A STOCHASTIC BURGERS EQUATION FROM A CLASS OF MICROSCOPIC INTERACTIONS

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ABSTRACT. We consider a class of nearest-neighbor weakly asymmetric mass conservative particle systems evolving on $\mathbb{Z}$, which includes zero-range and types of exclusion processes, starting from a perturbation of a stationary state. When the weak asymmetry is of order $O(n^{-\gamma})$ for $1/2 < \gamma \leq 1$, we show that the scaling limit of the fluctuation field, as seen across process characteristics, is a generalized Ornstein-Uhlenbeck process. However, at the critical weak asymmetry when $\gamma = 1/2$, we show that all limit points solve a martingale problem which may be interpreted in terms of a stochastic Burgers equation derived from taking the gradient of the KPZ equation. The proofs make use of a sharp 'Boltzmann-Gibbs' estimate which improves on earlier bounds.

1. INTRODUCTION

There has been much recent work on the classification of fluctuations of certain interfaces and currents, corresponding to mass conservative particle dynamics in one dimensional nearest-neighbor interacting particle systems such as simple exclusion and its variants, with respect to so-called Edwards-Wilkinson (EW) and Kardar-Parisi-Zhang (KPZ) classes (cf. [20] for a review and references). Following recent sensibilities, a $d = 1$ particle system is in the EW class if the standard deviation of the associated 'height' function $h_t(x)$ of the interface at time t and space point x, or the integrated current at time $t \geq 0$ across the space point $x \in \mathbb{R}$, is of order $t^{1/4}$, and also spatial correlations are nontrivial at range $t^{1/2}$. Examples in this class are independent random walk systems, random averaging, and reversible simple exclusion processes starting from a stationary state or even in non-stationary states [9], [23], [32], [39], [58].

On the other hand, a system is in the KPZ class if its 'height' function and integrated current have standard deviation of order $t^{1/3}$, and nontrivial spatial correlations at range $t^{2/3}$. A well-studied particle system model in this class is the asymmetric simple exclusion process starting from deterministic initial configurations such as step profile and alternating conditions, or from a stationary state (cf. [7], [8], [10], [16], [17], [25], [42], [49], [52], [63] and references therein).

These two classes can be seen in the study of the famous KPZ stochastic partial differential equation first mentioned in [35]:

$$\partial_t h_t(x) \;=\; D\Delta h_t(x) + a\big(\nabla h_t(x)\big)^2 + \sigma\dot{\mathcal{W}}_t(x) \tag{1.1}$$

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where $\dot{\mathcal{W}}_t(x)$ is a space-time white noise with unit variance. When $a = 0$ and $D, \sigma > 0$, then $h_t(x)$ is a generalized Ornstein-Uhlenbeck process in EW class. However, when $a \neq 0$ and $D, \sigma > 0$, a physical argument indicates that $h_t(x)$ is in the KPZ class (cf. [13], [35]). Also, in another sense, it has been shown that the 'Cole-Hopf' solution of the KPZ equation, starting from certain initial conditions, interpolates between the two classes when the centered solution is examined in different asymptotic scaling regimes, that is when normalized by $t^{1/3}$ as $t \uparrow \infty$ or when normalized by $t^{1/4}$ as $t \downarrow 0$, nontrivial limits are obtained (cf. [1], [12]).

Moreover, it is believed that in many 'critical' weakly asymmetric, $d = 1$ particle systems, that is when the weak asymmetry is scaled at a critical level, the diffusively scaled 'height' function or integrated current should converge to the solution of the KPZ equation with parameters depending on the structure of particle interactions and initial conditions. Recently much progress has been made in making clear this convergence. Part of the difficulty is that, since 'solutions' to the KPZ equation are expected to be distribution-valued, the nonlinear term in the equation does not make sense, and so the equation is ill-posed. Hence, what does it mean to solve the KPZ equation? And also, when properly interpreted, how to derive the KPZ equation from microscopic particle interactions?

One way to approach these questions is to observe that the Cole-Hopf transform $z_t(x) := \exp\{(a/D)h_t(x)\}$ linearizes the KPZ equation to a stochastic heat equation

$$\partial_t z_t(x) \;=\; D\Delta z_t(x) + (a\sigma/D) z_t(x)\dot{\mathcal{W}}_t(x) \tag{1.2}$$

which can be solved uniquely starting from a class of initial conditions and is also strictly positive for times $t > 0$ [47], [64]. Then, the 'Cole-Hopf' solution is defined as $h_t(x) := \log z_t(x)$. In [15], starting from near stationary measures in a certain weakly asymmetric simple exclusion process observed in diffusive scale, this sentiment was made rigorous. Namely, it was proved that the microscopic Cole-Hopf transform of the microscopic height function, using a clever device in [26] which linearizes the simple exclusion dynamics to a more manageable system, converges to the Cole-Hopf transform of the KPZ equation, the solution to the stochastic heat equation (1.2). More recently, in [1], [56] this notion of solution further gained traction in that the result in [15] was non-trivially generalized to step profile deterministic initial configurations. At the same time, in [31], it has been shown that $\log z_t(x)$ is the unique solution of a well-posed equation on a torus (the question on $\mathbb{Z}$ seems open), derived from a 'rough paths' approximation of (1.1), so that it is clear what sort of KPZ equation the 'Cole-Hopf' solution actually solves.

In this article, another approach is considered which allows to generalize the types of microscopic particle interactions considered, given that the device in [26] seems limited to simple exclusion and a few variants such as q-TASEP dynamics [19]. At the microscopic level, the height function $H_t(x)$, evaluated for $t \geq 0$ and $x \in \mathbb{Z}$, takes form

$$H_t(x) \;=\; \begin{cases} J_0(t) - \sum_{y=0}^{x-1} \eta_t(y) & \text{for } x \geq 1 \\ J_0(t) & \text{for } x = 0 \\ J_0(t) + \sum_{y=x}^{-1} \eta_t(y) & \text{for } x \leq -1 \end{cases} \tag{1.3}$$

where $J_y(t)$ is the current across bond $(y-1, y)$ and $\eta_t(y)$ is the particle number at $y \in \mathbb{Z}$ at time $t \geq 0$. Then, the discrete gradients of the microscopic height function are the particle numbers, $H_t(x+1) - H_t(x) = \eta_t(x)$, and the corresponding fluctuation field examined in diffusive scale, that is when time is scaled in terms of n^2 and space is scaled by n, is the particle density fluctuation field $\mathcal{Y}_t^n$. The guiding idea is that $\mathcal{Y}_t^n$ should converge to $\mathcal{Y}_t = \nabla h_t$ in some sense.

Formally, by carrying through the '∇' operation, $\mathcal{Y}_t$ satisfies a type of stochastic Burgers equation,

$$\partial_t \mathcal{Y}_t(x) \;=\; D\Delta \mathcal{Y}_t(x) + a\nabla\big(\mathcal{Y}_t(x)\big)^2 + \sigma\nabla\dot{\mathcal{W}}_t(x) \tag{1.4}$$

which again for the same reasons as for the original KPZ equation is ill-posed when $a \neq 0$. If $a = 0$, however, it is a type of Ornstein-Uhlenbeck equation which possesses a unique solution when starting from a large class of initial distributions (cf. [14], [64]).

A main contribution of the article is to understand the derived stochastic Burgers equation (1.4) in the context of a general class of nearest-neighbor weakly asymmetric interacting particle systems on $\mathbb{Z}$, starting from perturbations of the invariant measure ν_ρ. This class is composed of systems with 'gradient' dynamics, not necessarily product invariant measures, sufficient spectral gap and 'equivalence of ensembles' estimates among other technical conditions (cf. Subsection 2.1), which include in particular the already studied simple exclusion process, and also zero-range and exclusion models with kinetically constrained or speed-change interactions, which have varying and sometimes slow mixing behaviors. The initial distributions consist of 'bounded entropy' perturbations of the invariant measure ν_ρ (cf. Subsection 2.1 for a precise statement).

