A model for the prediction of the behaviour of continuous RC slabs flexurally strengthened with CFRP systems

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Keywords: NSM; CFRP; Flexural strengthening; Slabs; Analytical model.

SUMMARY
To predict the load-deflection response up to the collapse of statically indeterminate reinforced concrete (RC) structures, an analytical model was developed and its predictive performance was appraised by using the data obtained in experimental programs. The proposed approach is based on the force method by establishing a number of displacement compatibility equations that can provide the unknown variables. To determine the tangential flexural stiffness making part of these equations, moment-curvature relationships are determined for the cross sections representative of the structure. This model can be easily implemented according to a design format, and is applicable to statically determinate or indeterminate RC structures strengthened according to the near surface mounted (NSM) or externally bonded reinforcement (EBR) techniques. The predictive performance of the model was appraised by simulating two series of tests of RC slab strips strengthened with NSM carbon fibre reinforced polymer (CFRP) laminates.

1. INTRODUCTION
Indeterminate structures are being widely used since they can be more economic, safer and develop more ductile behaviour than statically determinate structures. In the case of indeterminate structures, the reactions, and consequently the internal forces, cannot be determined from direct application of the equilibrium equations. In such structures, the number of redundant supports exceeds the number of static equilibrium equations. There are, mainly, two methods for the analysis of statically indeterminate structures, whose designation is associated to the type of unknowns on the derived system of equations [1]:
1. Force method (also known as flexibility method), where a system of displacement compatibility equations is established, whose number is equal to the unknown redundant supports. Extra displacement compatibility equations can be established in sections where a certain displacement is intended to be obtained;
2. Displacement method (also known as stiffness matrix method), where a system of equilibrium equations is established, whose number is equal to the degrees of freedom of the structure.
In this work, an analytical model based on the force method is proposed for the evaluation of the force-deflection in RC slab strips flexurally strengthened with CFRP laminates according to the NSM technique. In this method, primary unknowns are reaction forces at the selected redundant supports. To determine, not only these forces but also the forces applied in the loaded sections (the tests were carried out under displacement control, so the imposed displacements are known values), extra displacement compatibility equations are established. By solving this system of equations, the forces at redundant supports and the loads at the loaded sections are determined. Once the redundant forces are calculated, the remaining reactions are evaluated by equilibrium equations, as well as the internal forces in the elements forming the structure [1].
2. ANALYTICAL MODEL

Figure 1 presents the slab strip used in the experimental program, which is statically indeterminate of one degree, e.g., a displacement compatibility equation corresponding to a reaction support should be established to determine the value of this reaction force. Assuming the principle of superposition of effects can be applied for each relatively small load increment, $\Delta F$, the structure is decomposed into a number of equilibrium configurations (each one is isostatic). In the present case, three displacement compatibility equations are established, two corresponding to the loaded sections, and the other to the intermediate support, in order to obtain the incremental forces ($\Delta F_1$ and $\Delta F_2$) and the incremental reaction ($\Delta R$) due to $\Delta F$ (Figure 2).

For each equilibrium configuration, the incremental forces ($\Delta F_1$ and $\Delta F_2$) corresponding to the imposed incremental displacements ($\Delta u_1$ and $\Delta u_2$) and the reaction $\Delta R$ are determined (Figure 2). The terms of the flexibility matrix, $f_{\Delta F_1, \Delta F_i}$, $f_{\Delta R, \Delta F_i}$, $f_{\Delta F_2, \Delta F_i}$, $f_{\Delta F_1, \Delta R}$, $f_{\Delta F_2, \Delta R}$, and $f_{\Delta F_1, \Delta R}$, with a generic representation, $f_{ij}$, that means the displacement in generalized $X_i$ force direction due to the application of an unit load in the $X_j$ direction. Each $f_{ij}$ is obtained by applying the principle of virtual work to the external and internal forces of the $X_i$ configuration (herein designated by $C_i$), in the external and internal displacements of the $X_j$ configuration (herein designated by $C_j$) [1]. The diagrams of bending moments for the three configurations of Figure 2 are represented in Figure 3. In the present work it is neglected the work due to axial and shear forces, since their contribution is assumed marginal for the present type of structures.

