Condensed exciton-polaritons in a 2D trap: elementary excitations and shaping by Gaussian pump beam

Y. Núñez Fernández, 1,2 M. I. Vasilevskiy, 2 C. Trallero-Giner, 3 and A. Kavokin 4

1Facultad de Física and ICTM, Universidad de La Habana, Vedado 10400, La Habana, Cuba
2Centro de Física and Departamento de Física, Universidade do Minho, Campus de Gualtar, Braga 4710-057, Portugal
3Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil
4Physics and Astronomy School, University of Southampton, Highfield, Southampton, SO17 1BJ, UK and Spin Optics Laboratory, State University of St-Petersburg, 1, Ulianovskaya, St-Petersburg, 198504, Russia

(Dated: February 23, 2013)

An exciton-polariton condensate (EPC) confined in a parabolic two-dimensional trap is considered theoretically. In the realistic limit of weakly interacting polaritons, the non-linear term in the Gross-Pitaevskii equation describing the properties of the condensate can be considered as a perturbation with respect to the trapping potential, which allows for a convenient analytical description of the EPC ground state and Bogolyubov–type elementary excitations around it. The excitation modes with the energies and wavefunctions depending on the polariton-polariton coupling strength are derived for the condensate neglecting interaction with uncondensed polaritons can be neglected. The energies of these modes are shown to be almost equidistant even for a rather strong polariton-polariton interaction inside the condensate. This makes lateral parabolic traps promising candidates for realization of bosonic cascade lasers based on exciton-polaritons. A different physical situation is also considered where the interaction with a reservoir of uncondensed polaritons is more important than that inside the EPC. In this case, it is shown that the condensate is “reshaped” by the repulsive interaction with the reservoir, namely, pushed out from the center of the trap in real space and blue-shifted in energy, in agreement with the results obtained in a number of recent experiments.

PACS numbers: 71.36.+c, 42.65.-k, 75.75.-c

I. INTRODUCTION

In the recent years, the Bose–Einstein condensation of microcavity exciton-polaritons has been demonstrated1 and its dramatic effects on the properties of this two-dimensional matter-light system have been shown. 2–10 Free electrons and holes in a semiconductor quantum well (QW) are created by pumping with a beam of photons, usually of the energy that is high above the band gap. The electrons and holes relax into lower energy states, form excitons that couple to the microcavity photons, and occupy the lowest exciton-polariton states. These mixed light-matter bosons can eventually condense. 1 Even though the short polariton lifetime permits only the formation of a quasi-equilibrium steady state, in which photons escaping from the cavity must be continuously replenished by the external pump, 8 exciton-polariton condensates (EPCs) can propagate over macroscopic distances outside the excitation area while preserving their spatial coherence. 6 The repulsive exchange interaction between QW excitons of the same spin orientation results in a repulsive potential acting on the condensate, induced by photo-generated excitons within the excitation area. 1 If the pumped spot is located within the condensate, the polaritons feel an outward force and the EPC expands. 10 On the contrary, if the reservoir of uncondensed excitons is spatially separated from the condensate, the repulsive potential allows for BEC localization in a trap with optically controlled dimensions. 6,9

Alternatively, Bose-Einstein condensate of exciton-polaritons in a semiconductor microcavity can be confined in a lateral trap. 11–14 A two-dimensional (2D) parabolic trap can be induced by local elastic strain shifting excitonic states downwards in energy. 11,12 Interestingly, the repulsive potential induced by uncondensed excitons created by two laser spots, appears also parabolic along the line between the spots, thus forming a (one-dimensional) parabolic trap for EPC. 10 Moreover, symmetric parabolic traps can be created for exciton polaritons by a ring-shaped optical excitation: repulsive interaction between the optically injected excitons on the ring and the condensate of exciton-polaritons is expected to induce the Bose-Einstein condensation in the bottom of the trap. A full optical control of the condensate would be achieved if the injection of polaritons into the trap could be provided by a second laser beam, additional to that creating the trap. In particular, such configuration is important for realization of Bosonic Cascade Lasers recently described theoretically. 15 Such experiments have not yet been realized, to the best of our knowledge. Here we aim at providing a theoretical analysis that would help designing new experiments on the realization of bosonic cascades in lateral polariton traps.

