similar. Hereby we present the most relevant data:

Table 2: Data collection of the module-based approaches.

Question/task	Collected Data
Euclid 1	All groups answered "no" and presented a graphic of a curve where the tangent line at a point cuts the curve at another point to justify their answer (below we present two examples of pupils' answers).  
Euclid 2	All groups presented a correct method to draw a tangent line to a circle, for example: - *Trace the radius through the given point;* - *Construct the perpendicular line to the radius through the given point.*
Archimedes	All groups answered "no" and presented an alternative definition. Bellow we present two examples of pupils' answers: ***Example 1:*** *Tangent line to a curve is a straight line that cuts the curve in only one point of the curve near the considered point.* ***Example 2:*** *A straight line is tangent to a curve if it touches the curve without cutting it at that point.* In fact, all the presented definitions had as common idea the fact that the tangent line touches the curve at the contact point and no other in a neighbourhood.
Roberval	The pupils presented many difficulties in the implementation of the proposed tasks, despite the examples presented in the 'historical package'. They just managed to perform the task after a few explanations of the teacher, mainly one related to the description of curves through movement.
Fermat 1	All groups, after some discussion and despite presenting some doubts about the method, were able to follow the method's directives and correctly determine the minimum of the function $f(x) = x^2 + 3x + 2$.
Fermat 2	Only one group was able to present the geometrical interpretation of Fermat's method, which was the following:

	 *The tangent line is horizontal.* To allow all pupils to fully understand the method, the teacher had to thoroughly explain it.
Leibniz	All groups recognized, by using similarity of triangles, that the slope of the tangent line at a given point is the derivative of the function at that point. All groups establish the correct relationship between the function monotony and the sign of the derivative and used that relationship to present a method that allowed the determination of extremes of a function.

The second phase of the experiment took place almost one year later and involved pupils of the 12th grade, ten of them (group A) that had participated in the first part of the experiment and twelve (group B) that were participating for the first time in the experiment. In this phase, it was used "the history-based approaches" (Jankvist, 2009, p. 246-247) by confronting the pupils with different definitions of the concept of tangent line and propositions about this concept, presented in a historical order, that were discussed and analysed with the pupils aiming at achieving a clearer and more complete concept image and concept definition of tangent line. We have chosen to work with the same mathematicians used in the first phase of the experiment, adding a new one: the Portuguese mathematician José Anastácio da Cunha (1744-1787) aiming at dealing with yet a different point of view on the concept alongside with a national/cultural component. The definitions/propositions analyzed are the following:

Table 3: Structure of the history-based approaches.

Author	Definition / Proposition
Euclid	*Elements III,* Definition 2: A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle. (Heath, 1956, p. 1)
Archimedes	*On spirals,* 13: If a straight line touch the spiral, it will touch it in one point only. (Heath, 2002, p. 167)
Roberval	*Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes curves*, Axiom or Invention Principle: The direction of the movement of one point which describes a curve is the tangent line of the curve at the position that the point occupies. (Roberval, 1736, p. 22)

Fermat [5]	Tangent line is the limit position of the secant when the two points of the curve tend to meet. (Eves, 1992, p. 391)
Leibniz	*Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, qua nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro illi Calculi Genus*: drawing the tangent line is to draw the straight line that joins two points on the curve that are distant from each other an infinitely small distance, or the side of a polygon of infinite angles that we consider equivalent to the curve. (Leibniz, 1684/1983, p. 7)
José Anastácio da Cunha	*Princípios Matemáticos II*, Definition III: If the sides of an angle meet at the vertex such as it is not possible to draw two straight lines between them it is said that the sides of the angle are tangent to each other. (Cunha, 1790, p. 13)

First the pupils were asked to write their comments and opinions on the definition validity, evolution and/or applicability and afterwards to debate their writings in the class. The most receptive and participative pupils were the ones who took part in the first phase of the experiment and almost all comments, in the class debate, were lead by those pupils. Hereby we present the most relevant data:

Table 4: Data collection of the history-based approaches.

Author	**Comments (group A)**	**Comments (group B)**	**Comments (class debate)**
Euclid	This definition is not valid for all curves.	This definition is correct and is the one generally used.	Group A tried to convince group B that this definition shouldn't be used today since it is not valid for all curves by using examples of curves where the tangent line cuts the curve.
Archimedes	Reference to the uniqueness of the tangency point that is not valid if the curve is a straight line.	Accepted that this property is valid for all curves.	Group A presented their example to group B.
Roberval	Pupils were able to establish, even though with some difficulties, the	Pupils weren't able to understand the relationship between the presented	An explanation on the part of the teacher was necessary in order to make the relationship between mathematics and

	connection with physics.	definition and their concept image of tangent line.	physics evident to all pupils.
Fermat	Recognized this as the generally used definition of tangent line.	Established a relationship with the concept of derivate.	The two groups' approaches were considered complementary.
Leibniz	Pupils referred to the last part of the definition as having no connection with the rest and so it is unnecessary.	Established a relationship with the concept of derivate.	Recognized the similarity with the treatment given by Fermat and discarded the last part of the definition.
José Anastácio da Cunha	Stated the difficulty of the definition and weren't able to understand the connection of the definition with the geometrical treatment of the functions they are used to do.		The teacher explained the presented definition, presenting Anastácio da Cunha's use of the definition.