Our results describe the limit points of the fluctuation field $\mathcal{Y}_t^{n,\gamma}$ in diffusive scale, in a reference frame moving with a process characteristic velocity $\upsilon_n(t) \sim n^{-1}\lfloor n^2(p_n - q_n)\upsilon t\rfloor$. Here $p_n - q_n$ is the difference of the single particle jump rates which identifies the strength of the weak asymmetry considered, and υ is a homogenized velocity parameter depending on the particle dynamics. Given the size of $p_n - q_n$, a dichotomy emerges in the form of the limits derived. Namely, for $p_n - q_n = O(n^{-\gamma})$, when $1/2 < \gamma \leq 1$, we show a 'crossover result' (Theorem 2.2) that $\mathcal{Y}_t^{n,\gamma}$ converges to an Ornstein-Uhlenbeck field with certain homogenized parameters. When $\gamma = 1$, convergence of $\mathcal{Y}_t^{n,\gamma}$ to the same Ornstein-Uhlenbeck field has been known for many particle systems since the work [18]. For discussions of 'crossover' results with respect to simple exclusion see [57], [29].

However, when $\gamma = 1/2$, a critical value, we prove (Theorem 2.3) that limit points of $\mathcal{Y}_t^{n,\gamma}$ solve a martingale problem, also with homogenized constants, which interprets the stochastic Burgers equation, namely the non-linear term in (1.4) is understood in terms of a certain Cauchy limit of a function of the fluctuation field acting on an approximation of a point mass as the approximation becomes more refined. In this context, we note [5] further clarifies the limit point found starting from the invariant state ν_ρ with respect to the simple exclusion process. Also, we note another martingale problem was given with respect to the Burgers equation in [3].

Convergence of $\mathcal{Y}_t^{n,\gamma}$ to a unique limit when $\gamma = 1/2$ is known with respect to the simple exclusion process (cf. [15]), although it has not been shown yet in our more general framework, an important open question. However, one may still try to characterize limit points of the height function across process characteristics, $H_t^{n,\gamma}(x) := n^{-1/2}H_{n^2t}\big(nx - nv_n(t)\big)$, via (1.3) given subsequential convergence of $\mathcal{Y}_t^{n,\gamma}$. Although this is not the purpose of this paper, we indicate how this might be accomplished to be more complete. Indeed, by (1.3) and $J_0(t) - J_{x+1}(t) = \sum_{y=0}^{x}\big(\eta_t(y) - \eta_0(y)\big)$, one has $H_t^{n,\gamma}(x) = n^{-1/2}J_{nx-nv_n}(n^2t) - n^{-1/2}\sum_{y=0}^{nx-nv_n(t)}\eta_0(y)$, say for $x > 0$. To write the current in terms of the fluctuation field, formally, $n^{-1/2}J_{nx-nv_n(t)}(n^2t) = \mathcal{Y}_t^{n,\gamma}(1_{[x,\infty)}) - \mathcal{Y}_0^{n,\gamma}(1_{[x,\infty)}) + o(1)$, although as there are an infinite number of particles and $1_{[x,\infty)}$ is not a compactly supported function some sort of truncation is needed to make a rigorous argument. Using the method in [55] and [32], one can approximate $n^{-1/2}J_{nx-nv_n(t)}(n^2t)$ by $\mathcal{Y}_t^{n,\gamma}(G_{k,x})$ for large k where $G_{k,x}(z) = (1-(z-x)/k)_+$, and so it is possible to take subsequential limits of $H_t^{n,\gamma}$. Finally, we remark if uniqueness of solution for the $\gamma = 1/2$ martingale problem were known in our more general framework, one should be able to identify the solution, modulo parameters, as the limit already identified for simple exclusion through the Cole-Hopf apparatus. In this way, one should be able to determine that the height function limits, with respect to a general class of interactions starting from nearly the invariant measure, are in the KPZ class for instance.

We now remark on the argument for Theorems 2.2 and 2.3. We take a stochastic differential of $\mathcal{Y}_t^{n,\gamma}$, namely

$$d\mathcal{Y}_t^{n,\gamma} \;=\; \Big(\partial_t\mathcal{Y}_t^{n,\gamma} + L_n\mathcal{Y}_t^{n,\gamma}\Big)dt + d\mathcal{M}_t^{n,\gamma}$$

where L_n is the system infinitesimal generator and $\mathcal{M}_t^{n,\gamma}$ is a martingale. We note, because the reference frame moves with velocity $v_n(t)$, the term $\partial_t\mathcal{Y}_t^{n,\gamma}$ does not vanish. Beginning in a perturbed invariant measure, the martingale term can be handled by an ergodic theorem. However, to write the drift term $\partial_t\mathcal{Y}_t^{n,\gamma} + L_n\mathcal{Y}_t^{n,\gamma}$, in terms of the fluctuation field itself and therefore 'close' the equation, is a more difficult task, and requires what has been known as a 'Boltzmann-Gibbs' principle. Such a principle, first proved in [18] when $\gamma = 1$, would replace in our context the expression

$$\int_0^t \frac{1}{n^{\gamma-1/2}}\sum_{x\in\mathbb{Z}}\nabla G(x/n)\tau_x V(\eta_{n^2s})ds$$

with

$$\frac{\varphi_V''(\rho)}{2}\int_0^t \frac{1}{n^{\gamma+1/2}}\sum_{x\in\mathbb{Z}}\nabla G(x/n) \times\Big[\mathcal{Y}_s^{n,\gamma}\Big(\frac{1}{2\varepsilon}1_{[x-\varepsilon,x+\varepsilon]}\Big)^2 - \mathbb{E}_{\nu_\rho}\Big[\mathcal{Y}_s^{n,\gamma}\Big(\frac{1}{2\varepsilon}1_{[x-\varepsilon,x+\varepsilon]}\Big)^2\Big]ds$$

in $L^2(\mathbb{P}_{\nu_\rho})$ as $n\uparrow\infty$ and $\varepsilon\downarrow 0$. Here, G is a function in the Schwarz class, τ_x is the x-shift operator, V is a mean-zero function with the property that the derivative of its 'tilted mean' $\varphi_V(z)$ vanishes at $z=\rho$ (cf. definition near (2.4)). Given such a replacement principle (cf. Subsection 3.2 for precise statements), one can prove the sequence $\mathcal{Y}_t^{n,\gamma}$ is tight and derive martingale problem characterizations of limit points as desired.

The case $\gamma = 1/2$ is the most difficult since there is no spatial averaging at all. However, there is much cancelation with respect to the time integral which helps to prove the replacement needed. We show the cases $1/2 < \gamma \leq 1$ would follow from the $\gamma = 1/2$ replacement. A similar replacement for symmetric simple exclusion, using specific duality methods, was performed in [4].

The method given here, in our general framework, is quite different. The main idea is to use an involved H_{-1} renormalization scheme to bound errors in the replacement. Such a scheme makes good use of three assumed ingredients (cf. precise statements (R), (G), (EE) in Subsection 2.1): First, the measure ν_ρ is invariant with respect to all asymmetric and symmetric versions of the process, the main reason for the 'gradient dynamics' condition. Second, a spectral gap lower bound for the symmetric process localized on a interval Λ_ℓ with width ℓ and $\sum_{x\in\Lambda_\ell}\eta(x)$ particles which, after averaging with respect to ν_ρ, is of order $O(\ell^{-2})$. Also, third, an 'equivalence of ensembles' estimate holds with respect to canonical $\nu_\rho(\cdot|\sum_{|x|\leq\ell}\eta(x) = k)$ and grand canonical ν_ρ measures.