In this work we consider a polariton condensate excited non-resonantly in an external parabolic trap. We analyze the Bogolyubov-type elementary excitations of the condensate and its shape modified by a Gaussian pump beam focused in the center of the trap. Because of the parabolic confinement of the condensate, the elementary excitations are also localized, i.e. they are not character-
ized by a certain wavenumber, in contrast with those considered for unconfined EPCs (see, for example Ref. 16). While in most of the previous works the theoretical approach was based on numerical solution of a generalized Gross-Pitaevskii equation, here we use a semi-analytical perturbation theory approach. In the realistic limit of weakly interacting polaritons, the non-linear term in the Gross–Pitaevskii equation (GPE) can be considered as a perturbation with respect to the trap potential, which allows for a convenient analytical description of the EPC ground state. In the next Section, the generalized time-dependent GPE and its simplified limiting forms are discussed. In Sec. III, Bogoliubov-type elementary excitations in a parabolically confined EPC are considered. The effect of the repulsive potential produced by uncondensed excitons generated by a pump beam focused in the center of the trap is considered and discussed in Sec. IV. The last section is devoted to conclusions.

II. MEAN FIELD DESCRIPTION OF EPC

Within the mean field theory framework, the steady state and dynamics of an exciton-polariton condensate confined in a potential trap are ruled by the time-dependent GPE, which includes the loss ($\mathcal{L}$) and generation ($R$) terms, and the repulsive interaction with uncondensed excitons (reservoir).\textsuperscript{16}

\[
\begin{align*}
\imath \hbar \partial_t \Psi &= -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + \\
&\quad + \imath \hbar \left[ R - \mathcal{L} + g |\Psi|^2 + V_{\text{res}}(\mathbf{r}) \right] \Psi .
\end{align*}
\]

(1)

Here $g$ is the polariton-polariton interaction parameter for the condensate, $m$ is the polariton mass and $V(\mathbf{r}) = \frac{1}{2} m \omega_0^2 (x^2 + y^2)$ is the parabolic trap potential.\textsuperscript{20} The last term in Eq. (1) describes the repulsive interaction of the condensate with the reservoir close to the pump spot, which is proportional to the number of uncondensed polaritons ($N_r$). We assume that they are generated by a laser beam focused in the center of the trap, so this potential is axial-symmetric.

Assuming that $(R - \mathcal{L})$ is constant and using the transformation,

\[
\Psi = \exp \left[ (R - \mathcal{L}) \frac{\imath t}{\hbar} \right] \psi ,
\]

we obtain:

\[
\begin{align*}
\imath \hbar \partial_t \psi &= -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + \\
&\quad + g \exp \left[ 2(R - \mathcal{L}) \frac{\imath t}{\hbar} \right] |\psi|^2 + V_{\text{res}} \right] \psi .
\end{align*}
\]

(2)

It is important to note that the stability condition of the condensate described by Eq. (2) requires that $R \leq \mathcal{L}$. Equation (2) can be conveniently re-scaled by introducing $l_0 = \sqrt{\hbar/m \omega_0}$, $\Lambda = gm N_p/\hbar^2$, $(x,y) = (\xi,\eta)$ and $\psi = \sqrt{N_p} \Phi /l_0$, with $N_p$ denoting the number of polaritons in the condensate.

Under stationary conditions, $R \approx \mathcal{L}$ in Eq. (2). If the number of polaritons in the reservoir is small enough, $N_r \ll N_p$, e.g., because the pump power is low, then $V_{\text{res}} \approx 0$ and we recover the usual stationary GPE by substituting $\psi(\rho,t) = \exp(-\imath \mu t/\hbar) \Phi_0(\rho)$,

\[
\left[ \hat{H}_0 + \Lambda |\Phi_0|^2 - \mu \right] \Phi_0 = 0 .
\]

(3)

Here $\hat{H}_0 = -\frac{\hbar^2}{2} \nabla^2 + \frac{1}{2} \mu^2$, the operator $\nabla$ is defined in terms of the dimensionless coordinates $\xi$ and $\eta$, $\mu^2 = \xi^2 + \eta^2$, $\mu = \mu/\hbar \omega_0$ and $\mu$ is the chemical potential. This situation will be considered in the next section where the Bogoliubov-type elementary excitations in EPC uncoupled from the reservoir will be derived in terms of the dimensionless interaction parameter $\Lambda$.

