The Moore-Penrose inverse of 2×2 matrices over a certain $*$-regular ring

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Abstract

In this paper, we study representations of the Moore-Penrose inverse of a 2×2 matrix M over a $*$-regular ring with two term star-cancellation. As applications, some necessary and sufficient conditions for the Moore-Penrose inverse of M to have different types are given.

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Moore-Penrose inverse, $*$-regular ring, two term star-cancellation
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1. Introduction

Representations for the Moore-Penrose inverse (abbr. MP-inverse) of matrices over various settings attract wide interest from many scholars. For instance, Cline [1, 2] derived the representations for the MP-inverse of a partitioned complex matrix. Hung and Markham [7, 8] obtained the explicit formula for the MP-inverse of an $m \times n$ partitioned matrix. Recently, Hartwig and Patrício [6] obtained new expressions for the MP-inverse of the matrix $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$ over a $*$-regular ring, extending some well known results for complex matrices.

This article is motivated by the papers [5, 6]. We investigate the MP-inverse of $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ over a $*$-regular ring satisfying some additional con-

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ditions. As applications, some necessary and sufficient conditions for the matrix M to have various types are obtained. Some results in [5, 6] are generalized.

Let R be a unital $*$-ring, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$. By $R_{m\times n}$ we denote the set of $m \times n$ matrices over R. The involution on R induces a map $R_{m\times n} \to R_{n\times m}$, $(a_{ij}) \mapsto (a^*_{ji})$ denoted still by $*$. A matrix $A \in R_{m\times n}$ is said to have an *MP-inverse* if there exists $B \in R_{n\times m}$ such that the following equations hold [10]:

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB \quad \text{and} \quad (BA)^* = BA.$$

Any element $B \in R_{n\times m}$ satisfying the equations above is called an MP-inverse of A. If such a B exists, it is unique and is denoted by $A^\dagger$.

Following [4], a $*$-ring R is said to satisfy the *k-term star-cancellation law* (SC_k) if

$$a_1^*a_1 + \cdots + a_k^*a_k = 0 \Rightarrow a_1 = \cdots = a_k = 0$$

for any $a_1, \cdots, a_k \in R$. Note that a $*$-ring satisfying SC_1 is known as a $*$-*cancellable* ring. A ring is said to be $*$-*regular* if it is regular and $*$-cancellable (see, e.g., [9]). It is well-known that R is a $*$-regular ring if and only if every element in R is MP-invertible, and that $R_{2\times 2}$ is a $*$-regular ring if and only if R is a regular $*$-ring satisfying SC_2 (see, e.g., [6, p.182]).

2. Main results

Throughout this article we assume that R is a regular $*$-ring satisfying SC_2, an assumption that plays an essential role in Theorem 2.1 and Theorem 2.7. (See Examples 2.2 and 2.8.). In particular, the rings R and $R_{2\times 2}$ are $*$-regular rings and every matrix $M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ in $R_{2\times 2}$ is MP-invertible. Note that $M^\dagger = M^*(MM^*)^\dagger$ in this case (see [10, p.407]), a result that will be widely-used in the sequel.

If $ab^* + cd^* = 0$, as $MM^* = \begin{bmatrix} aa^*+cc^* & ab^*+cd^* \\ ba^*+dc^* & bb^*+dd^* \end{bmatrix}$ then

$$M^\dagger = \begin{bmatrix} a^*(aa^*+cc^*)^\dagger & b^*(bb^*+dd^*)^\dagger \\ c^*(aa^*+cc^*)^\dagger & d^*(bb^*+dd^*)^\dagger \end{bmatrix}.$$

Next theorem shows that the condition $ab^* + cd^* = 0$ is also necessary for such a decomposition to hold.

As usual, we denote the right annihilator of an element a in a ring R by a^0. That is, $a^0 = \{r \in R \mid ar = 0\}$.

