# Generalized invertibility in two semigroups of a ring *


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**Abstract**

In *Linear and Multilinear Algebra*, 1997, Vol.43, pp.137-150, R. Puystjens and R. E. Hartwig proved that given a regular element $t$ of a ring $R$ with unity 1, then $t$ has a group inverse if and only if $u = t^2t^- + 1 - tt^-$ is invertible in $R$ if and only if $v = t^-t^2 + 1 - t^-t$ is invertible in $R$. There, R. E. Hartwig posed the pertinent question whether the inverse of $u$ and $v$ could be directly related. Similar equivalences appear in the characterization of Moore-Penrose and Drazin invertibility, and therefore analogous questions arise. We present a unifying result to answer these questions not only involving classical invertibility, but also some generalized inverses as well.



Keywords: Generalized invertibility, corner rings, matrices over rings, semigroups.



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# 1 Introduction

Let $R$ be an arbitrary ring with unity 1, $Mat\,(R)$ the category of all matrices over $R$, $\mathcal{M}_{m\times n}\,(R)$ the set of $m\times n$ matrices and $\mathcal{M}_m\,(R)$ the ring of $m\times m$ matrices over $R$. Let $^*$ be an involution on the matrices over $R$. That is,

$$(A^*)^* = A, (AB)^* = B^*A^*, (A+B)^* = A^* + B^*,$$

with $A, B \in Mat\,(R)$ and whenever the operations are well defined.

Given an $m\times n$ matrix $A$ over $R$, $A$ is *(von Neumann) regular* if there exists an $n\times m$ matrix $A^-$ such that

$$AA^-A = A.$$

The set of von Neumann inverses of $A$ will be denoted by $A\,\{1\}$. That is,

$$A\,\{1\} = \{X \in \mathcal{M}_{n\times m}\,(R) : AXA = A\}\,.$$

$A$ is said to be *Moore-Penrose invertible* with respect to $^*$ if there exists a (unique) $n\times m$ matrix $A^\dagger$ such that:

$$\begin{aligned} AA^\dagger A &= A, \\ A^\dagger AA^\dagger &= A^\dagger, \\ \left(AA^\dagger\right)^* &= AA^\dagger, \\ \left(A^\dagger A\right)^* &= A^\dagger A. \end{aligned}$$

Necessary and sufficient conditions for the existence as well as expressions for $A^\dagger$ can be found in [10], [11], [13] and [14].

Also, if $m = n$, then the *group inverse* of $A$ exists if there is a (unique) $A^\#$ such that

$$\begin{aligned} AA^\# A &= A, \\ A^\# AA^\# &= A^\#, \\ AA^\# &= A^\# A. \end{aligned}$$

Necessary and sufficient conditions for the existence as well as expressions for $A^\#$ can be found in [16].

The *Drazin inverse of index* $k$ of $A$ exists if $k$ is the smallest natural number such that there is a (unique) $A^{D_k}$ for which

$$\begin{aligned} A^kA^{D_k}A &= A^k, \\ A^{D_k}AA^{D_k} &= A^{D_k}, \\ AA^{D_k} &= A^{D_k}A. \end{aligned}$$

Necessary and sufficient conditions for the existence as well as expressions for $A^{D_k}$ can be found in [15].

A motivation for this research appeared in [16]. There, the authors proved that given a regular element $t$ of a ring $R$ with unity 1 , then $t$ has a group inverse if and only if $u = t^2t^- + 1 - tt^-$ is invertible in $R$ if and only if $v = t^-t^2 + 1 - t^-t$ is invertible in $R$. R. E. Hartwig posed the pertinent question whether the inverse of $u$ and $v$ could be directly related. Similar equivalences appear in the characterization of Moore-Penrose (see [10]) and Drazin (see [15]) invertibility, and therefore analogous questions arise. The equivalence of the invertibility of the elements $u$ and $v$ was not proved directly. A direct proof of the equivalence follows now from Proposition 3. In Propositions 4,5 we show that similar equivalences can be proved directly for von Neumann and Drazin (and in particular for group) inverses. For the Moore-Penrose case, we remark that a similar result is not valid in general but we give a sufficient condition for such equivalence to hold.

These considerations leaded to a remarkable behavior between the generalized inverses of elements in the two semigroups $eRe$ and $eRe+1-e$ of a ring $R$, where $e^2 = e$, given in Theorem 1.

## 2 Generalized invertibility in a corner ring

In this section, $\mathcal{R}$ is a ring with unity 1 and $e \in \mathcal{R}$ is an idempotent. Moreover, and when appropriate, $\star$ is an involution in $\mathcal{R}$. Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$, we set

$$\begin{aligned} \mathcal{A} + \mathcal{B} &= \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}, \\ \mathcal{A}\mathcal{B} &= \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}. \end{aligned}$$

In the case one of the sets is a singleton, then we will drop the brackets in the notation. For instance,

$$e\mathcal{R}e + 1 - e = \{exe + 1 - e : x \in \mathcal{R}\}.$$

It should be stressed that this set is a (multiplicative) semigroup. The subrings of the form $e\mathcal{R}e$ are called *corner rings*.

Definitions of von Neumann, group, Drazin and Moore-Penrose inverses are similar to those given for matrices (see also [6], [9]).