As the pump beam intensity increases, the number of uncondensed polaritons grows and the interaction between the EPC and the reservoir cannot be neglected any more. Under the conditions $N_r \gg N_p$ and $R < \mathcal{L}$, the non-linear term in Eq. (2) can be neglected in comparison with $V_{\text{res}}$, after a certain time, $t \sim |R - \mathcal{L}|^{-1}$. In this case the EPC interacting with uncondensed polaritons in the reservoir can be described by a linear partial differential equation including the 2D parabolic confinement. The effect of this interaction on the EPC shape will be considered in Sec. IV.

III. BOGOLIUBOV EXCITATIONS

Applying Bogoliubov’s method to the time dependent GPE (2) with $V_{\text{res}} = 0$, the collective excitations with frequencies $\omega$ are obtained by expressing the solution as

\[
\psi(\rho,t) = \exp \left[ -\frac{\imath \mu t}{\hbar} \right] \{ \Phi_0(\rho) + \delta \Phi [u,v^*] \}
\]

(4)

and linearizing it in terms of the amplitudes $u(\rho,t)$ and $v^*(\rho,t)$. Owing to the axial symmetry, the excitation modes can be classified according to the $z-$component of the angular momentum, $m_z$, and the principal quantum number, $N$. Also, by virtue of the inversion symmetry, the space of solutions of the linearized equations can be split into two independent subspaces, $I$ and $II$ for $m_z$ even and odd, respectively.

Since in real experiments exciton-polariton condensates can be considered as a weakly interacting gas, one can obtain approximate solutions of the stationary GPE (3) and the corresponding Bogoliubov–de Gennes equations for the $u$ and $v^*$ components using a perturbation theory approach. Considering the term $\Lambda |\Phi_0^2|$ as a perturbation with respect to the harmonic trap potential, $V(\rho) = \frac{1}{2} \mu^2$, the ground state wave function (or order parameter), $\Phi_0(\rho)$, and the amplitudes $u(\rho,t)$ and $v^*(\rho,t)$,
can be sought in terms of the complete set of the 2D harmonic oscillator wave functions, $\varphi_{N,m}$. (see Appendix A). In Ref. 18, compact solutions for the reduced chemical potential, $\overline{\mu}$, and the dimensionless order parameter, $\Phi_0(\rho)$, are presented up to the 2-nd and the 1-st order in $\Lambda$, respectively. Following the procedure described in Ref. 18 (the details will be published elsewhere 21), it is possible to obtain the Bogolyubov excitation mode’s frequencies, $\omega_{N,m}$. Here we limit ourselves by considering the EPC states with zero angular momentum, $m_z = 0$, which refers to the macroscopic motion of the condensate. In experiments, EPCs are probed through the (spatially resolved) emission that escapes from the cavity. That related to the $m_z = 0$ states corresponds to a Gaussian beam and can be distinguished from the higher order (Laguerre-Gaussian) modes (which have been used to detect vortices in EPC5). For this case, we have:

$$\frac{\omega_{N,0}}{\omega_0} = N + \left[ \frac{N!}{2^{N-1} \left( \frac{N}{2} \right)^2} - 1 \right] \frac{\Lambda}{2\pi} + \beta_{N,0}\Lambda^2,$$

where $N = 2, 4, \ldots$ and $\beta_{N,0}$ are some numbers.22

The perturbation of the order parameter, $\delta \Phi_{N,0}$, to the first order in $\Lambda$, is given by

$$\delta \Phi_{N,0} \equiv \left( \begin{array}{c} u_{N,0}(\rho) \exp\left( -i\omega_{N,0}t \right) \\ v_{N,0}(\rho) \exp\left( i\omega_{N,0}t \right) \end{array} \right),$$

with

$$\begin{pmatrix} u_{N,0} \\ v_{N,0} \end{pmatrix} = \frac{\exp\left( -\rho^2/2 \right)}{\sqrt{\pi}} \begin{pmatrix} L_{N/2}(\rho^2) + 2\Lambda F_N^{-}(\rho) \\ -\Lambda F_N^{+}(\rho) \end{pmatrix},$$

and the functions $F_N^{\pm}(\rho)$ are given in Appendix A.