Theorem 2.1. *Let R be a regular $*$-ring satisfying SC_2 and $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] \in R_{2\times 2}$. Pose $k = aa^* + cc^*$, $l = bb^* + dd^*$ and $m = ab^* + cd^*$. Then $M^\dagger = \begin{bmatrix} a^*k^\dagger & b^*l^\dagger \\ c^*k^\dagger & d^*l^\dagger \end{bmatrix}$ if and only if $m = 0$.*

PROOF. We need only to prove the "only if" part.

First, we show that $l^0 \subseteq (b^*)^0$.

Let $x \in l^0$, i.e., $(bb^* + dd^*)x = 0$. Then $(b^*x)^*b^*x + (d^*x)^*d^*x = 0$. Since R satisfies SC_2, we have $b^*x = 0$, i.e., $x \in (b^*)^0$.

Since $1 - l^\dagger l \in l^0$, it follows that $b^* = b^*l^\dagger l$ and hence $b = l^*(l^*)^\dagger b = ll^\dagger b$. Similarly, $d = ll^\dagger d$.

As $\begin{bmatrix} kk^\dagger a+ml^\dagger b & kk^\dagger c+ml^\dagger d \\ m^*k^\dagger a+ll^\dagger b & m^*k^\dagger c+ll^\dagger d \end{bmatrix} = MM^\dagger M = M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$, one can see that $m^*k^\dagger a = 0 = m^*k^\dagger c$, which implies $m^*k^\dagger aa^* = 0 = m^*k^\dagger cc^*$. Hence $m^*k^\dagger k = 0$.

Again, SC_2 guarantees that $k^0 \subseteq (m^*)^0$ and hence $m^* = m^*k^\dagger k = 0$. Consequently, $m = 0$.

The next example shows that the assumption "R is a regular $*$-ring satisfying SC_2" plays an essential role in Theorem 2.1.

Example 2.2. Let $R = \mathbb{Z}/2\mathbb{Z}$ with $*$ given by the identity map. Then R is a regular $*$-ring but it does not fulfil SC_2 as $1^*1 + 1^*1 = 0$ but $1 \neq 0$. Let $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Then $m = ab^* + cd^* = 0$ but $M^\dagger$ does not exist.

Hartwig and Patrício [6] expressed the flipped MP-inverse of $M = \left[\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right]$. Among others, they gave a necessary and sufficient condition under which $M^\dagger$ is of (2, 1, 0) type, i.e., the (2, 1) entry of $M^\dagger$ is 0. Taking $c = 0$ in Theorem 2.1, we obtain a special case in which $M^\dagger$ is of (2, 1, 0) type.

Corollary 2.3. *Let R be a regular $*$-ring satisfying SC_2 and $M = \left[\begin{smallmatrix} a & 0 \\ b & d \end{smallmatrix}\right] \in R_{2\times 2}$. Then $M^\dagger = \begin{bmatrix} a^\dagger & b^*(bb^*+dd^*)^\dagger \\ 0 & d^*(bb^*+dd^*)^\dagger \end{bmatrix}$ if and only if $ab^* = 0$.*

Theorem 2.4. *A ring R is a regular $*$-ring satisfying SC_n if and only if every $n\times 1$ matrix* $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ *over R is MP-invertible.*

PROOF. "$\Leftarrow$" We first prove that R has the SC_n property. Assume $a_1^*a_1+\cdots+a_n^*a_n=0$ and $\alpha=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. It follows that $A=[\alpha,0]\in R_{n\times n}$ and $A^*A=0$. As $\alpha^\dagger$ exists, $A^\dagger=\begin{bmatrix} \alpha^\dagger \\ 0 \end{bmatrix}$. Note that $A=(AA^\dagger)^*A=(A^\dagger)^*A^*A=0$. We see that R has the SC_n property.

For $a\in R$, let $\begin{bmatrix} a \\ \vdots \\ 0 \end{bmatrix}^\dagger=[c_1,\cdots,c_n]$. Then c_1 is the MP-inverse of a by a direct check.

Therefore, R is a regular $*$-ring satisfying SC_n.