For all nonzero idempotents $e$ of $\mathcal{R}$ we can consider the group $H_e$ of $e$-units in the corner ring $e\mathcal{R}e$. This is given by

$$\begin{aligned} H_e &= \{exe | exe\mathcal{R} = e\mathcal{R}, \mathcal{R}exe = \mathcal{R}e\} \\ &= \{x \in \mathcal{R} | x\mathcal{R} = e\mathcal{R}, \mathcal{R}x = \mathcal{R}e\}. \end{aligned}$$

If $exe \in H_e$ then its unique $e$-unit will be denoted by $(exe)^{-1_e}$. If $exe$ is regular in $\mathcal{R}$ then it also has a von Neumann inverse in $e\mathcal{R}e$, namely $eye$ given $y \in exe\{1\}$. An arbitrary von Neumann inverse of $exe$ still belonging to $e\mathcal{R}e$ will be denoted by $(exe)^{-e}$. We note in passing that for group and Drazin inverses we will keep the usual notation as $(exe)^{\#}$, $(exe)^{D}$ both belong to $e\mathcal{R}e$ if they exist in $\mathcal{R}$. The same reasoning applies to Moore-Penrose inverses when we assume in addition $e^{\star} = e$.

In [8], the relation between invertible elements of $e\mathcal{R}e$ and $e\mathcal{R}e + 1 - e$ was investigated. In the following result, similar equivalences are given involving some generalized inverses. This theorem will play an important role in the forthcoming section. In its proof, we will use the following facts:

1. If $ab = 0 = ba$ and $a^{D_p}, b^{D_q}$ exist then $a + b$ is Drazin invertible and
$$(a+b)^{D_l} = a^{D_p} + b^{D_q},$$
where $l = \max\{p, q\}$.

2. If $a^{\star}b = 0 = ab^{\star}$ and $a, b$ are Moore-Penrose invertible then $a + b$ is Moore-Penrose invertible and
$$(a+b)^{\dagger} = a^{\dagger} + b^{\dagger}.$$

**Theorem 1.** *Let $\mathcal{R}$ be a ring with unity 1 and $e$ an idempotent in $\mathcal{R}$. Then for all $x$ in $\mathcal{R}$, the following hold:*

1. *$exe + 1 - e \in H_1$ iff $exe \in H_e$, in which case*
$$(exe)^{-1_e} = e\,(exe + 1 - e)^{-1}\,e \in e\mathcal{R}e$$
*and*
$$(exe + 1 - e)^{-1} = (exe)^{-1_e} + 1 - e \in e\mathcal{R}e + 1 - e.$$

2. *$exe + 1 - e$ is regular in $\mathcal{R}$ iff $exe$ is regular in the ring $e\mathcal{R}e$, in which case*
$$e\,(exe + 1 - e)^{-}\,e \in exe\{1\}$$
*and*
$$(exe)^{-e} + 1 - e \in (exe + 1 - e)\{1\} \cap e\mathcal{R}e + 1 - e.$$

3. *$exe + 1 - e$ is group invertible in $\mathcal{R}$ iff $exe$ is group invertible in the ring $e\mathcal{R}e$, in which case*
$$(exe)^{\#} = e\,(exe + 1 - e)^{\#}\,e \in e\mathcal{R}e$$
*and*
$$(exe + 1 - e)^{\#} = (exe)^{\#} + 1 - e \in e\mathcal{R}e + 1 - e.$$

4. *$exe + 1 - e$ has Drazin index $k$ in $\mathcal{R}$ iff $exe$ has Drazin index $k$ in the ring $e\mathcal{R}e$ (with $k \geq 1$), in which case*

$$(exe)^{D_k} = e\,(exe + 1 - e)^{D_k}\, e \in e\mathcal{R}e$$

*and*

$$(exe + 1 - e)^{D_k} = (exe)^{D_k} + 1 - e \in e\mathcal{R}e + 1 - e.$$

5. *If $\mathcal{R}$ has an involution $^\star$ and $e = e^\star$, then $exe + 1 - e$ is Moore-Penrose invertible in $\mathcal{R}$ w.r.t. $^\star$ iff $exe$ is Moore-Penrose invertible in the ring $e\mathcal{R}e$ w.r.t. $^\star$, in which case*

$$(exe)^\dagger = e\,(exe + 1 - e)^\dagger\, e \in e\mathcal{R}e$$

*and*

$$(exe + 1 - e)^\dagger = (exe)^\dagger + 1 - e \in e\mathcal{R}e + 1 - e.$$

*Proof.* (1) was proved in [8].

(2) : Assume first that $exe + 1 - e$ is regular in $\mathcal{R}$, i.e.,

$$(exe + 1 - e)\,(exe + 1 - e)^-\,(exe + 1 - e) = exe + 1 - e.$$

Multiplying on the left and on the right by $e$,

$$exe\,(exe + 1 - e)^-\,exe = exe,$$

and therefore $e\,(exe + 1 - e)^-\,e$ is a von Neumann inverse of $exe$ in $e\mathcal{R}e$. Conversely, it is clear that if

$$exe\,(exe)^{-_e}\,exe = exe$$

then $(exe)^{-_e} + 1 - e$ is a von Neumann inverse of $exe + 1 - e$ in $\mathcal{R}$.