The evaluation of each term of the flexibility matrix (only considering the internal work due to bending) is obtained from [1]:

$$f_{ij} = \int_0^L \frac{M_i M_j}{EI} dL = \sum_{k=1}^{\text{mol}} \left[ \frac{M_{i,k} M_{j,k}}{(EI)_k} \right] dL_k$$

(1)
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Figure 3: Physical meaning of the terms of the flexibility matrix, based on the displacements for each equilibrium configuration: a) $\Delta F_1$, b) $\Delta F_2$, and c) $\Delta R$.

where $L$ is the length of the beam, $EI$ is the tangential flexural stiffness of the element of length $dL$, $M_j$ is the bending moment installed in this element corresponding to the equilibrium configuration $C_j$, and $M_i$ is the moment in this element in the equilibrium configuration $C_i$. The integral is transformed in the evaluation of the $(EI)_{ij} dL$ in the number of elements ($nel$) adopted to discretize the beam of length $L$. The $(EI)_{kk}$ is obtained from the moment-curvature relationship for the cross section of the element $k$. From Figure 3 the following three equations of displacements compatibility can be established by applying the principle of superposition effects:

$$\Delta u_1 = f_{SF1,AF1} \times \Delta F_1 + f_{SF1,AF2} \times \Delta F_2 + f_{SF1,AR} \times \Delta R$$
$$\Delta u_2 = f_{SF2,AF1} \times \Delta F_1 + f_{SF2,AF2} \times \Delta F_2 + f_{SF2,AR} \times \Delta R$$
$$0 = f_{AR,AF1} \times \Delta F_1 + f_{AR,AF2} \times \Delta F_2 + f_{AR,AR} \times \Delta R$$

or

$$\begin{bmatrix}
    f_{SF1,AF1} & f_{SF1,AF2} & f_{SF1,AR} \\
    f_{SF2,AF1} & f_{SF2,AF2} & f_{SF2,AR} \\
    f_{AR,AF1} & f_{AR,AF2} & f_{AR,AR}
\end{bmatrix}
\begin{bmatrix}
    \Delta F_1 \\
    \Delta F_2 \\
    \Delta R
\end{bmatrix}
= \begin{bmatrix}
    \Delta u_1 \\
    \Delta u_2 \\
    0
\end{bmatrix}$$

that can get the following format:

$$\begin{bmatrix}
    f
\end{bmatrix}
\begin{bmatrix}
    \Delta F
\end{bmatrix}
= \begin{bmatrix}
    \Delta U
\end{bmatrix}$$

where $f$ is the tangent flexibility matrix, $\Delta F$ is the vector of unknown applied forces ($\Delta F_1$, $\Delta F_2$) and reaction $\Delta R$, and $\Delta U$ is the vector of the imposed incremental displacements in the directions of
\( \Delta F_1, \Delta F_2 \) and \( \Delta R \) (the displacement corresponding to \( \Delta R \) is null). By solving equation (4) in terms of the vector of the unknown incremental forces the \( \Delta F \) is obtained:

\[
[\Delta F] = [f]^{-1} [\Delta U] \tag{5}
\]

or

\[
\begin{bmatrix}
\Delta F_1 \\
\Delta F_2 \\
\Delta R
\end{bmatrix} =
\begin{bmatrix}
f_{s1,F1} & f_{s1,F2} & f_{s1,SR} \\
f_{s2,F1} & f_{s2,F2} & f_{s2,SR} \\
f_{R,F1} & f_{R,F2} & f_{R,SR}
\end{bmatrix}^{-1}
\begin{bmatrix}
\Delta u_1 \\
\Delta u_2 \\
0
\end{bmatrix} \tag{6}
\]

Therefore, imposing for each loading step the increment of displacements adopted in the experimental tests (\( \Delta u_1 \) and \( \Delta u_2 \), where \( \Delta u_1 = \Delta u_2 \)), and solving the equation (5), the unknown incremental forces \( \Delta F_1, \Delta F_2 \) and \( \Delta R \) are obtained. Knowing these values, the updated diagrams of internal resultant stresses are determined for each loading step by applying fundamental principles of statics. The developed algorithm is described in Figure 4. After the initialization of the accumulative variables of the formulation (for the vector of the incremental and total forces, \( F^0 = \Delta F^0 = 0 \), an unitary vector was assumed), block (1), a loop on the incremental displacement is executed, \( \Delta u^q \), up to an assumed maximum deflection, \( u_{max} \). The bending moments are updated in block (2), and the tangential flexibility matrix is determined in block (3) by evaluating Eq. (1). The vector of the incremental forces is determined in block (4) by applying Eq. (5) and the total force vector is calculated in block (5). The norm of residual forces is evaluated in block (6) and if this norm is lower than an adopted tolerance (Toler), the total deflection is updated (block (7)), otherwise a new iteration is executed adopting for the new \( \Delta F \) in the equation of the block (2) the vector determined previously in the block (4). The incremental procedure in terms of deflection terminates when the total deflection attains \( u_{max} \).