In Fig. 1a the eigenfrequencies of the Bogolyubov collective oscillations for $N = 2$ and $4$ are plotted against the interaction parameter $\Lambda$. Also, the ground state energy of the condensate per particle, $\epsilon$, is displayed. It should be pointed out that the chemical potential and the ground state energy (per particle) are not the same.23 The (dimensionless) energy per particle can be calculated as

$$\epsilon = \int_V \Phi_0 \left( \hat{H}_0 + \frac{1}{2} \Lambda |\Phi_0|^2 \right) \Phi_0 d\mathbf{r},$$

which differs from $\overline{\mu}$ by the 1/2 factor in the nonlinear term. Thus, $\overline{\mu}$ and $\epsilon$ coincide only when the particles do not interact. It is possible to obtain an identity relating the energy per particle and the chemical potential, that is:24

$$\epsilon = \frac{1}{\Lambda} \int_0^\Lambda \overline{\mu}(z) dz.$$

Substituting in Eq. (7) the previously obtained expression18 for the chemical potential, we have:

$$\epsilon = 1 + \frac{1}{4\pi} \Lambda - \frac{\ln 4/3}{8\pi^2} \Lambda^2.$$

From Fig. 1a it follows that the excitation modes are only weakly dependent on $\Lambda$, thus, the total energy of the excited state, $E_{N,0}(\Lambda) = N_\epsilon(\Lambda) + \omega_{N,0}(\Lambda)/\omega_0$, shows almost the same blue-shift dependence on $\Lambda$ as the ground state energy, $N_\epsilon(\Lambda)$. Using Eq. (5), a direct calculation of the level spacing between the EPC excited states $(N + 2, 0)$ and $(N, 0)$ yields $\Delta E_{N,m=0} \simeq 2$, only very slightly decreasing with $\Lambda$. Therefore, the spectrum of the Bogolyubov-type excitations is nearly equidistant even for rather large values of the polariton-polariton interaction parameter.

As known,16,19 the spectrum of elementary excitations around the stationary state of a spatially homogeneous Bose-Einstein condensate can be described as a function of their wavevector multiplied by so called healing length, $\sqrt{\hbar/(m\mu)}$. For small values of this product $<< 1$, a linear (phonon-type) dispersion of the Bogolyubov’s excitations is expected and its observation for a very weakly confined EPC has been reported.2 In the present case of laterally confined EPC, the wavevector is not a good quan-
condensed excitons. As explained in the Appendix A, represent a complete set for the Hamiltonian $\hat{H} = \hat{H_0} + \nu_{\text{res}}$. Therefore, the solution of Eq. (10) can be cast in the form

$$\Phi_{\text{con-res}}(\rho) = \sum_{N=0}^{\infty} c_N \varphi_{N,0}(\rho).$$

(11)

The set $\{\varphi_{N,0}(\rho)\}$ ensures the convergence, at least in mean, of the series (11) to the solution $\Phi_{\text{con-res}}$. It takes place if and only if the coefficients $\{c_N\}$ obey the relation, which is obtained by inserting Eq. (11) into Eq. (10),

$$[(N + 1 - E) \mathbf{I} + \Lambda_{\text{res}} \mathbf{M}] \mathbf{C} = 0,$$

(12)

where $\mathbf{I}$ is the unity matrix, $\mathbf{C} = (c_0, c_1, \ldots)$, and $\mathbf{M}$ is defined in Appendix B.

We solved Eq. (12) in a finite basis of dimension $N_{\text{max}}$, requiring that $|E_N^{(N_{\text{max}})} - E_N^{(N_{\text{max}}-1)}| < \delta_N^{(N_{\text{max}})}$, where $\delta_N^{(N_{\text{max}})}$ is the desired accuracy for the energy $E_N$. To warrant the accuracy of $\delta = 10^{-6}$ for all considered excited states in the range of values $0 \leq \Lambda_{\text{res}} \leq 15$, a basis set smaller than 50 oscillator wave functions $\{\varphi_{N,0}(\rho)\}$ is sufficient. Thus, the order of the symmetric matrix $\mathbf{M}$ is lower than $50 \times 50$.