(3) : If $(exe)^\#$ exists then it also belongs to the corner ring $e\mathcal{R}e$ and it follows easily that $(exe)^\# + 1 - e$ is the group inverse of $exe + 1 - e$ which belongs to the semigroup $e\mathcal{R}e + 1 - e$. Conversely, if $exe + 1 - e$ is group invertible then $e\,(exe + 1 - e)^\#\,e$ is a von Neumann inverse of $exe$ in $e\mathcal{R}e$ and

$$(exe + 1 - e)\,(exe + 1 - e)^\# = (exe + 1 - e)^\#\,(exe + 1 - e)$$

implies, multiplying on the left and on the right by $e$, that

$$exe\,(exe + 1 - e)^\#\,e = e\,(exe + 1 - e)^\#\,exe.$$

That is, $e\,(exe + 1 - e)^\#\,e$ is a von Neumann inverse of $exe$ which commutes with $exe$. Consequently,

$$e\,(exe + 1 - e)^\#\,exe\,(exe + 1 - e)^\#\,e$$

is the group inverse of $exe$. So, the existence of $(exe+1-e)^{\#}$ implies the existence of $(exe)^{\#} \in e\mathcal{R}e$, and this is sufficient for

$$(exe+1-e)^{\#} \in e\mathcal{R}e+1-e.$$

Therefore,

$$(exe)^{\#} = e\,(exe+1-e)^{\#}\,e.$$

(4) : It is known that $t \in \mathcal{R}$ has Drazin index $k$ iff $k$ is the smallest natural number such that $t^k$ is group invertible (see [4], [15]). If $exe+1-e$ has Drazin index $k$ (with $k \geq 1$), then $k$ is the smallest natural number such that

$$\begin{aligned}(exe+1-e)^k &= (exe)^k+1-e \\ &= e\left[x\,(ex)^{k-1}\right]e+1-e\end{aligned}$$

is group invertible, and therefore $e\left[x\,(ex)^{k-1}\right]e = (exe)^k$ is group invertible. We remark that $k$ is the smallest natural number such that $(exe)^k$ is group invertible, and therefore $exe$ has Drazin index $k$ in the ring $e\mathcal{R}e$. For the expression of $(exe)^{D_k}$, using [4], [15],

$$\begin{aligned}(exe)^{D_k} &= (exe)^{k-1}\left[(exe)^k\right]^{\#} \\ &= (exe)^{k-1}\,e\left[(exe)^k\right]^{\#} \\ &= \left((exe)^{k-1}+1-e\right)e\left[(exe)^k\right]^{\#} \\ &= \left((exe)^{k-1}+1-e\right)e\left((exe)^k+1-e\right)^{\#}e \\ &= e\,(exe+1-e)^{k-1}\left[(exe+1-e)^k\right]^{\#}e \\ &= e\,(exe+1-e)^{D_k}\,e.\end{aligned}$$

Conversely, and as $exe\,(1-e) = (1-e)\,exe = 0$ and $(1-e)^{\#}, (exe)^{D_k}$ exist, it follows that

$$(exe+1-e)^{D_k} = (exe)^{D_k}+1-e.$$

(5) : If $(exe)^{\dagger}$ is the Moore-Penrose inverse of $exe$ in $e\mathcal{R}e \subseteq \mathcal{R}$ then

$$\begin{aligned}(exe+1-e)^{\dagger} &= (exe)^{\dagger}+(1-e)^{\dagger} \\ &= (exe)^{\dagger}+1-e\end{aligned}$$

since $(exe)^{*}\,(1-e) = 0 = exe\,(1-e)^{*}$.

Conversely, if $exe+1-e$ is Moore-Penrose invertible then $e\,(exe+1-e)^{\dagger}\,e$ is a von Neumann inverse of $exe$ in $e\mathcal{R}e$ and

$$\left((exe+1-e)\,(exe+1-e)^{\dagger}\right)^{\star} = (exe+1-e)\,(exe+1-e)^{\dagger}.$$

Multiplying on the left and on the right by $e^\star = e = e^2$,

$$\left(exe\,(exe+1-e)^\dagger\, e\right)^\star = exe\,(exe+1-e)^\dagger\, e.$$

Moreover,

$$\left((exe+1-e)^\dagger\,(exe+1-e)\right)^\star = (exe+1-e)^\dagger\,(exe+1-e)\,,$$

and multiplying on the left and on the right by $e^\star = e = e^2$,

$$\left(e\,(exe+1-e)^\dagger\, exe\right)^\star = e\,(exe+1-e)^\dagger\, exe.$$

Therefore,

$$(exe)^\dagger = e\,(exe+1-e)^\dagger\, exe\,(exe+1-e)^\dagger\, e.$$

Since $(exe)^\dagger + 1 - e \in e\mathcal{R}e + 1 - e$ then

$$(exe)^\dagger = e\,(exe+1-e)^\dagger\, e.$$

□

**Corollary 2.** *Given $e^2 = e \in \mathcal{R}$ then, and in case the elements exist,*

1. $(exe+1-e)^{-1} \in e\mathcal{R}e + 1 - e$,
2. *there is a von Neumann inverse of* $exe+1-e$ *also belonging to the semigroup* $e\mathcal{R}e + 1 - e$,
3. $(exe+1-e)^{D_k} \in e\mathcal{R}e + 1 - e$,
4. $(exe+1-e)^\# \in e\mathcal{R}e + 1 - e$,
5. *and if in addition* $e^\star = e$, *then* $(exe+1-e)^\dagger \in e\mathcal{R}e + 1 - e$.