We note the current article is an evolution of the arXiv paper [28], encompassing the work there on a type of exclusion model starting from a Bernoulli product invariant measure and a model specific Boltzmann-Gibbs principle. See also [5] for a different type of resolvent method specific to simple exclusion. In this context, the current article is a nontrivial generalization to more diverse models, starting from perturbations of the stationary state, using a more general H_{-1} renormalization scheme. We remark that part of this improvement, of its own interest, is that the Boltzmann-Gibbs principle (Theorem 3.2) shown here does not rely on the independence structure of a product measure, or on a sharp spectral gap estimate, or on a process 'duality'. Finally, we note elements of our H_{-1} renormalization scheme go back to [27] and [61] in different contexts.

We now give the structure of the article. In Section 2, the general class of models studied, results, and specific systems satisfying the class assumptions are discussed. Then, in Section 3, we outline the proof of the main results, Theorems 2.2 and 2.3, stating the form of 'Boltzmann-Gibbs' principle used. In Section 4, this principle is proved. In Section 5, we prove for a class of systems, including the specific processes discussed in Section 2, the 'equivalence of ensembles' estimate assumed for the proofs in Section 3. Finally, in Section 6, we show that the field convergences in Theorems 2.2 and 2.3 can be restricted to a Hermite Hilbert space of functions which strictly contains the Schwarz space.

2. Abstract Framework, Results, and Models

We now discuss the abstract framework we work with in Subsection 2.1, and state results in this framework in Subsection 2.2. This framework covers a wide class of models such as zero-range models and different types of exclusion processes which we detail in Subsections 2.3 - 2.5. A reader focusing on one of these models, might skip to its subsection while referring to Subsection 2.1, and then proceed to results in Subsection 2.2.

2.1. **Notation and Assumptions.** We consider a sequence of 'weakly asymmetric' nearest-neighbor 'mass conservative' particle systems $\{\eta^n_t : t \geq 0\}$ on the state space $\Omega = \mathbb{N}_0^{\mathbb{Z}}$ where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The configuration of the system $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ is a collection of occupation numbers $\eta_t(x)$ which counts the numbers of particles at sites $x \in \mathbb{Z}$ at time $t \geq 0$. In some of the examples we will consider, the occupation number is bounded by 1, in which case the effective state space reduces to $\{0,1\}^{\mathbb{Z}}$.

'Gradient' dynamics. The dynamics will be of 'gradient' type. That is, we suppose there are functions $\{b^{R,n}_x\}_{n\geq 1}$ and $\{b^{L,n}_x\}_{n\geq 1}$ satisfying the following conditions (R1) and (R2). Let τ_x be the shift operator where $(\tau_x\eta)(z) = \eta(x+z)$ and $\tau_x f(\eta) = f(\tau_x\eta)$ for $x \in \mathbb{Z}$. Let also $\Lambda_k = \{j : |j| \leq k\} \subset \mathbb{Z}$ for $k \geq 1$.

(R1) For all $n \geq 1$, $b^{R,n}_x = \tau_x b^{R,n}_0$ and $b^{L,n}_x = \tau_x b^{L,n}_0$ are nonnegative finite-range functions on Ω such that $b^{R,n}_0$ and $b^{L,n}_0$ are supported on $\{\eta(y) : y \in \Lambda_R\}$ for some $R \geq 1$. We suppose uniformly in n that $|b^{R,n}_0(\eta)| + |b^{L,n}_0(\eta)| \leq C\sum_{y\in\Lambda_R}\eta(y)$. Moreover, there are nonnegative functions $c^n_x = \tau_x c^n$ on Ω, supported on $\{\eta(x) : x \in \Lambda_R\}$ such that

$$b^{R,n}_x(\eta) - b^{L,n}_x(\eta) \;=\; c^n_x(\eta) - c^n_{x+1}(\eta).$$

In addition, suppose there are fixed functions b^R_0, b^L_0 and c_0 such that configurationwise

$$\lim_{n\uparrow\infty} b^{R,n}_0(\eta) = b^R_0(\eta), \quad \lim_{n\uparrow\infty} b^{L,n}_0(\eta) = b^L_0(\eta), \quad \text{and} \quad \lim_{n\uparrow\infty} c^n_0(\eta) = c_0(\eta).$$

In some of the models considered, such as zero-range processes in Subsection 2.3, the functions $b^{R,n}_0 = b^R_0$, $b^{L,n}_0 = b^L_0$ and $c^n_0 = c_0$ are fixed and do not depend on the parameter n. However, for the kinetically constrained exclusion models in Subsection 2.4, the rates do depend on n.

(R2) With respect to a fixed measure ν_ρ on Ω, for all $n \geq 1$, we have

$$b^{R,n}_x(\eta^{x+1,x})\frac{d\nu^{x+1,x}_\rho}{d\nu_\rho}(\eta) \;=\; b^{L,n}_x(\eta)$$

where $\nu^{x+1,x}_\rho$ is the measure of the variable $\zeta = \eta^{x+1,x}$ under ν_ρ.

We also define $b^n_x(\eta) = b^{R,n}_x(\eta) + b^{L,n}_x(\eta)$, $b^n(\eta) = b^n_0(\eta)$ and $c^n(\eta) = c^n_0(\eta)$ to simplify notation.

We now specify the process generator. For $a \in \mathbb{R}$ and $\gamma > 0$, let

$$p_n = \frac{1}{2} + \frac{a}{2n^\gamma} \quad \text{and} \quad q_n = 1 - p_n = \frac{1}{2} - \frac{a}{2n^\gamma}.$$

Let also n_0 be such that $0 \leq p_{n_0}, q_{n_0} \leq 1$, and $T > 0$ be a fixed time.

(M) Suppose, for each $a \in \mathbb{R}$ and $\gamma > 0$, that $\{\eta^n_t : t \in [0,T]\}$ is a $L^2(\nu_\rho)$ Markov process with strongly continuous Markov semigroup P^n_t and Markov generator L_n (cf. [43][Chapter I; Section IV.4]) with a core composed of local $L^2(\nu_\rho)$ functions on which

$$L_n f(\eta) \;=\; n^2\sum_{x\in\mathbb{Z}}\Big\{b^{R,n}_x(\eta)p_n\nabla_{x,x+1}f(\eta) + b^{L,n}_x(\eta)q_n\nabla_{x+1,x}f(\eta)\Big\} \tag{2.1}$$

where $\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta)$, and $\eta^{x,y}$ is the configuration obtained from η by moving a particle from x to y:

$$\eta^{x,y}(z) \ = \ \begin{cases} \eta(y)+1 & \text{when } z=y \\ \eta(x)-1 & \text{when } z=x \\ \eta(z) & \text{otherwise.} \end{cases}$$

The role of $a \in \mathbb{R}$ and $\gamma > 0$ is to control the strength of the 'weak asymmetry' in the model.

Invariant measure ν_ρ. We now specify some technical properties which ν_ρ should satisfy. Define for a probability measure κ, the path measure $\mathbb{P}_\kappa$ governing the process $\{\eta^n_t : t \in [0,T]\}$ with initial configurations η_0 distributed according to κ. Let then E_κ and $\mathbb{E}_\kappa$ denote expectations with respect to κ and $\mathbb{P}_\kappa$ respectively.