Conversely, let $\alpha=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\in R_{n\times 1}$ and $A=[\alpha,0]\in R_{n\times n}$. By hypothesis $A^\dagger$ exists and set $A^\dagger=\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$. It is easy to see $\alpha^\dagger=\beta_1$.

Cline [2, Theorem 2] provided the presentation for the MP-inverse of $A+C$, where A and C are complex matrices such that $AC^*=0$. His formula indeed holds in the ring case, i.e., for any $a,c\in R$ with $ac^*=0$,

$$(a+c)^\dagger=a^\dagger+(1-a^\dagger c)[u^\dagger+(1-u^\dagger u)vc^*(a^\dagger)^*a^\dagger(1-cu^\dagger)],$$

where $u=(1-aa^\dagger)c$, $w=a^\dagger c(1-u^\dagger u)$ and $v=(1+w^*w)^{-1}$. Note that the invertibility of $1+w^*w$ is guaranteed by our assumption at the beginning of this section (see [6, p. 182]).

Hartwig and Patrício [6, p.183] simplified the above formula to

$$(a+c)^\dagger=(1+y^*)(1+yy^*)^{-1}s+u^\dagger,$$

where $u=(1-aa^\dagger)c$, $s=a^\dagger(1-cu^\dagger)$ and $y=a^\dagger c(1-u^\dagger u)$. In addition, they proved the following result.

Lemma 2.5. [6, p.186] *Let R be a regular $*$-ring satisfying SC_2 and let $A,C\in R_{2\times 2}$ with $AC^*=0$. If $I+YY^*$ is invertible then*

$$(A+C)^\dagger=(I+Y^*)(I+YY^*)^{-1}S+U^\dagger,$$

where $U=(I-AA^\dagger)C$, $S=A^\dagger(I-CU^\dagger)$ *and* $Y=A^\dagger C(I-U^\dagger U)$.

Lemma 2.6. *Given* $a\in R$, $\begin{bmatrix}1\\a\end{bmatrix}$ *is MP-invertible if and only if* $1+a^*a$ *is invertible.*

Proof. "$\Rightarrow$" Let $\begin{bmatrix}1\\a\end{bmatrix}^\dagger=[b,c]$. As $(\begin{bmatrix}1\\a\end{bmatrix}[b,c])^*=\begin{bmatrix}1\\a\end{bmatrix}[b,c]$, we have $(ac)^*=ac$, $b^*=b$ and $c^*=ab$. As $\begin{bmatrix}1\\a\end{bmatrix}=\begin{bmatrix}1\\a\end{bmatrix}[b,c]\begin{bmatrix}1\\a\end{bmatrix}$, we get $b+ca=1$. So, $(1+a^*a)b=b^*+a^*c^*=(b+ca)^*=1$, and hence $b^*(1+a^*a)=1$.

Conversely, pose $y=(1+a^*a)^{-1}[1,a^*]$. It is easy to check that y is the MP-inverse of $\begin{bmatrix}1\\a\end{bmatrix}$.

By virtue of Lemma 2.5, we can now prove our main theorem of this paper. To calculate simply, we introduce the following notations

$$\begin{array}{llll} e=a^*a+b^*b, & f=a^*c+b^*d, & g=c-ae^\dagger f, & h=d-be^\dagger f,\\ j=g^*g+h^*h, & k=e^\dagger f(1-j^\dagger j), & l=e^\dagger(a^*-fj^\dagger g^*) \text{ and } & m=e^\dagger(b^*-fj^\dagger h^*). \end{array}$$

Theorem 2.7. *Let* R *be a regular* $*$*-ring satisfying* SC_2 *and* $M=\begin{bmatrix}a&c\\b&d\end{bmatrix}\in R_{2\times 2}$.