As a remark, it should be strongly pointed out that *not all* von Neumann inverses of $exe+1-e$ (in case they exist) need to belong to $e\mathcal{R}e+1-e$. In fact, if $\mathcal{R} = \mathcal{M}_2(\mathbb{C})\,, E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then

$$E\mathcal{R}E + I - E = \left\{ \begin{bmatrix} z & 0 \\ z-1 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})\,|z \in \mathbb{C} \right\}.$$

Calculations show that $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \in E\mathcal{R}E + I - E$, $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\{1\}$, but still $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \notin E\mathcal{R}E + I - E$.

Nevertheless, given a von Neumann inverse $y$ of $exe+1-e$, then there is $z \in e\mathcal{R}e + 1 - e$ that is also a von Neumann inverse of $exe+1-e$, namely $eye + 1 - e$.

## 3 Generalized invertibility in two matrix semigroups

Let $E \in \mathcal{M}_m(R)$ be such that $E^2 = E$. In the previous section, we related some generalized inverses between the semigroup $E\mathcal{M}_m(R)E + I_m - E$ and the corner ring $E\mathcal{M}_m(R)E$, assuming $E$ is also symmetric when considering Moore-Penrose inverses. We refer to the previous section for notation. If $A \in \mathcal{M}_{m\times n}(R), A^-, A^= \in A\{1\}$, then $AA^-$ and $A^=A$ are two idempotents. In this section, we will relate some generalized inverses and the classical inverse between the semigroup

$$AA^-\mathcal{M}_m(R)AA^- + I_m - AA^-$$

and the semigroup

$$A^=A\mathcal{M}_n(R)A^=A + I_n - A^=A,$$

using Theorem 1. It should be remarked that in the Moore-Penrose inverse case, the symmetry of the idempotents $AA^-$ and $A^=A$ is assumed, or equivalently, the existence of $A^\dagger$.

**Proposition 3.** *Let $A \in \mathcal{M}_{m\times n}(R)$ be a regular matrix with von Neumann inverses $A^-$ and $A^=$, and $B \in \mathcal{M}_m(R)$. Then the following conditions are equivalent:*

1. $\Gamma = AA^-BAA^- + I_m - AA^-$ *is an invertible matrix.*
2. $\Omega = A^=AA^-BA + I_n - A^=A$ *is an invertible matrix.*

*Moreover,*

$$\Omega^{-1} = A^=AA^-\Gamma^{-1}A + I_n - A^=A$$

*and also*

$$\Gamma^{-1} = A\Omega^{-1}A^=AA^- + I_m - AA^-.$$

*Proof.* If $AA^-BAA^-+I_m-AA^-$ is invertible in $\mathcal{M}_m(R)$ then it follows from Theorem 1 (1) that $AA^-BAA^-$ is invertible in the ring $AA^-\mathcal{M}_m(R)AA^-$. Therefore, there exists an $X \in AA^-\mathcal{M}_m(R)AA^-$ such that

$$AA^-BAA^-X = XAA^-BAA^- = AA^-.$$

Multiplying on the left by $A^=$ and on the right by $A$, and as $AA^-X = XAA^- = X$, then

$$\left[(A^=A)A^-BA(A^=A)\right]\left[(A^=A)A^-XA(A^=A)\right] = A^=A$$

and

$$\left[(A^=A)A^-XA(A^=A)\right]\left[(A^=A)A^-BA(A^=A)\right] = A^=A.$$

Hence, $(A^{=}A)\,A^{-}BA\,(A^{=}A)$ is invertible in the ring $A^{=}A\mathcal{M}_n\,(R)\,A^{=}A$ and thus $A^{=}AA^{-}BA+I_n-A^{=}A$ is an invertible matrix.

The converse is analogous.

To prove $A^{=}AA^{-}\Gamma^{-1}A+I_n-A^{=}A$ is the inverse of $\Omega$, we remark that

$$\begin{aligned}\Omega\left(A^{=}AA^{-}\Gamma^{-1}A+I_n-A^{=}A\right) &= A^{=}AA^{-}BAA^{-}\Gamma^{-1}A+I_n-A^{=}A\\ &= A^{=}AA^{-}\Gamma\Gamma^{-1}A+I_n-A^{=}A\\ &= I_n\\ &= \left(A^{=}AA^{-}\Gamma^{-1}A+I_n-A^{=}A\right)\Omega.\end{aligned}$$

The expression of the inverse of $\Gamma$ can be verified analogously. □

**Remarks.**

1. If $AA^{-}BAA^{-}+I_m-AA^{-}$ is invertible then there exists $X\in AA^{-}\mathcal{M}_m\,(R)\,AA^{-}$ such that $AA^{-}BAA^{-}X=XAA^{-}BAA^{-}=AA^{-}$. Then, given $A^{=}\in A\,\{1\}\,,$

$$\begin{aligned}AA^{=} &= AA^{=}AA^{-}BAA^{=}XAA^{=}\\ &= AA^{=}XAA^{=}AA^{=}AA^{-}BAA^{=}\end{aligned}$$

   and therefore $AA^{=}AA^{-}BAA^{=}$ is invertible over $AA^{=}\mathcal{M}_m\,(R)\,AA^{=}$. That is to say, if $AA^{-}BAA^{-}+I_m-AA^{-}$ is invertible for a particular $A^{-}\in A\,\{1\}\,,$ then, and *for every* $A^{=}\in A\,\{1\}\,,$ the invertibility of

$$AA^{-}BAA^{=}+I_m-AA^{=}$$

   holds.