(IM1) Suppose ν_ρ is a translation-invariant measure which is 'spatially mixing'. That is, for local $L^2(\nu_\rho)$ functions f and h,

$$\lim_{|x|\uparrow\infty} E_{\nu_\rho}[f(\eta)\tau_x h(\eta)] \ = \ E_{\nu_\rho}[f] E_{\nu_\rho}[h].$$

In addition, suppose the mean $E_{\nu_\rho}[\eta(0)] = \rho$, and moment-generating function $E_{\nu_\rho}[e^{\lambda\eta(0)}] < \infty$ for $|\lambda| \leq \lambda^*$ for a $\lambda^* > 0$.

Although product measures ν_ρ are considered in most of the examples, we note, in Subsection 2.5, a non-product measure ν_ρ corresponding to an exponentially mixing ergodic Markov chain is used.

Now, the measure ν_ρ, by (IM1) and the 'gradient dynamics' conditions (R1) and (R2), is an invariant measure with respect to L_n for all $a \in \mathbb{R}$ and $\gamma > 0$. Indeed, let ϕ be a local $L^2(\nu_\rho)$ function supported with respect to sites in Λ_k. Then, for $\ell > k$, we have

$$\begin{aligned} E_{\nu_\rho}[L_n\phi] &= -E_{\nu_\rho}\Big[\sum_{|x|\leq \ell} (p_n - q_n)\phi(\eta)\big[c^n_x(\eta) - c^n_{x+1}(\eta)\big]\Big] \\ &= -(p_n - q_n)E_{\nu_\rho}\big[\phi(\eta)\big(c^n_{-\ell}(\eta) - c^n_{\ell+1}(\eta)\big)\big]. \end{aligned}$$

The limit as $\ell \uparrow 0$ vanishes, by translation-invariance and the spatial mixing assumption in (IM1).

One can also compute that the $L^2(\nu_\rho)$ adjoint L^*_n is the generator with parameter $-a$, that is when the jump probability is reversed. Define $S_n = (L_n + L^*_n)/2$. Then, the Dirichlet form $D_{\nu_\rho,n}(f) := E_{\nu_\rho}[f(-L_n f)] = E_{\nu_\rho}[f(-S_n f)]$ on local $L^2(\nu_\rho)$ functions, is given by

$$D_{\nu_\rho}(f) \ = \ \frac{1}{2}\sum_{x\in\mathbb{Z}} E_{\nu_\rho}\big[b^{R,n}_x(\eta)\big(\nabla_{x,x+1}f(\eta)\big)^2\big]. \tag{2.2}$$

We remark when $a = 0$, S_n is the generator of the process and ν_ρ is a reversible measure.

Consider now the empirical measure

$$\mathcal{Y}^n_0 \ = \ \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}}(\eta(x) - \rho)\delta_{x/n}$$

and its covariance under measure κ,

$$\mathcal{C}^n_\kappa(G,H) \ = \ E_\kappa\Big[\Big(\mathcal{Y}^n_0(G) - E_\kappa\big[\mathcal{Y}^n_0(G)\big]\Big)\Big(\mathcal{Y}^n_0(H) - E_\kappa\big[\mathcal{Y}^n_0(H)\big]\Big)\Big]$$

with respect to compactly supported functions G and H.

(IM2) We assume, starting from ν_ρ, that $\mathcal{Y}^n_0$ converges weakly to a spatial Gaussian process with covariance $\mathcal{C}_{\nu_\rho}(G,H) := \lim_{n\uparrow\infty}\mathcal{C}^n_{\nu_\rho}(G,H)$ such that $\mathcal{C}_{\nu_\rho}(G,G) \leq C(\rho)\|G\|^2_{L^2(\mathbb{R})}$. Also, suppose the fourth moment bound $\sup_{\ell\geq 1} E_{\nu_\rho}\big[\big(\frac{1}{\sqrt{\ell}}\sum_{x=1}^{\ell}(\eta(x)-\rho)\big)^4\big] < \infty$.

It will be convenient to define the variances

$$\sigma_n^2(\rho) \ := \ \mathcal{C}^n_{\nu_\rho}(H,H) \ = \ E_{\nu_\rho}\Big[\Big(\frac{1}{\sqrt{2n+1}}\sum_{x\in\Lambda_n}(\eta(x)-\rho)\Big)^2\Big]$$

and $\sigma^2(\rho) = \mathcal{C}_{\nu_\rho}(H,H) = \lim_{n\uparrow\infty}\sigma^2_n(\rho)$ when $H(x) = 1_{[-1,1]}(x)$.

When ν_ρ is sufficiently mixing, the case of our examples, (IM2) holds with $\mathcal{C}_{\nu_\rho}(G,H) = \sigma^2(\rho)\langle G,H\rangle_{L^2(\mathbb{R})}$.

Now, for $\lambda \in (-\lambda^*,\lambda^*)$, consider the tilted measure ν_ρ^λ with 'tilt' or 'chemical potential' λ given by

$$\frac{d\nu_\rho^\lambda}{d\nu_\rho}\Big(\eta(x) = e(x), x\in\Lambda_\ell\Big|\eta(x) = \xi(x), x\not\in\Lambda_\ell\Big) \ = \ \frac{e^{\lambda\sum_{x\in\Lambda_\ell}(e(x)-\rho)}}{Z(\lambda,\ell,\xi)} \tag{2.3}$$

where $e,\xi\in\Omega$ and $Z(\lambda,\ell,\xi)$ is the normalization.

(D1) We will assume the measures $\{\nu_\rho^\lambda : |\lambda| < \lambda^*\}$ are well-defined on Ω, that is a limit of the finite-dimensional measure in (2.3) as $\Lambda_\ell \nearrow \mathbb{Z}$ can be taken which does not depend on ξ. Also, we assume that the measures can be indexed by density, that is $E_{\nu_\rho^\lambda}[\eta(0)]$ is strictly increasing in λ for $|\lambda|\leq\lambda^*$.

These assumptions hold of course when ν_ρ is a product measure. They also hold when ν_ρ is an FKG measure corresponding to an ergodic Markov chain on a finite alphabet, the case for the exclusion with speed-change model in Subsection 2.5.

The measures $\{\nu_\rho^\lambda : |\lambda| < \lambda^*\}$ are translation-invariant since ν_ρ is assumed translation-invariant (IM1). Also, given exponential moments of ν_ρ (IM1), $E_{\nu_\rho^\lambda}[\eta(0)]$ is continuous in λ for $|\lambda| < \lambda^*$. Hence, by the strict increasing assumption in (D1), one can reparametrize $\{\nu_\rho^\lambda\}$ in terms of density: Let $z \in (\rho_*,\rho^*)$ where $\rho_* = \lim_{\lambda\downarrow-\lambda^*}E_{\nu_\rho^\lambda}[\eta(0)]$ and $\rho^* = \lim_{\lambda\uparrow\lambda^*}E_{\nu_\rho^\lambda}[\eta(0)]$. Let $\lambda(z)\in(-\lambda^*,\lambda^*)$ be the parameter such that $E_{\nu_\rho^{\lambda(z)}}[\eta(0)] = z$. Then, we will define $\nu_z = \nu_\rho^{\lambda(z)}$.

Define also, for a local $L^2(\nu_\rho)$ function f, the 'tilted mean' function $\varphi_f(z) : (\rho_*,\rho^*)\to\mathbb{R}$ where

$$\varphi_f(z) \ = \ E_{\nu_z}\big[f(\eta)\big].$$

We define the derivatives of $\varphi_f(z)$ as the formal limits of the derivatives of $E_{\nu_z}[f(\eta)|\eta(x) = \xi(x), x\in\Lambda_\ell]$ as $\ell\uparrow\infty$ which take form as

$$\begin{aligned}
\varphi'_f(z) \ &:= \ \lambda'(z)E_{\nu_z}[(f(\eta)-E_{\nu_z}[f])(\sum_{x\in\mathbb{Z}}(\eta(x)-z))] \\
\varphi''_f(z) \ &:= \ (\lambda'(z))^2E_{\nu_z}[(f(\eta)-E_{\nu_z}[f])(\sum_{x\in\mathbb{Z}}(\eta(x)-z))^2] \\
&\quad +\lambda''(z)E_{\nu_z}[(f(\eta)-E_{\nu_z}[f])(\sum_{x\in\mathbb{Z}}(\eta(x)-z))].
\end{aligned} \tag{2.4}$$

For the 0th derivative, we set $\varphi^{(0)}_f(z) := E_{\nu_z}[f]$.