Then $M^\dagger=\begin{bmatrix}p&r\\q&s\end{bmatrix}$, *where*

$$\begin{array}{ll} p=(1+kk^*)^{-1}l, & r=(1+kk^*)^{-1}m,\\ q=j^\dagger g^*+k^*(1+kk^*)^{-1}l \text{ and } & s=j^\dagger h^*+k^*(1+kk^*)^{-1}m. \end{array}$$

Proof. Let $A=\begin{bmatrix}a&0\\b&0\end{bmatrix}$, $C=\begin{bmatrix}0&c\\0&d\end{bmatrix}$, $U=(I-AA^\dagger)C$, $S=A^\dagger(I-CU^\dagger)$ and $Y=A^\dagger C(I-U^\dagger U)$. As $M=A+C$ and $AC^*=0$, then $M^\dagger=(I+Y^*)(I+YY^*)^{-1}S+U^\dagger$.

It is straightforward to check that $A^\dagger=(A^*A)^\dagger A^*=\begin{bmatrix}e^\dagger a^*&e^\dagger b^*\\0&0\end{bmatrix}$ and $U=(I-AA^\dagger)C=\begin{bmatrix}0&g\\0&h\end{bmatrix}$. Similarly, $U^\dagger=(U^*U)^\dagger U^*=\begin{bmatrix}0&0\\j^\dagger g^*&j^\dagger h^*\end{bmatrix}$. Hence

$$Y=A^\dagger C(I-U^\dagger U)=\begin{bmatrix}e^\dagger a^*&e^\dagger b^*\\0&0\end{bmatrix}\begin{bmatrix}0&c\\0&d\end{bmatrix}\begin{bmatrix}1&0\\0&1-j^\dagger j\end{bmatrix}=\begin{bmatrix}0&k\\0&0\end{bmatrix}$$

and

$$S = A^\dagger(I - CU^\dagger) = \begin{bmatrix} e^\dagger a^* & e^\dagger b^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1-cj^\dagger g^* & -cj^\dagger h^* \\ -dj^\dagger g^* & 1-dj^\dagger h^* \end{bmatrix} = \begin{bmatrix} l & m \\ 0 & 0 \end{bmatrix}.$$

According to Theorem 2.4 and Lemma 2.6, it follows that $1 + kk^*$ is invertible and hence $I + YY^*$ is invertible. Now, we have

$$(I + Y^*)(I + YY^*)^{-1}S = \begin{bmatrix} 1 & 0 \\ k^* & 1 \end{bmatrix} \begin{bmatrix} (1+kk^*)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l & m \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1+kk^*)^{-1}l & (1+kk^*)^{-1}m \\ k^*(1+kk^*)^{-1}l & k^*(1+kk^*)^{-1}m \end{bmatrix}.$$

Therefore, the result follows by Lemma 2.5.

The next example shows that the assumption "R is a regular $*$-ring satisfying SC_2" is also essential for Theorem 2.7.

Example 2.8. Let $R = \mathbb{Z}/2\mathbb{Z}$ be as in Example 2.2. The following table exhibits two cases in which $M^\dagger = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$ does not hold.
Table

M	$M^\dagger$	$1 + kk^*$	$\begin{bmatrix} p & r \\ q & s \end{bmatrix}$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	does not exist	1	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	1	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

In the remainder of this section, we give some applications of Theorem 2.7.

Corollary 2.9. *Under the hypothesis of Theorem 2.7, the following statements are equivalent:*

(1) $M^\dagger = \begin{bmatrix} (1+kk^*)^{-1}e^\dagger a^* & (1+kk^*)^{-1}e^\dagger b^* \\ k^*(1+kk^*)^{-1}e^\dagger a^* & k^*(1+kk^*)^{-1}e^\dagger b^* \end{bmatrix}$.

(2) $j = 0$.

Proof. (2)⇒(1) is obvious.