2. If $A$ and $B$ commute or if $B=AX$ is a consistent matrix equation, the invertibility $AA^{-}BAA^{-}+I_m-AA^{-}$ for a particular choice of $A^{-}\in A\,\{1\}$ is sufficient for its invertibility for *any* choice of $A^{-}$.

3. Analogously, if $A^{=}AA^{-}BA+I_n-A^{=}A$ is invertible for a particular choice of $A^{=}\in A\,\{1\}\,,$ then it is invertible for *all* choices of $A^{=}$.

4. If $A$ and $B$ commute or if $B=AX$ is a consistent matrix equation, the invertibility $A^{-}AA^{-}BA+I_n-A^{-}A$ for a particular choice of $A^{-}\in A\,\{1\}$ is sufficient for its invertibility for *any* choice of $A^{-}$.

5. As in the previous remarks, from Proposition 3 can be derived the interesting case when $A$ and $B$ commute or when $B=AX$ is a consistent matrix equation. That is, $BAA^{-}+I_m-AA^{-}$ is invertible for one, and hence for all choices of $A^{-}\in A\,\{1\}$ if and only if $A^{-}BA+I_n-A^{-}A$ is invertible for one, and hence for all choices of $A^{-}\in A\,\{1\}\,.$

6. If $B = A$, it follows that
$$A^2 A^- + I_m - AA^-$$
is invertible for one, and hence all choices of $A^-$, if and only if
$$A^- A^2 + I_m - A^- A$$
is invertible for one, and hence all choices of $A^-$ , which gives an answer to R. E. Hartwig's question.

We now give direct proofs of similar equivalences for generalized inverses. Similar remarks can also be stated for the considered generalized inverses.

**Proposition 4.** *Let $A \in \mathcal{M}_{m\times n}(R)$ be a regular matrix with von Neumann inverses $A^-$ and $A^=$, and $B \in \mathcal{M}_m(R)$. Then the following conditions are equivalent:*

1. *$\Gamma = AA^- BAA^- + I_m - AA^-$ is a von Neumann regular matrix.*
2. *$\Omega = A^= AA^- BA + I_n - A^= A$ is a von Neumann regular matrix.*

*Moreover,*
$$A^= AA^- \Gamma^- A + I_n - A^= A \in \Omega\{1\}$$
*and also*
$$A\Omega^- A^= AA^- + I_m - AA^- \in \Gamma\{1\}.$$

*Proof.* If $\Gamma$ is von Neumann regular, then
$$\begin{aligned}
\Gamma^- \in \Gamma\{1\} \quad &\Rightarrow \quad AA^-\Gamma^- AA^- \in AA^- BAA^-\{1\} \\
&\Rightarrow \quad AA^- BAA^- \Gamma^- AA^- BAA^- = AA^- BAA^- \\
&\Rightarrow \quad A^= AA^- BA\left(A^- \Gamma^- A\right) A^= AA^- BA = A^= AA^- BA \\
&\Rightarrow \quad A^= AA^- \Gamma^- A \in A^= AA^- BA\{1\} = A^= A\Omega A^= A\{1\} \\
&\Rightarrow \quad A^= AA^- \Gamma^- A + I_n - A^= A \in \Omega\{1\}.
\end{aligned}$$
Conversely, if $\Omega$ is von Neumann regular, then
$$\begin{aligned}
\Omega^- \in \Omega\{1\} \quad &\Rightarrow \quad A^= A\Omega^- A^= A \in A^= AA^- BA\{1\} \\
&\Rightarrow \quad A^= AA^- BA\Omega^- A^= AA^- BA = A^= AA^- BA \\
&\Rightarrow \quad AA^- BAA^-\left(A\Omega^- A^=\right) AA^- BAA^- = AA^- BAA^- \\
&\Rightarrow \quad A\Omega^- A^= AA^- \in AA^- \Gamma AA^-\{1\} \\
&\Rightarrow \quad A\Omega^- A^= AA^- + I_m - AA^- \in \Gamma\{1\}.
\end{aligned}$$
$\square$

**Proposition 5.** *Let $A \in \mathcal{M}_{m\times n}(R)$ be a regular matrix with von Neumann inverses $A^-$ and $A^=$, and $B \in \mathcal{M}_m(R)$. Then the following conditions are equivalent:*

1. *$\Gamma = AA^-BAA^- + I_m - AA^-$ is Drazin invertible with index $k$ (group invertible if $k = 1$).*
2. *$\Omega = A^=AA^-BA + I_n - A^=A$ is Drazin invertible with index $k$ (group invertible if $k = 1$).*

*Moreover,*

$$\Omega^{D_k} = A^=AA^-\Gamma^{D_k}A + I_n - A^=A$$

*and also*

$$\Gamma^{D_k} = A\Omega^{D_k}A^=AA^- + I_m - AA^-.$$

*Proof.* Let us first consider the case $k = 1$, i.e., the group invertibility case.