(D2) For local $L^2(\nu_\rho)$ functions f, suppose the limits (2.4) are well-defined and $|\varphi'_f(\rho)|, |\varphi''_f(\rho)| \leq C(\rho)\|f\|_{L^2(\nu_\rho)}$; already, $|\varphi_f(\rho)| \leq \|f\|_{L^2(\nu_\rho)}$. Also, suppose

$$\lim_{n\uparrow\infty} \varphi'_{f_n}(\rho) = \varphi'_f(\rho) \text{ and } \lim_{n\uparrow\infty} \varphi''_{f_n}(\rho) = \varphi''_f(\rho)$$

when $\{f_n\}$ and f are local functions such that $\lim_{n\uparrow\infty} f_n(\eta) = f(\eta)$ and $f_n(\eta) \leq \hat{f}(\eta)$ configurationwise for each n where $\hat{f} \in L^2(\nu_\rho)$.

When, $\{\nu_x\}$ are product or rapidly mixing Markov measures, again the case for our examples, this condition also holds by calculation with (2.3).

Spectral gap. We now give a 'spectral gap' condition. For $\ell \geq 1$, recall Λ_ℓ is the box of size $2\ell + 1$, namely $\Lambda_\ell := \{x \in \mathbb{Z} : |x| \leq \ell\}$. Let also, for $k \geq 0$ and $\xi \in \Omega$, $\mathcal{G}_{k,\ell,\xi} = \{\eta : \sum_{x\in\Lambda_\ell} \eta(x) = k, \eta(y) = \xi(y) \text{ for } y \notin \Lambda_\ell\}$ be the hyperplane of configurations on Λ_ℓ with k particles which equal ξ outside Λ_ℓ. Denote by $\nu_{k,\ell,\xi}$ the canonical measure on $\mathcal{G}_{k,\ell,\xi}$, namely

$$\nu_{k,\ell,\xi}(\cdot) \ := \ \nu_\rho\Big(\cdot \,|\, \sum_{x\in\Lambda_\ell} \eta(x) = k, \eta(y) = \xi(y) \text{ for } y \notin \Lambda_\ell\Big).$$

Consider now the process, restricted to the hyperplane $\mathcal{G}_{k,\ell,\xi}$ with generator

$$\mathcal{S}_{n,\mathcal{G}_{k,\ell,\xi}} f(\eta) \ = \ \frac{1}{2} \sum_{\substack{|x-y|=1\\ x,y\in\Lambda_\ell}} b^n_x(\eta) \nabla_{x,y} f(\eta).$$

This is a finite-state Markov process with reversible invariant measure $\nu_{k,\ell,\xi}$. Denote by $\lambda_{k,\ell,\xi,n}$ the spectral gap, that is the second largest eigenvalue of $-\mathcal{S}_{n,\mathcal{G}_{k,\ell,\xi}}$ (with 0 being the largest). Let $W(k,\ell,\xi,n)$ denote the reciprocal of $\lambda_{k,\ell,\xi,n}$, which is set to ∞ if $\lambda_{k,\ell,\xi,n} = 0$. Then, the associated Poincaré-inequality reads as

$$\mathrm{Var}(f, \nu_{k,\ell,\xi}) \ \leq \ W(k,\ell,\xi,n)\mathcal{D}_n(f, \nu_{k,\ell,\xi}) \tag{2.5}$$

where $\mathrm{Var}(f, \nu_{k,\ell,\xi})$ is the variance of f with respect to $\nu_{k,\ell,\xi}$ and the canonical Dirichlet form $\mathcal{D}_n(f, \nu_{k,\ell,\xi})$ is given by

$$\mathcal{D}_n(f, \nu_{k,\ell,\xi}) \ := \ \frac{1}{2} \sum_{x,x+1\in\Lambda_\ell} E_{\nu_{k,\ell,\xi}}\big[b^{R,n}_x(\eta)\big(\nabla_{x,x+1} f(\eta)\big)^2\big].$$

When $W(k,\ell,\xi,n) < \infty$, the process is ergodic and $\nu_{k,\ell,\xi}$ is the unique invariant measure.

Denote the 'outside variables' by $\eta^c_\ell = \{\eta(x) : x \notin \Lambda_\ell\}$. We will assume the following condition on $W(k,\ell,\xi,n)$.

(G) Suppose there is a constant $C = C(\rho)$ such that, for $n \geq 1$, we have

$$E_{\nu_\rho}\left[W\left(\sum_{x\in\Lambda_\ell} \eta(x), \ell, \eta^c_\ell, n\right)^2 \right] \ \leq \ C\ell^4.$$

We remark a sufficient condition to verify (G) would be the uniform bound $\sup_{k,\xi,n} \ell^{-2} W(k,\ell,\xi,n) < \infty$, which holds for some types but not all of the specific models discussed.

Equivalence of ensembles. We will also assume an 'equivalence of ensembles' estimate between the canonical and grand-canonical measures. Define, for $\ell \geq 1$ and $\eta \in \Omega$, the empirical average

$$\eta^{(\ell)} \;=\; \frac{1}{2\ell+1}\sum_{y\in\Lambda_\ell}\eta(y).$$

(EE) For local $L^5(\nu_\rho)$ functions f, supported on $\{\eta(x) : x \in \Lambda_{\ell_0}\}$, such that $\varphi_f(\rho) = \varphi'_f(\rho) = 0$, and $\ell \geq \ell_0$, there exist constants $\alpha_0 > 0$ and $C = C(\rho,\alpha_0)$ where

$$\Big\| E_{\nu_\rho}\Big[f|\eta^{(\ell)},\eta^c_\ell\Big] - \frac{\varphi''_f(\rho)}{2}\Big[(\eta^{(\ell)}-\rho)^2 - \frac{\sigma^2_\ell(\rho)}{2\ell+1}\Big]\Big\|_{L^4(\nu_\rho)} \;\leq\; \frac{C\|f\|_{L^5(\nu_\rho)}}{\ell^{1+\alpha_0/2}}.$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\Big\| E_{\nu_\rho}\Big[f|\eta^{(\ell)},\eta^c_\ell\Big] - \varphi'_f(\rho)\big(\eta^{(\ell)}-\rho\big)\Big\|_{L^4(\nu_\rho)} \;\leq\; \frac{C\|f\|_{L^5(\nu_\rho)}}{\ell^{1/2+\alpha_0/2}}.$$

We remark, a weaker version, where the $L^2(\nu_\rho)$ norm, instead of the $L^4(\nu_\rho)$ norm of the difference, is say less than the same right-hand side expressions with $\|f\|_{L^3(\nu_\rho)}$ in place of $\|f\|_{L^5(\nu_\rho)}$ would be sufficient for our purposes if there is a uniform bound on the inverse gap $\sup_{k,\xi,n}\ell^{-2}W(k,\ell,\xi,n) < \infty$.

Usually, such estimates follow from a local central limit theorem. In Proposition 5.1, we show, when ν_ρ is a nondegenerate product measure, that (EE) holds with $\alpha_0 = 1$. In Proposition 5.2, with respect to a Markovian measure, we prove (EE) holds with $\alpha_0 = 1-\varepsilon$ for any fixed $0 < \varepsilon < 1$. These two propositions cover the examples discussed in the article.