In the end pupils were asked to present their own definition of a tangent line to a curve and they have chosen the definition referred as Fermat's.

CONCLUSIONS

The use of original sources to teach the concept of tangent line has proved to be not only important for us to realize many positive aspects, but also to identify some obstacles to overcome.

The first major difficulty was faced when constructing the historical package. The choice of the sources to be used, the translation of the original sources into Portuguese that have raised problems of terminology and notation, and the choice of the questions/tasks made the construction of the module to be time-consuming and scientifically intense. It is only possible to develop this type of work if the teacher has some teaching experience and a profound knowledge of the concept/subject to teach together with historical interest and practice. We are sure that we might never develop such competence if it was not for our own process of finding, disclosing and intense debate on many details on the studied sources. This calls our attention to the need of introducing/consolidating history of mathematics in teachers' training as well as the need of a real collaborative work among colleagues that are mathematics teachers, to share materials, experiences, ideas that will make less time-consuming the preparation of this sort of activities.

During the three-class periods where the history package was used, it was hard to get the commitment and stimulate pupils' interest in this type of activity. The first arguments used by the pupils were that mathematics is not history, they do not attend history classes anymore and that they do not like history. Pupils were not, at all, aware of the importance of the historical development of mathematics for understanding mathematics itself, so they discard this sort of activities even before they begin working with them. To overcome this pupils' arguments/attitude history of mathematics might be introduced into mathematics teaching early and often and history classes and mathematics classes might even become, in some aspects, coordinated disciplines.

After the first refusal and the beginning of the task, pupils faced the complexity of the presented texts. The texts had been previously prepared by ourselves but, even so, since Portuguese pupils are not used to read mathematical texts, it was difficult for them to understand the presented historical material. This barrier might only be overcome if pupils are more often faced with reading and analysing mathematical texts. In fact, if we decide to use this historical package again, it might be necessary to prepare its application, by presenting earlier to pupils small excerpts of original sources so that students gradually become familiar with that sort of texts.

Pupils also presented difficulties with the establishment of connections between mathematics and physics. Although Portuguese mathematics curriculum explicitly defends the interdisciplinarity of mathematics and physics, that is rarely a fact in teaching practices. This experiment shows us that a true collaborative work between the teachers of the two disciplines is necessary, for example, for pupils to realise the importance of mathematics to explain physic phenomena.

Another difficulty was related to the purpose of the task. In Portugal, history of mathematics is not a topic evaluated in final examinations, so pupils of these levels (prior to being selected for universities) tend to discard anything that will not be part of national examinations and to concentrate only in "model questions" which might prepare them to the final exam. Pupils showed only interest in the questions concerning to Leibniz's work, since it seemed to them that this might be useful for their exam preparation and they were not able to see any more immediate advantages in any other part of the work.

After using the historical package with the pupils we realised that some changes are necessary. The first one is the awareness that it is necessary more time for pupils to fulfil the tasks presented. We also noticed that it is necessary to introduce more information concerning Roberval or Fermat's method for pupils can perform the tasks and understand it. The use of secondary sources (such as the work of Eves concerning Fermat's definition of a tangent line or the differential triangle, in the Leibniz method) may seem awkward but justified by the need of additional information that would help pupils fully understand the original sources.

The real advantages for pupils with the use of original sources came to be fully acknowledged by them in the second year of the experiment. When asked to comment the definitions/propositions presented, the pupils who took part in the first phase of the experiment were the most receptive and participative ones. They also presented less doubts and less difficulties. This shows that, even though they had some initial concerns, the work made in the previous year developed the inwardness of concepts that gave them the knowledge to perform easily the 2nd phase tasks. At the end it was clear to all participants that the pupils involved in the two years of the experiment had a more solid formation than the others, showing the utility of this sort of tasks to develop a profound and more complete knowledge of the concept. When faced with counterintuitive examples such as the tangent line to a straight line or the tangent line to a curve at an inflection point, pupils showed little difficulty and did not present the problems that are often reported (Vinner, 1991; Tall, 1990) concerning to the concept of tangent line.

It is our conviction that the contact with the historical evolution of the concept allowed our pupils to fully understand it: they internalized the concept, they constructed clearer concept images, they achieved a correct concept definition in their minds. They even performed better in their final examinations.

NOTES

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2. All translations into English were made by the authors.
3. We have analyzed the textbooks Matemática A 12º, Novo Espaço 12 and Xeqmat 12 which are the most used in the country.
4. All texts were translated into Portuguese by the authors.
5. Although Fermat does not present a definition of tangent line in his work he has presented a method from which Eves has composed the definition used in our experiment.

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