If $\Gamma^\#$ exists, then by Theorem 1 and Proposition 4,

$$A^=AA^-\Gamma^\# A \in A^=A\Omega A^=A\{1\} = A^=AA^-BA\{1\},$$

and furthermore

$$\begin{aligned} A^=A\Omega A^=A\left(A^=AA^-\Gamma^\# A\right) &= A^=AA^-BA\left(A^=AA^-\Gamma^\# A\right) \\ &= A^=AA^-\Gamma\Gamma^\# A \\ &= A^=AA^-\Gamma^\#\Gamma A \\ &= \left(A^=AA^-\Gamma^\# A\right)A^=AA^-BA \\ &= \left(A^=AA^-\Gamma^\# A\right)A^=A\Omega A^=A. \end{aligned}$$

Thus,

$$\begin{aligned} (A^=A\Omega A^=A)^\# &= A^=AA^-\Gamma^\# A\Omega A^=AA^-\Gamma^\# A \\ &= A^=AA^-\Gamma^\#\Gamma AA^-\Gamma^\# A \\ &= A^=AA^-\Gamma^\# A \end{aligned}$$

since $AA^-\Gamma^\# = \Gamma^\# AA^-$. In fact, using Corollary 2 (4), it follows that

$$\Gamma^\# \in AA^-\mathcal{M}_m(R)\,AA^- + I_m - AA^-,$$

and hence $AA^-\Gamma^\# = \Gamma^\# AA^-$. Therefore,

$$\Omega^\# = A^=AA^-\Gamma^\# A + I_n - A^=A.$$

Conversely, if $\Omega^{\#}$ exists then

$$A\Omega^{\#}A^{=}AA^{-} \in AA^{-}\Gamma AA^{-}\{1\}$$

and also

$$\begin{aligned}
\left(AA^{-}\Gamma AA^{-}\right)\left(A\Omega^{\#}A^{=}AA^{-}\right) &= \left(AA^{-}BAA^{-}\right)\left(A\Omega^{\#}A^{=}AA^{-}\right) \\
&= AA^{-}BA\Omega^{\#}A^{=}AA^{-} \\
&= A\Omega\Omega^{\#}A^{=}AA^{-} \\
&= A\Omega^{\#}\Omega A^{=}AA^{-} \\
&= A\Omega^{\#}A^{=}AA^{-}BAA^{-} \\
&= \left(A\Omega^{\#}A^{=}AA^{-}\right)\left(AA^{-}\Gamma AA^{-}\right).
\end{aligned}$$

So,

$$\begin{aligned}
\left(AA^{-}\Gamma AA^{-}\right)^{\#} &= A\Omega^{\#}\Omega A^{=}A\Omega^{\#}A^{=}AA^{-} \\
&= A\Omega^{\#}A^{=}AA^{-}
\end{aligned}$$

since $A^{=}A\Omega^{\#} = \Omega^{\#}A^{=}A$, using

$$\Omega^{\#} \in A^{=}A\mathcal{M}_n(R)A^{=}A + I_n - A^{=}A$$

by Corollary 2 (4). Therefore,

$$\Gamma^{\#} = A\Omega^{\#}A^{=}AA^{-} + I_m - AA^{-}.$$

For the general case, suppose $\Gamma$ has index $k$, i.e., $\Gamma^{D_k}$ exists. Then $\left(\Gamma^k\right)^{\#} = \left(AA^{-}\left(BAA^{-}\right)^k + I_m - AA^{-}\right)^{\#}$ exists. Using the first part of the proof and keeping in mind that $B$ is arbitrary,

$$\Omega^k = A^{=}AA^{-}\left(BAA^{-}\right)^k A + I_n - A^{=}A$$

is group invertible. Thus, $\Omega^{D_k}$ exists. Moreover, and using [4], [15],

$$\begin{aligned}
\Omega^{D_k} &= \Omega^{k-1}\left(\Omega^k\right)^{\#} \\
&= \Omega^{k-1}\left(A^{=}AA^{-}\left(BAA^{-}\right)^k A + I_n - A^{=}A\right)^{\#} \\
&= \Omega^{k-1}\left(A^{=}AA^{-}\left(\Gamma^k\right)^{\#} A + I_n - A^{=}A\right) \\
&= A^{=}AA^{-}\left(BAA^{-}\right)^{k-1}AA^{-}\left(\Gamma^k\right)^{\#} A + I_n - A^{=}A \\
&= A^{=}AA^{-}\Gamma^{k-1}\left(\Gamma^k\right)^{\#} A + I_n - A^{=}A \\
&= A^{=}AA^{-}\Gamma^{D_k}A + I_n - A^{=}A.
\end{aligned}$$