### 3. ASSESSMENT OF THE PREDICTIVE PERFORMANCE OF THE MODEL CASE STUDY

#### 3.1 Short description of the experimental program and relevant results

To assess the predictive performance of the developed model, it was applied to the simulations of the tests of two experimental programs composed of RC slab strips flexurally strengthened according to the CFRP NSM technique [2, 3]. The experimental program was composed of eight 120×375×5875 mm\(^3\) RC slab strips (Figure 1). Three of them were unstrengthened RC slabs forming a control set (SL15-HS, SL30-HS and SL45-HS), and the other five slabs were strengthened with CFRP strips according to the NSM technique (SL15s25-HS, SL30s25-HS, SL30s50-HS, SL45s25-HS and SL45s50-HS) applied in both sagging and hogging regions (HS Series). The amount and disposition of the steel bars were designed to assure a moment redistribution percentages of 15\%, 30\% and 45\%. The NSM CFRP systems applied in the flexurally strengthened RC slabs were designed to increase in 25\% and 50\% the load carrying capacity of the reference slab. From the obtained results, it was verified that the strengthening configurations composed by laminates only applied in the hogging region did not attain the target increase of the load carrying capacity. In fact, when the CFRP laminates were applied in the hogging region, an increase of the load carrying capacity of 8.02\%, 19.76\%, 5.93\%, 9.15\%, 2.86\% and 8.46\% was registered for the slab strips SL15s25-H, SL15s50-H, SL30s25-H, SL30s50-H, SL45s25-H and SL45s50-H, respectively. These values were obtained when a concrete compressive strain of 3.5‰ was recorded in the sagging regions. When applying CFRP laminates in both sagging and hogging regions (HS series), an increase of the load carrying capacity of 36.11\%, 29.84\%, 49.44\%, 24.42\% and 37.24\% was attained for the slab strips SL15s25-HS, SL30s25-HS, SL30s50-HS, SL45s25-HS and SL45s50-HS, respectively. Thus, the target increase of the load carrying capacity was attained.
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Figure 4: Algorithm to derive the force-deflection relationship.

Loop of displacement increments, \( \Delta u^y \), up to \( u^{max} \)

Update the bending moments, \( M^y \)

Evaluation of the tangential flexibility matrix, \( f^y \)

Determine \( \Delta F^y \)

Update \( F^y \)

\[ M_y = 0 \quad \Delta F_0 = I \]
\[ F_2 = \Delta F_0 \quad \mu = 0 \]
\[ \Delta u^y = \Delta u_n \quad u_{max} \]

\[ M^y = M^{y-1} + \begin{bmatrix} \Delta F_1 & \Delta F_2 & \Delta R \end{bmatrix} \begin{bmatrix} M_{AF_1} \\ M_{AF_2} \\ M_{AF_3} \end{bmatrix} \]

\[ f^y = \int_0^L M^y \frac{M^y}{EI} d_l = \sum_{k=1}^{n} \left[ M_{AF_k} \frac{M^y}{(EI)_k} d_{l_k} \right] \]

\[ \Delta F^y = \left( f^y \right)^{-1} \Delta u^y \]

\[ F^y = F^{y-1} + \Delta F^y \]

\[ \frac{\| F^y - F^{y-1} \|}{\| F^{y-1} \|} < Toler \]

\[ u^y = u^{y-1} + \Delta u^y \]

\[ u^y < u^{max} \]

END
3.2 Simulations

Due to space limitation, the applicability of the model is described for the SL15-H slab, but this methodology is extensive to the other slabs [3]. The SL15-H is a statically indeterminate RC slab strip designed to assure a moment redistribution percentage, $\eta$, of 15% [2]. The arrangement of the top and bottom longitudinal steel reinforcement is presented in Figure 5. The tangential flexural stiffness of each $k$ element that discretize the slab, $(EI)_k$, was determined from the moment-curvature relationship, $M - \chi$, obtained from a cross section layer model described in [4]. For this purpose, each span of the slab strip was discretized in six different cross-sections, as shown in Figure 6, in order to take into account the different reinforcement arrangements along the slab (Figure 7). The $M - \chi$ of the cross sections was evaluated with the DOCROS computer program [4, 5]. According to the model implemented in DOCROS, a cross section is discretized in layers that can have distinct constitutive laws for the characterization of the behaviour of the materials that constitute these layers. It should be noted that the cross section can be composed of plain concrete and can include steel and FRP laminates/bars. A detailed description of DOCROS can be found elsewhere [5]. Tables 1 and 2 present a brief resume of the relevant material properties of the slabs simulated in the present work. The letter “H” and “HS” in the designation of the slabs mean that the slab is only strengthened with CFRP laminates at the hogging region and at both hogging and sagging regions (Figure 1), respectively [3].