Initial conditions. We will start from initial measures $\{\mu^n\}$ which have bounded relative entropy $H(\mu^n;\nu_\rho)$ with respect to ν_ρ.

(BE) Suppose $\{\mu^n\}$ satisfies

$$\sup_n H(\mu^n;\nu_\rho) \;=\; \sup_n E_{\nu_\rho}\Big[\frac{d\mu^n}{d\nu_\rho}\log\frac{d\mu^n}{d\nu_\rho}\Big] \;<\; \infty.$$

In addition, we presume a diffusive initial limit starting from $\{\mu^n\}$.

(CLT) Under initial measures $\{\mu^n\}$, we suppose $\mathcal{Y}^n_0$ converges weakly to a spatial Gaussian process $\bar{\mathcal{Y}}_0$ with covariance $\mathcal{C}(G,H) = \lim_{n\uparrow\infty}\mathcal{C}^n_{\mu^n}(G,H)$ for compactly supported functions G, H.

Of course, if $\mu^n \equiv \nu_\rho$, (BE) and (CLT) trivially hold with $\mathcal{C}(G,H) = \mathcal{C}_{\nu_\rho}(G,H)$. When ν_ρ is a product measure, a possible way to get non-trivial examples of measures $\{\mu^n\}$ satisfying (BE) and (CLT) is the following. For simplicity, we consider the case on which ν_ρ is a Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$. Let $\{\kappa^n_x : x \in \mathbb{Z}\}$ be a given bounded sequence and define μ^n as the non-homogeneous Bernoulli product measure satisfying

$$\mu_n(\eta(x)=1) \;=\; \rho + \frac{\kappa^n_x}{\sqrt{n}}.$$

A simple computation shows that

$$H(\mu^n;\nu_\rho) \;\leq\; \frac{C(\|\kappa\|_{\ell^\infty})}{n}\sum_{x\in\mathbb{Z}}(\kappa^n_x)^2.$$

Therefore, taking $\kappa_x^n = \kappa(x/n)$, where $\kappa : \mathbb{R} \to \mathbb{R}$ is bounded and in $L^2(\mathbb{R})$, we see that $\sup_n H(\mu^n; \nu_\rho) < \infty$, and (BE) is satisfied. On the other hand, since the measure μ^n is product, a simple computation shows that, under $\{\mu^n\}$, the process $\mathcal{Y}_0^n$ converges in distribution to $\bar{\mathcal{Y}}_0 + \kappa$, where $\bar{\mathcal{Y}}_0$ is a white noise with variance $\rho(1-\rho)$. In [51], the Cole-Hopf solution of KPZ is considered starting from such initial conditions.

One may relate probabilities of events A under μ^n with those under ν_ρ by an application of the entropy inequality:

$$\mathbb{P}_{\mu^n}(A) \ \le \ \frac{\log 2 + H(\mu^n; \nu_\rho)}{1 + \log \mathbb{P}_{\nu_\rho}(A)}. \tag{2.6}$$

For instance, let $r \in L^2(\nu_\rho)$ be a local function. By the spatial mixing assumption (IM2), under ν_ρ, we have the convergence in probability,

$$\lim_{n\to\infty} \int_0^T \frac{1}{2n+1} \sum_{x \in \Lambda_n} \tau_x r(\eta_s^n) ds \ = \ E_{\nu_\rho}[r(\eta)]. \tag{2.7}$$

Then, by the entropy relation, also under $\{\mu^n\}$, the same limit also holds in probability.

Of course, given that we begin from nearly the invariant measure ν_ρ, (2.7) is a trivial case of 'hydrodynamics'. Formally, starting from more general measures, the hydrodynamic equation for the limiting empirical density $\rho = \rho(x,t)$ would read

$$\partial_t \rho(x,t) + \frac{a}{2} \nabla \varphi_b\big(\rho(x,t)\big) \ = \ \frac{1}{2} \Delta \varphi_c\big(\rho(x,t)\big). \tag{2.8}$$

In a sense, the main results of the paper are on the different fluctuations from the law of large numbers (2.7) which arise for different regimes of the strength asymmetry parameters a and γ.

2.2. **Results.** Denote by $\mathbb{S}(\mathbb{R})$ the standard Schwarz space of rapidly decreasing functions equipped with the usual metric, and let $\mathbb{S}'(\mathbb{R})$ be its dual, namely the set of tempered distributions in $\mathbb{R}$, endowed with the uniform weak-$*$ topology. Denote the density fluctuation field acting on functions $H \in \mathbb{S}(\mathbb{R})$ as

$$\mathcal{Y}_t^n(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x}{n}\Big)(\eta_t^n(x) - \rho).$$

Denote by $D([0,T], \mathbb{S}'(\mathbb{R}))$ and $C([0,T], \mathbb{S}'(\mathbb{R}))$ the spaces of right continuous functions with left limits and continuous functions respectively from $[0,T]$ to $\mathbb{S}'(\mathbb{R})$.

We now state a result from the literature which has been proved for some processes (cf. [24], [36][Chapter 11] for zero-range processes with bounded rate, [53], [22] for simple exclusion processes, and [62][Section II.2.10] for exclusion systems with speed-change), sometimes from more general initial conditions, when the asymmetry is of order $O(n^{-1})$.

Proposition 2.1. *For $\gamma = 1$, starting from $\{\mu^n\}$, the sequence $\{\mathcal{Y}_t^n; n \ge 1\}$ converges in the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the process $\mathcal{Y}_t$ which solves the Ornstein-Uhlenbeck equation*

$$\partial_t \mathcal{Y}_t \ = \ \frac{1}{2}\varphi_c'(\rho)\Delta \mathcal{Y}_t + \frac{a}{2}\varphi_b'(\rho)\nabla \mathcal{Y}_t + \sqrt{\frac{1}{2}\varphi_b(\rho)}\nabla \dot{\mathcal{W}}_t, \tag{2.9}$$

where $\dot{\mathcal{W}}_t$ is a space-time white noise with unit variance, and $\mathcal{Y}_0 = \bar{\mathcal{Y}}_0$, the field given in (CLT).

The Ornstein-Uhlenbeck equation (2.9) has a drift term coming from the weak asymmetry of the jump rates. The drift, as is well known, can be understood in terms of a characteristic velocity $\upsilon = (a/2)\varphi_b'(\rho)$ from considering the linearization of the hydrodynamic equation (2.8) (cf. [62][Chapter II.2]). However, it can be removed from the limit field by observing the density fluctuation field in the frame of an observer moving along the process characteristics. Define

$$\mathcal{Y}_t^{n,\gamma}(H) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} H\Big(\frac{x}{n} - \frac{1}{n}\Big\{\frac{a\varphi_{b^n}'(\rho)tn^2}{2n^\gamma}\Big\}\Big)(\eta_t^n(x)-\rho).$$

If $\gamma = 1$, Proposition 2.1 is equivalent to the statement that $\mathcal{Y}_t^{n,\gamma}$ converges in the uniform topology on $D([0,T],\mathbb{S}'(\mathbb{R}))$ to $\mathcal{Y}_t$, the unique solution of the drift-removed Ornstein-Uhlenbeck equation

$$\partial_t\mathcal{Y}_t = \frac{1}{2}\varphi_c'(\rho)\Delta\mathcal{Y}_t + \sqrt{\frac{1}{2}\varphi_b(\rho)}\nabla\dot{\mathcal{W}}_t. \tag{2.10}$$

This equation of course corresponds to (2.9) with $a = 0$, is well-posed and has a unique solution (cf. [64]).