The converse is analogous. For the expression of $\Gamma^{D_k}$,

$$\begin{aligned}
\Gamma^{D_k} &= \Gamma^{k-1}\left(\Gamma^k\right)^{\#} \\
&= \Gamma^{k-1}\left(AA^-\left(BAA^-\right)^k AA^- + I_m - AA^-\right)^{\#} \\
&= \Gamma^{k-1}\left(A\left(\Omega^k\right)^{\#} A^{=}AA^- + I_m - AA^-\right) \\
&= AA^-\left(BAA^-\right)^{k-1} A\left(\Omega^k\right)^{\#} A^{=}AA^- + I_m - AA^- \\
&= A\Omega^{k-1}\left(\Omega^k\right)^{\#} A^{=}AA^- + I_m - AA^- \\
&= A\Omega^{D_k}A^{=}AA^- + I_m - AA^-.
\end{aligned}$$

$\square$

These propositions suggest that a similar equivalence would hold concerning Moore-Penrose inverses. That is, the conditions

(1) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ is Moore-Penrose invertible

(2) $\Omega = A^\dagger BA + I_n - A^\dagger A$ is Moore-Penrose invertible

would be equivalent. But taking $B = \begin{bmatrix} 1 & 0 \\ i & 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ over the field of complexes and transposition as the involution, we already see that (1) $\Leftrightarrow$ (2) does *not* hold in general.

In order to give a sufficient condition for (1) $\Leftrightarrow$ (2), let us introduce some more notation and definitions.

Let $\mathcal{X}$ be a ring with involution $\iota$ and $\mathcal{Y}$ a ring with involution $\tau$. We say that $\psi : \mathcal{X} \to \mathcal{Y}$ is a $\iota,\tau$-*invariant homomorphism* if $\psi$ is a ring homomorphism and $\psi\left(x^\iota\right) = \left(\psi\left(x\right)\right)^\tau$, for all $x \in \mathcal{X}$. If $\iota$ and $\tau$ coincide, then we will write $\iota$-*invariant* for short, which is equivalent to say that $\iota$ and $\psi$ commute.

Let $A \in \mathcal{M}_{m\times n}\left(R\right)$, and $\phi_A : AA^\dagger \mathcal{M}_m\left(R\right) AA^\dagger \to A^\dagger A\mathcal{M}_n\left(R\right) A^\dagger A$ defined by

$$\phi_A\left(AA^\dagger XAA^\dagger\right) = A^\dagger XA.$$

We will say $A$ is $*$-*invariant* if $\phi_A$ is $*$-invariant. Some calculations show that $\phi_A$ is actually an isomorphism and preserves invertible, von Neumann regular, Drazin and group invertible elements. But it may *not* preserve Moore-Penrose invertible elements. However, we will show that if $\phi_A$ is $*$-invariant then it also preserves Moore-Penrose inverses. Obviously, if $A^* \in A\left\{1\right\}$ and thus $A^\dagger = A^*$, i.e., $A$ is a partial isometry, then $\phi_A$ is $*$-invariant. That is, partial isometries are $*$-invariant, but not conversely. This can be shown by the following example.

**Example** Take $R = \mathbb{F}$ any field such that $char(\mathbb{F}) > 2, n = 2, A = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ with $x \neq 0, 1-1$, and the transposition $^T$ as an involution in $\mathcal{M}_2(\mathbb{F})$. We notice that $\phi_A$ is $^T$-invariant iff, for all $Y \in \mathcal{M}_2(\mathbb{F})$, the equality $A^\dagger Y A = A^T Y \left(A^\dagger\right)^T$ holds. Now,

$$A^\dagger = \begin{bmatrix} x^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

and if $Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{aligned} A^\dagger Y A &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \\ &= AYA^\dagger \\ &= A^T Y \left(A^\dagger\right)^T, \end{aligned}$$

and so $\phi_A$ is $^T$-invariant. However, $A^* = A \notin A\{1\}$.

We will now give a sufficient condition for $(1) \Leftrightarrow (2)$.

**Proposition 6.** *Let $A \in \mathcal{M}_{m\times n}(R)$ and $B \in \mathcal{M}_m(R)$. Consider the following conditions:*

1. *$\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ is Moore-Penrose invertible.*
2. *$\Omega = A^\dagger BA + I_n - A^\dagger A$ is Moore-Penrose invertible.*

*If $A$ is $*$-invariant then $(1) \Leftrightarrow (2)$, in which case*

$$\Gamma^\dagger = A\Omega^\dagger A^\dagger + I_m - AA^\dagger$$

*and*

$$\Omega^\dagger = A^\dagger \Gamma^\dagger A + I_n - A^\dagger A.$$

*Proof.* If $\Gamma$ is Moore-Penrose invertible then $AA^\dagger BAA^\dagger$ has a Moore-Penrose inverse $\Gamma_0^\dagger$ in $AA^\dagger \mathcal{M}_m(R) AA^\dagger$. As $AA^\dagger BAA^\dagger \Gamma_0^\dagger AA^\dagger BAA^\dagger = AA^\dagger BAA^\dagger$ then

$$A^\dagger BA \left(A^\dagger \Gamma_0^\dagger A\right) A^\dagger BA = A^\dagger BA.$$

Also,

$$\left(A^\dagger \Gamma_0^\dagger A\right) A^\dagger BA \left(A^\dagger \Gamma_0^\dagger A\right) = A^\dagger \Gamma_0^\dagger A.$$