Figure 1b presents the renormalized energy, $E$, versus $\Lambda_{\text{res}}$ for the non-degenerate states ($m_z = 0$) of the condensate with the principal quantum number $N$ ranging from 0 to 8 and for two different values of the dimensionless beam radius $\tilde{a}$. (Remember that $\Lambda_{\text{res}} = 0$ corresponds to the 2D harmonic oscillator energy levels). The influence of the pumping spot size is clearly seen in the figure. Two limiting cases of Eq. (12) can be distinguished. If $\tilde{a} \to 0$, the matrix $\mathbf{M} \to 0$ and the energy levels tend to the harmonic oscillator eigenvalues, $E_N = N + 1$, while in the opposite case of $\tilde{a} \to \infty$ we have $\mathbf{M} \to \delta_{N,N_1}$ (see Appendix B) and $E_N \approx N + 1 + \Lambda_{\text{res}}$. In the latter case, the level spacings, $\Delta E_N = E_{N+2} - E_N$, show a strong dependence on the laser spot size. For example, if $\tilde{a} = 0.2$, $\Delta E_N \approx 2$, as for the 2D harmonic potential. On the contrary, if $\tilde{a}$ increases, $\Delta E_{N,mz=0} \not= 2$ and it depends on the number of polaritons in the reservoir.

Figure 2 shows the influence of the reservoir size, $\tilde{a}$, on the EPC density, $|\Phi_{\text{con-res}}(\rho)|^2$, for several values of the dimensionless coupling constant $\Lambda_{\text{res}}$. From this figure it can be seen that the position of the density maximum is pushed away from the origin as $\Lambda_{\text{res}}$ increases. This effect is linked to the repulsive interactions produced by

\begin{align*}
\hat{H}_0 + \nu_{\text{res}} \Phi_{\text{con-res}} &= E \Phi_{\text{con-res}}. \\
(10)
\end{align*}
FIG. 3: Spatial distribution of the density, $|\Phi_{\text{con-res}}(\rho)|^2$, of EPC coupled to the reservoir for the excited states $N = 1, 2$ and 3 ($m_z = 0$). The influence of $\Lambda_{\text{res}}$ on the density profile is shown for the values of 0 (no interaction), 6 and 25 ($\tilde{a} = 0.2l_0$).

the Gaussian density profile of uncondensed polaritons created in the trap. The condensate is repelled from the origin as the number of uncondensed excitons created in the trap. The condensate is repelled from the Gaussian density profile of uncondensed polaritons around it. We analytically describe the condensate’s ground state and general trap. This approach allows for a convenient analytical approach to the Gross-Pitaevskii equation describing a Bose-Einstein condensate of exciton-polaritons in a semiconductor microcavity, confined in a parabolic lateral trap. Moreover, the blue shift of the condensate emission associated with the increase of the EPC ground state due to the action of $V_{\text{res}}$ has been observed in laterally confined condensates. It is also clear from Fig. 2 that the condensate becomes more delocalized as the spot width increases because its ground state energy grows, an effect characteristic of “soft” parabolic confinement.

Using Eqs. (11) and (12), we can also obtain the dependence of the condensate density profile on $\Lambda_{\text{res}}$ for the excited states. These profiles are shown in Fig. 3 for the states with $N = 1, 2$ and 3. Indeed, the condensation of polaritons in several quantized states of a trap has been observed by taking their snapshots in real and reciprocal space. The shape of the emission pattern observed in these works qualitatively corresponds to the dependence of the condensate density on $\rho$ (oscillations in real space) and $E$ (several spectral peaks under intense pumping) that comes out from our calculated results, even though the form of the lateral confinement potential is different.

V. CONCLUSIONS

In summary, we applied the perturbation theory approach to the Gross-Pitaevskii equation describing a Bose-Einstein condensate of exciton-polaritons in a semiconductor microcavity, confined in a parabolic lateral trap. This approach allows for a convenient analytical description of the condensate’s ground state and Bogolyubov-type elementary excitations around it. We derived phonon-type modes with the energies and wavefunctions depending on the polariton–polariton interaction parameter. This set of states can be used to describe the dynamics of the polariton BEC, for instance, vortices and their interaction, which is planned for a future work. We point out that the spectrum of these Bogolyubov–type excitations in a condensate whose interaction with uncondensed polaritons can be neglected, is almost equidistant even for rather larger values of the polariton-polariton interaction parameter inside the condensate. This makes polariton parabolic traps promising candidates for realization of bosonic cascade lasers.