(1)⇒(2). As $k^*(1 + kk^*)^{-1} = (1 + k^*k)^{-1}k^*$, then

$$M^\dagger = \begin{bmatrix} (1+kk^*)^{-1}e^\dagger a^* & (1+kk^*)^{-1}e^\dagger b^* \\ (1+k^*k)^{-1}k^*e^\dagger a^* & (1+k^*k)^{-1}k^*e^\dagger b^* \end{bmatrix}.$$

Hence

$$(1+kk^*)^{-1}e^\dagger a^* \;=\; (1+kk^*)^{-1}[e^\dagger(a^*-fj^\dagger g^*)] \tag{2.1}$$

and

$$(1+k^*k)^{-1}k^*e^\dagger a^* \;=\; j^\dagger g^* + (1+k^*k)^{-1}k^*[e^\dagger(a^*-fj^\dagger g^*)] \tag{2.2}$$

by Theorem 2.7. From (2.1) one can obtain $e^\dagger fj^\dagger g^* = 0$. Combining this with (2.2), we get $j^\dagger g^* = 0$.

Similarly, it follows that $j^\dagger h^* = 0$. Therefore, $j = jj^\dagger j = jj^\dagger(g^*g + h^*h)$ $= 0$.

A matrix $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ with coefficients in R is said to be of $(i,j,0)$ type if the (i,j) entry of M is zero. Note in [3, Corollary 2.7] that $aa^\dagger = a^\dagger a$ for any $a \in R^\dagger$ such that $aa^* = a^*a$. It is easy to see that $ee^\dagger = e^\dagger e$ since $e = a^*a + b^*b$.

If $M^\dagger$ is of $(1,1,0)$ type, then $p = 0$ reduces to $e^\dagger a^* {=} e^\dagger fj^\dagger g^*$ and hence $ea^* = efj^\dagger g^*$. This implies $ae = gj^\dagger f^*e$. We hence obtain the following corollary.

Corollary 2.10. *Let $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. Then $M^\dagger$ is of $(1,1,0)$ type if and only if $ae = gj^\dagger f^*e$. In this case, we have*
$M^\dagger = \left[\begin{smallmatrix} 0 & (1+kk^*)^{-1}m \\ j^\dagger g^* & j^\dagger h^* + k^*(1+kk^*)^{-1}m \end{smallmatrix}\right]$.

Corollary 2.11. *Let $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. Then $M^\dagger$ is of $(1,2,0)$ type if and only if $be = hj^\dagger f^*e$. In this case, we have*
$M^\dagger = \left[\begin{smallmatrix} (1+kk^*)^{-1}l & 0 \\ j^\dagger g^* + k^*(1+kk^*)^{-1}l & j^\dagger h^* \end{smallmatrix}\right]$.

If $M^\dagger$ is of $(2,1,0)$ type, then $q = j^\dagger g^* {+} k^*(1{+}kk^*)^{-1}l = 0$. By multiplying the above equations by $1 - j^\dagger j$ on the left, it follows that $(1-j^\dagger j)k^*(1+kk^*)^{-1}l = 0$, that is $k^*(1+kk^*)^{-1}l = 0$. Hence $k^*l = 0$ since $k^*(1+kk^*)^{-1} = (1{+}k^*k)^{-1}k^*$. By substituting $k^*l = 0$ back into q, then follows that $j^\dagger g^* = 0$. As $(1+kk^*)^{-1} = 1 - (1+kk^*)^{-1}kk^*$, we have

Corollary 2.12. *Let $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. Then $M^\dagger$ is of $(2,1,0)$ type if and only if $j^\dagger g^* = k^*l = 0$. In this case, we have*
$M^\dagger = \left[\begin{smallmatrix} l & (1+kk^*)^{-1}m \\ 0 & j^\dagger h^* + k^*(1+kk^*)^{-1}m \end{smallmatrix}\right]$.

Corollary 2.13. *Let* $M = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$. *Then* $M^\dagger$ *is of* $(2,2,0)$ *type if and only if* $j^\dagger h^* = k^* m = 0$. *In this case, we have*

$$M^\dagger = \left[\begin{smallmatrix} (1+kk^*)^{-1}l & m \\ j^\dagger g^* + k^*(1+kk^*)^{-1}l & 0 \end{smallmatrix}\right].$$

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