![Figure 5: Arrangement of the longitudinal steel reinforcement of the SL15-H slab strip.](image)

![Figure 6: Discretization of the slab strip SL15-H.](image)
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For the flexural strengthening of the RC slab strips two types of CFRP laminates were used. One with a cross section of 1.4×10mm² and the other with a cross section of 1.4×20mm². The ultimate tensile strain and the modulus of elasticity for these two types of CFRP laminates were, respectively: 17.67‰ and 159.30 GPa, and 17.76‰ and 156.69 GPa. These laminates were installed into grooves opened on the concrete cover of the RC slabs and bonded to the concrete substrata using epoxy adhesive. The strengthening procedures and the properties of the adhesive are described in [3]. The values of the parameters that define the constitutive models used in the DOCROS to simulate the nonlinear behaviour of concrete and steel are included in Tables 1 and 2, respectively. Figure 8 represents the $M - \chi$ relationships for the cross sections representative of the SL15-H slab strip.

Table 1: Concrete properties used in DOCROS to simulate the $M - \chi$ of the representative cross sections of the simulated slabs.

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Initial Young's modulus (GPa)</td>
<td>33.36</td>
<td>29.43</td>
<td>32.30</td>
<td>30.09</td>
<td>33.69</td>
<td>33.93</td>
</tr>
<tr>
<td>Strain at peak compressive stress (%)</td>
<td>-2.20</td>
<td>-2.20</td>
<td>-2.20</td>
<td>-2.20</td>
<td>-2.20</td>
<td>-2.20</td>
</tr>
<tr>
<td>Compressive strength (MPa)</td>
<td>40.07</td>
<td>26.37</td>
<td>35.99</td>
<td>28.40</td>
<td>41.41</td>
<td>42.38</td>
</tr>
<tr>
<td>Post-peak strain on the compression envelope (‰)</td>
<td>-7.69</td>
<td>-6.75</td>
<td>-7.44</td>
<td>-6.91</td>
<td>-7.77</td>
<td>-7.83</td>
</tr>
<tr>
<td>Tensile strength (MPa)</td>
<td>3.03</td>
<td>2.09</td>
<td>2.76</td>
<td>2.24</td>
<td>3.11</td>
<td>3.17</td>
</tr>
<tr>
<td>Post-peak strain on the tension envelope curve (‰)</td>
<td>0.58</td>
<td>0.53</td>
<td>0.57</td>
<td>0.54</td>
<td>0.58</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Table 2: Properties of the steel bars used in DOCROS to simulate the $M - \chi$ of the representative cross sections of the simulated slabs.

<table>
<thead>
<tr>
<th>Steel bar diameter</th>
<th>Yielding strain (-); stress (MPa)</th>
<th>Hardening strain (-); stress (MPa)</th>
<th>Ultimate strain (-); stress (MPa)</th>
<th>Es [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ø 8mm</td>
<td>(1.30x10^-3; 461.84)</td>
<td>(4.05x10^-2; 550.00)</td>
<td>(1.50x10^-1; 578.00)</td>
<td>200.80</td>
</tr>
<tr>
<td>Ø 10mm</td>
<td>(2.70x10^-3; 481.24)</td>
<td>(5.00x10^-2; 500.00)</td>
<td>(1.25x10^-1; 546.25)</td>
<td>178.24</td>
</tr>
<tr>
<td>Ø 12mm</td>
<td>(2.40x10^-3; 476.06)</td>
<td>(5.00x10^-2; 500.00)</td>
<td>(1.25x10^-1; 537.98)</td>
<td>198.36</td>
</tr>
</tbody>
</table>
Figure 8: $M - \chi$ relationship for the cross section of the SL15-H slab strip.

By applying the algorithm described in Figure 4, the force-deflection response was obtained, which are compared in Figure 9 with the ones registered experimentally. In this figure $F_{(522)}$ and $F_{(123)}$ is the force recorded experimentally in the loaded section at the left and right span of the slab strip, respectively (Figure 1). Figure 9 evidences that the developed model is capable of predicting the response of these type of structures with high accuracy up to a very high deflection level.

5. CONCLUSIONS
In this work a model was proposed to evaluate the force-deflection relationship of statically indeterminate reinforced concrete (RC) strips flexurally strengthened with carbon fibre reinforced polymer (CFRP) laminates applied according to the near surface mounted technique (NSM). This model is based on the force method by establishing a set of compatibility equations to derive the unknown variables (incremental applied loads and incremental reaction on the redundant support for the imposed increment of deflections). To determine the tangential flexibility matrix subjacent to this set of equations, the flexural stiffness of the representative cross sections of the slab strip is determined from a layer model capable of deriving the moment-curvature relationship of these cross sections. By simulating the force-deflection response recorded in two experimental programs, the good predictive performance of this analytical/numerical strategy was demonstrated.
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Figure 9: Experimental and analytical/numerical force-deflection relationship for the simulated tested slab strips.

ACKNOWLEDGMENTS
This study is part of the research program “PRELAMI - PTDC/ECM/114945/2009” supported by FCT. The second author would like to acknowledge the National Council for Scientific and Technological Development (CNPq) for scholarship (GDE 200953/2007-9).

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