We also considered a different physical situation where the interaction with a reservoir of uncondensed polaritons is more important than that inside it. In this case, we obtained a semi-analytical solution for the ground and excited states of the condensate, which shows how it is “reshaped” by the repulsive interaction with the reservoir, namely, pushed out from the center of the trap in real space and blue-shifted in energy. Our results are in agreement with those obtained in a number of recent experiments and numerical simulation studies. In particular, we show that the level spacings between the condensate states increase with the pump power (Fig. 1b), similar to the recent experimental observation. It can imply that the experimentally observed emission patterns in confined condensates, pumped through the polariton reservoir are not related to the Bogolyubov-type elementary excitations in the condensate itself; rather they are determined by the repulsive condensate-reservoir interaction reshaping the density profile of the former.

Acknowledgments

Financial support from the COMPETE Programme (FEDER) and the Portuguese Foundation for Science and Technology (FCT) through Projects PEst-C/FIS/UI0607/2011 and PTDC/FIS/113199/2009 is gratefully acknowledged. YNF thanks the Erasmus Mundus Programme for supporting his stay in Portugal.

Appendix A: Complete basis set and related functions

The 2D harmonic oscillator functions, solutions of the equation $\hat{H}_0 \varphi_{N,m_z} = \varepsilon_N \varphi_{N,m_z}$, in cylindrical coordinates are:

$$\varphi_{N,m_z} = \frac{1}{\sqrt{2\pi} r^m m_z!} R_{N,m}(r) \rho^m L^{(m)}(\rho^2),$$

Here

$$R_{N,m}(r) = \frac{1}{\sqrt{N_N,m}} e^{-\frac{r^2}{\rho}} \rho^m L^{(m)}(\rho^2),$$

$m_z = 0, \pm 1, \pm 2, ..., m = |m_z|$, $n_r = 0, 1, 2, ...$, is the radial quantum number, $N = 2n_r + m$, $\varepsilon_N = N + 1$ are the
it is possible to show that

\[ V. B. Timofeev, A. V. Gorbunov, and D. A. Demin, Low \]

\[ R. Balili, B. Nelsen, D. W. Snoke, L. Pfeifer, and K. West, \]

\[ 11 \]


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\[ V. B. Timofeev, A. V. Gorbunov, and D. A. Demin, Low \]

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\[ with N = 2, 4, \ldots \]

\[ \text{Appendix B: Matrix elements} \]

\[ The elements of the matrix } M \text{ introduced in Eq. (12) are defined as } \]

\[ M_{NN_1} = \langle N | \exp(-\rho^2/\bar{a}^2) | N_1 \rangle \]

\[ = \int_0^\infty L_{N/2}(t)L_{N_1/2}(t) \exp \left[ - \left( 1 + \frac{\bar{a}^2}{t} \right) \right] dt \]

\[ = \left( \frac{N + N_1}{\frac{\bar{a}^2}{2}} \right) \frac{\bar{a}^2}{(\bar{a}^2 + 1)^{\frac{N + N_1}{2} + 1}} \]

\[ \times F \left( \frac{N}{2}, -\frac{N_1}{2}; -\frac{N + N_1}{2}, 1 - \bar{a}^2 \right), \] (B1)

\[ \text{where } N \text{ and } N_1 \text{ are even numbers and } F(a, b; c, z) \text{ is the hypergeometric function.} \]

\[ \text{From Eq. (B1) we have the following properties:} \]

\[ i) \ M_{NN_1} = M_{N_1 N}; \]

\[ ii) \text{ For } N = 0, \]

\[ M_0N_1 = \frac{\bar{a}^2}{(\bar{a}^2 + 1)^{\frac{N_1}{2} + 1}}; \] (B2)

\[ \text{iii) If } \bar{a} = 0, \ M_{NN_1} = 0; \]

\[ iv) \text{ If } \bar{a} \to \infty, \text{ the function } \exp(-\rho^2/\bar{a}^2) \to 1 \text{ and } M_{NN_1} = \langle N | N_1 \rangle = \delta_{NN_1}. \]
β₂₀ = 0.0101520 and β₄₀ = 0.0119756, while the general expressions for βₙₘ=₀ will be reported elsewhere.²¹


That is, \[ \int_{0}^{\infty} \left| \Phi_{\text{con-}res}(\rho) - \sum_{N=0}^{N_{\text{max}}} c_{N,0}(\rho) \right|^2 \rho \, d\rho < \epsilon \] and \( \epsilon(N_{\text{max}}) \to 0 \) if \( N_{\text{max}} \to \infty \) (see also Ref. 27).