Since $\phi_A$ is $*$-invariant then, for *all* $Y$,

$$A^\dagger Y A = A^* Y A^{\dagger\,*}.$$

As

$$\left(AA^\dagger BAA^\dagger \Gamma_0^\dagger\right)^* = AA^\dagger BAA^\dagger \Gamma_0^\dagger,$$

then multiplying on the left by $A^*$,

$$\begin{aligned}\left(AA^\dagger BAA^\dagger \Gamma_0^\dagger A\right)^* &= A^* AA^\dagger BAA^\dagger \Gamma_0^\dagger \\ &= A^* BA^{\dagger\,*} A^* \Gamma_0^\dagger \\ &= A^\dagger BAA^* \Gamma_0^\dagger.\end{aligned}$$

Multiplying on the right by $A^{\dagger\,*}$,

$$\begin{aligned}\left(A^\dagger BAA^\dagger \Gamma_0^\dagger A\right)^* &= A^\dagger BAA^* \Gamma_0^\dagger A^{\dagger\,*} \\ &= A^\dagger BAA^\dagger \Gamma_0^\dagger A.\end{aligned}$$

Moreover, and similarly, $\left(\Gamma_0^\dagger AA^\dagger BAA^\dagger\right)^* = \Gamma_0^\dagger AA^\dagger BAA^\dagger$ implies that

$$\begin{aligned}\left(A^\dagger \Gamma_0^\dagger AA^\dagger BAA^\dagger\right)^* &= \Gamma_0^\dagger AA^\dagger BAA^\dagger A^{\dagger\,*} \\ &= \Gamma_0^\dagger AA^\dagger BA^{\dagger\,*} \\ &= \Gamma_0^\dagger A^{\dagger\,*} A^* BA^{\dagger\,*} \\ &= \Gamma_0^\dagger A^{\dagger\,*} A^\dagger BA\end{aligned}$$

and therefore

$$\begin{aligned}\left(A^\dagger \Gamma_0^\dagger AA^\dagger BA\right)^* &= A^* \Gamma_0^\dagger A^{\dagger\,*} A^\dagger BA \\ &= A^\dagger \Gamma_0^\dagger AA^\dagger BA.\end{aligned}$$

So, $A^\dagger \Gamma_0^\dagger A$ is the Moore-Penrose inverse of $A^\dagger A \Omega A^\dagger A$ in $A^\dagger A \mathcal{M}_n\left(R\right) A^\dagger A$, and hence

$$\Omega^\dagger = A^\dagger \Gamma_0^\dagger A + I_n - A^\dagger A.$$

As $\Gamma_0^\dagger = AA^\dagger \Gamma^\dagger AA^\dagger$, it follows that

$$\Omega^\dagger = A^\dagger \Gamma^\dagger A + I_n - A^\dagger A.$$

Analogously, if $\Omega^\dagger$ exists then $\Omega_0^\dagger = A^\dagger A \Omega^\dagger A^\dagger A$ is the Moore-Penrose inverse of $A^\dagger BA = A^\dagger A \Omega A^\dagger A$ in the ring $A^\dagger A \mathcal{M}_n\left(R\right) A^\dagger A$. As in the previous case, $A\Omega_0^\dagger A^\dagger$ is the Moore-Penrose inverse of $AA^\dagger \Gamma AA^\dagger$ in $AA^\dagger \mathcal{M}_m(R) AA^\dagger$, and therefore

$$\begin{aligned} \Gamma^\dagger &= A\Omega_0^\dagger A^\dagger + I_m - AA^\dagger \\ &= A\Omega^\dagger A^\dagger + I_m - AA^\dagger. \end{aligned}$$

□

**Remarks.**

1. The $*$-invariance of $A$ is *not* necessary for $(1) \Leftrightarrow (2)$. Indeed, consider $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ over the field $\mathbb{C}$ of complexes and let $*$ be the involution defined as the transposed conjugate. Clearly, $(1) \Leftrightarrow (2)$ since *every* matrix has a Moore-Penrose inverse. Now, $\phi_A$ is $*$-invariant iff, for all $X \in \mathcal{M}_2(\mathbb{C})$,
$$A^\dagger X A = A X A^\dagger,$$
where $A^\dagger = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Taking an arbitrary $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\phi_A$ is $*$-invariant iff
$$\begin{bmatrix} a & \frac{2b}{3} \\ \frac{3c}{2} & d \end{bmatrix} = \begin{bmatrix} a & \frac{3b}{2} \\ \frac{2c}{3} & d \end{bmatrix}.$$
Therefore, $\phi_A$ is not $*$-invariant.

2. Propositions 3,4,5 generalize the fact that similarity between matrices is an equivalence relation which preserves classical, von Neumann, group and Drazin invertibility. Proposition 6 shows also the known fact that the same does not happen with respect to Moore-Penrose invertibility. But if $A$ is unitary, i.e., $A^* = A^{-1}$, then $\phi_A(X) = A^\dagger X A$ is $*$-invariant and therefore $A$ is $*$-invariant, and Proposition 6 gives the known fact that $B$ is Moore-Penrose invertible iff $A^* B A$ is Moore-Penrose invertible.

3. It will be of interest to extend our results on the (generalized) Drazin inverse of the sum $a + b$ under one-sided condition $ab = 0$, see [2], [7], if the generalized Drazin inverse cannot be characterized by the invertibility of an element of the form $exe + 1 - e$ with $e = e^2$.

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