Now we increase the strength of the asymmetry in the jump rates by decreasing the value of γ. We show for $1/2 < \gamma < 1$, starting from the measures $\{\mu^n\}$, that there is no effect in the convergence result of the fluctuation field.

Theorem 2.2 (Crossover fluctuations). *For $1/2 < \gamma < 1$, starting from initial measures $\{\mu^n\}$, the sequence $\{\mathcal{Y}_t^{n,\gamma}; n \geq 1\}$ converges in the uniform topology on $D([0,T],\mathbb{S}'(\mathbb{R}))$ to the process $\mathcal{Y}_t$ which is the solution of the Ornstein-Uhlenbeck equation* (2.10) *with initial condition $\mathcal{Y}_0 = \bar{\mathcal{Y}}_0$ given in (CLT).*

However, for $\gamma = 1/2$, which is a threshold, a much different qualitative limit behavior is obtained as the strength of the weak asymmetry in the jump rates is big enough to influence the limit field. As mentioned in the introduction, the limit field $\mathcal{Y}_t$ should satisfy, in some sense, a stochastic Burgers equation, written in our framework as

$$\partial_t\mathcal{Y}_t \ = \ \frac{\varphi_c'(\rho)}{2}\Delta\mathcal{Y}_t + \frac{a}{2}\varphi_b''(\rho)\nabla\mathcal{Y}_t^2 + \sqrt{\frac{1}{2}\varphi_b(\rho)}\nabla\dot{\mathcal{W}}_t, \tag{2.11}$$

although it is ill-posed.

We now detail in what sense we mean to 'solve' (2.11) in terms of a martingale problem. Let $\iota : \mathbb{R} \to [0,\infty)$ be the function $\iota(z) = (1/2)1_{[-1,1]}(z)$. Also, for $0 < \varepsilon \leq 1$, define $\iota_\varepsilon(z) = \varepsilon^{-1}\iota(\varepsilon^{-1}z)$ and let $G_\varepsilon : \mathbb{R} \to [0,\infty)$ be a smooth compactly supported function in $\mathbb{S}(\mathbb{R})$ which approximates ι_ε: That is, $\|G_\varepsilon\|_{L^2(\mathbb{R})}^2 \leq 2\|\iota_\varepsilon\|_{L^2(\mathbb{R})}^2 = \varepsilon^{-1}$ and

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1/2}\|G_\varepsilon - \iota_\varepsilon\|_{L^2(\mathbb{R})} \ = \ 0.$$

Such choices can be readily found by convoluting ι_ε with smooth kernels. Also, for $x \in \mathbb{R}$, define the shift τ_x so that $\tau_x G_\varepsilon(z) = G_\varepsilon(x+z)$.

Consider now an $\mathbb{S}'(\mathbb{R})$-valued process $\{\mathcal{Y}_t; t \in [0,T]\}$ and for $0 \le s \le t \le T$ let

$$\mathcal{A}^{\varepsilon}_{s,t}(H) \;=\; \int_s^t \int_{\mathbb{R}} \nabla H(x)\Big[\mathcal{Y}_u(\tau_{-x}G_\varepsilon)\Big]^2 dxdu.$$

We say the process $\mathcal{Y}_\cdot$ satisfies the *Cauchy energy condition* if for each $H \in \mathbb{S}(\mathbb{R})$,

$$\{\mathcal{A}^{\varepsilon}_{s,t}(H)\} \text{ is Cauchy in probability as } \varepsilon \downarrow 0 \tag{2.12}$$

and the limit in probability does not depend on the particular smoothing family $\{G_\varepsilon\}$. This limit defines the process $\{\mathcal{A}_{s,t}; 0 \le s \le t \le T\}$ given by

$$\mathcal{A}_{s,t}(H) \;:=\; \lim_{\varepsilon\downarrow 0} \mathcal{A}^{\varepsilon}_{s,t}(H),$$

which is $\mathbb{S}'(\mathbb{R})$ valued (cf. [64][p. 364-365; Theorem 6.15 and Corollary 6.16]).

We will say that $\{\mathcal{Y}_t; t \in [0,T]\}$ is a *Cauchy energy solution* of (2.11) if the following conditions hold.

(i) Initially, $\mathcal{Y}_0$ is a spatial Gaussian process with covariance $\mathcal{C}(G,H)$ for $G,H \in \mathbb{S}(\mathbb{R})$.

(ii) The process $\{\mathcal{Y}_t; t \in [0,T]\}$ satisfies the Cauchy energy condition (2.12).

(iii) Then, the $\mathbb{S}'(\mathbb{R})$ valued process $\{\mathcal{M}_t : t \in [0,T]\}$ where

$$\mathcal{M}_t(H) \;:=\; \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{\varphi_c'(\rho)}{2}\int_0^t \mathcal{Y}_s(\Delta H)ds - \frac{a\varphi_b''(\rho)}{2}\mathcal{A}_{0,t}(H) \tag{2.13}$$

is a continuous martingale with quadratic variation

$$\langle \mathcal{M}_t(H)\rangle \;=\; \frac{\varphi_b(\rho)t}{2}\|\nabla H\|^2_{L^2(\mathbb{R})}.$$

In particular, condition (iii) specifies, by Levy's theorem, that $\mathcal{M}_t(H)$ is a Brownian motion with variance $(\varphi_b(\rho)/2)t\|\nabla H\|^2_{L^2(\mathbb{R})}$.

We also define a stronger notion of solution to (2.11) which may be verified in some cases. We say that $\mathcal{Y}_t$ satisfies the L^2 *energy condition* if in (2.12), instead of in the probability sense, we assert $\{\mathcal{A}^{\varepsilon}_{s,t}(H)\}$ is Cauchy in L^2 with respct to the underlying probability measure, and $\mathcal{A}_{s,t}(H)$ is its L^2 limit. Then, we say $\mathcal{Y}_t$ is an L^2 *energy solution* of (2.11) if (i) holds as before, (ii)' the L^2 energy condition holds, and (iii) holds with respect to the L^2 limit $\mathcal{A}_{s,t}(H)$.

Theorem 2.3 (KPZ fluctuations). *For $\gamma = 1/2$, starting from initial measures $\{\mu^n\}$, the sequence of processes $\{\mathcal{Y}^{n,\gamma}_t : n \ge n_0\}$ is tight in the uniform topology on $D([0,T],\mathbb{S}'(\mathbb{R}))$. Moreover, any limit point of $\mathcal{Y}^{n,\gamma}_t$ is a Cauchy energy solution with respect to* (2.11) *with initial field $\bar{\mathcal{Y}}_0$ given in (CLT).*

If the initial measure is $\mu^n \equiv \nu_\rho$, any limit point of $\mathcal{Y}^{n,\gamma}_t$ is an L^2 energy solution of (2.11) *with initial field $\bar{\mathcal{Y}}_0$ given in (CLT).*

Remark 2.4. We now make the following comments.

1. Formally, equation (2.13) corresponds to the stochastic Burgers equation (2.11) where the nonlinear term is represented by $\mathcal{A}_{0,t}$. We remark, as in [5], by taking a fast subsequence in ε, one may write $\mathcal{A}_{0,t}$ as a function of $\{\mathcal{Y}_u : u \le t\}$, and form an equation which $\mathcal{Y}_t$ satisfies (2.11) a.s. on a type of negative order Hermite Hilbert space.

2. We also remark, as alluded to in the introduction, if there were a unique Cauchy (or L^2) energy solution, that is uniqueness of process in the associated

'martingale problem', since with respect to simple exclusion the fluctuation field limit is in terms of the 'Cole-Hopf' solution of the KPZ equation [15], not only could one conclude a unique fluctuation field limit in Theorem 2.3 in the framework of particle systems considered, but also identify it in terms of the 'Cole-Hopf' apparatus.

3. In addition, the space $\mathbb{S}'(\mathbb{R})$, in the results Theorems 2.2 and 2.3 and notions of Cauchy and L^2 energy solutions, can be relaxed to a Hermite Hilbert space $\mathcal{H}_{-k}$ with $k \geq 4$ which is strictly contained in $\mathbb{S}'(\mathbb{R})$. [With respect to Proposition 2.1, starting from ν_ρ, this has already been proved in the models mentioned just before the proposition statement.] To make this improvement, since all arguments with $G \in \mathbb{S}(\mathbb{R})$ use only properties of functions in $\mathcal{H}_4$, we need only show $\mathcal{Y}_t^{n,\gamma}$ is tight in the uniform topology on $D([0,T];\mathcal{H}_{-k})$ with $k \geq 4$. In particular, in Section 6, we define these Hermite spaces and show how to improve the simpler tightness argument on $\mathbb{S}'(\mathbb{R})$ given in the main argument.

4. We also note that the statement of Theorem 2.3 is non-trivial when $a \neq 0$ and b is such that

$$\varphi_b''(\rho) \neq 0. \tag{2.14}$$

Otherwise, when $\varphi_b''(\rho) = 0$, the limit field $\mathcal{Y}_t$ satisfies the Ornstein-Uhlenbeck equation (2.10). Examples, fitting in our framework, where the second derivative vanishes include types of zero-range, that is independent particle systems where $\varphi_b(\rho) = 2\rho$ which are in the EW class.

2.3. **Model 1: Zero-Range Processes.** The one-dimensional weakly asymmetric zero-range process η_t^n, on the state space $\Omega := \mathbb{N}_0^{\mathbb{Z}}$, consists of a collection of random walks which interact in that the jump rate of a particle at vertex x only depends on the number of particles at x. More precisely, the generator is in form (2.1) where

$$b_x^{R,n}(\eta) \;=\; g(\eta(x)) \;\text{ and }\; b_x^{L,n}(\eta) = g(\eta(x+1))$$

do not depend on n and are fixed with respect to a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and g is Lipschitz,

(LIP) $\sup_{k\geq 0} |g(k+1) - g(k)| < \infty$.

Under this specification, a Markov process η_t^n can be constructed (on a subset of Ω) [2]. Hence, (R1) holds and we identify the fixed function $c^n \equiv c$ as

$$c(\eta) \;=\; g(\eta(0)).$$

The zero-range process possesses a family of invariant measures which are fairly explicit product measures. For $\alpha \geq 0$, define

$$\mathcal{Z}(\alpha) := \sum_{k\geq 0} \frac{\alpha^k}{g(1)...g(k)}.$$

Let α^* be the radius of convergence of this power series and notice that $\mathcal{Z}$ increases on $[0,\alpha^*)$. Fix $0 \leq \alpha < \alpha^*$ and let $\bar{\nu}_\alpha$ be the product measure on $\mathbb{N}^{\mathbb{Z}}$ whose marginal at the site x is given by

$$\bar{\nu}_\alpha\{\eta : \eta(x) = k\} = \begin{cases} \frac{1}{\mathcal{Z}(\alpha)} \frac{\alpha^k}{g(1)...g(k)} & \text{when } k \geq 1 \\ \frac{1}{\mathcal{Z}(\alpha)} & \text{when } k = 0. \end{cases}$$

It will be useful to reparametrize these measures in terms of the 'density'. Let $\rho(\alpha) := E_{\bar\nu_\alpha}[\eta(0)] = \alpha\mathcal{Z}'(\alpha)/\mathcal{Z}(\alpha)$. By computing the derivative, we obtain that $\rho(\alpha)$ is strictly increasing on $[0,\alpha^*)$. Then, let $\alpha(\cdot)$ denote its inverse. Now, we define

$$\nu_\rho(\cdot) := \bar\nu_{\alpha(\rho)}(\cdot),$$

so that $\{\nu_\rho : 0 \le \rho < \rho^*\}$ is a family of invariant measures parameterized by the density. Here, $\rho^* = \lim_{\alpha\uparrow\alpha^*}\rho(\alpha)$, which may be finite or infinite depending on whether $\lim_{\alpha\to\alpha^*}\mathcal{Z}(\alpha)$ converges or diverges.

Note, since ν_ρ is a product measure, that $\nu_\rho^{\lambda(z)} = \nu_z$ for $0 \le z < \rho^*$, and condition (D) holds.

One can readily check that (R2) holds:

$$g\big(\eta^{x+1,x}(x)\big)\frac{d\nu_\rho^{x+1,x}}{d\nu_\rho} \;=\; g(\eta(x)+1))\frac{g(\eta(x))!g(\eta(x+1))!}{g(\eta(x)+1)!g(\eta(x+1)-1)!} \;=\; g(\eta(x+1)).$$

Also, by the construction in [59], which extends the construction in [2] to an $L^2(\nu_\rho)$ process, we have that L_n is a Markov $L^2(\nu_\rho)$ generator whose core can be taken as the space of all local $L^2(\nu_\rho)$ functions. [Indeed, in [59], a core of bounded Lipschitz functions is identified; however, since any local $L^2(\nu_\rho)$ function is a limit of bounded Lipschitz functions, and the formula (2.1) is well defined and bounded for a local $L^2(\nu_\rho)$ function, by dominated convergence the core can be extended.] It follows that the measures $\{\nu_\rho : 0 \le \rho < \rho^*\}$ are invariant for the zero-range process. Also, (IM) holds as ν_ρ is a product measure whose marginal has some exponential moments. In addition, one can check that (EE) holds by Proposition 5.1.

We now address the spectral gap properties of the system. Since the measures are product measures, the gap does not depend on the outside variables ξ. However, the gap depends on g, as it should since g controls the rate of jumps. We identify three types of rates for which a spectral gap bound has been proved.

• If g is not too different from the independent case, for which the gap is of order $O(\ell^{-2})$ uniform in k, one expects similar behavior as for a single particle. This has been proved for $d \ge 1$ in [40] under assumptions (LIP) and

(U) There exists x_0 and $\varepsilon_0 > 0$ such that $g(x+x_0) - g(x) \ge \varepsilon_0$ for all $x \ge 0$.

• If g is sublinear, that is $g(x) = x^\gamma$ for $0<\gamma<1$, then it has been shown the spectral gap depends on the number of particles k, namely the gap for $d \ge 1$ is $O((1+\beta)^{-\gamma}\ell^{-2})$ where $\beta = k/(2\ell+1)^d$ [48].

• If $g(x) = 1(x \ge 1)$, then it has been shown in $d \ge 1$ that the gap is $O((1+\beta)^{-2}\ell^{-2})$ where $\beta = k/(2\ell+1)$ [46]. In $d=1$, this is true because of the connection between the zero-range and simple exclusion processes for which the gap estimate is well-known [50]: The number of spaces between consecutive particles in simple exclusion correspond to the number of particles in the zero-range process.

In all these cases, (G) follows readily by straightforward moment calculations.

2.4. **Model 2: Kinetically Constrained Exclusion Systems.** We consider a version of the exclusion process, developed in [30][see also references therein], in one dimension on $\Omega = \{0,1\}^{\mathbb{Z}}$ where particles more likely hop to unoccupied nearest-neighbor sites when at least $m-1 \geq 1$ other neighboring sites are full. When $m = 2$, the rates are in the form

$$\begin{aligned} b_x^{R,n}(\eta;\theta) &= \eta(x)(1-\eta(x+1))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big] \\ b_x^{L,n}(\eta;\theta) &= \eta(x+1)(1-\eta(x))\Big[\eta(x-1)+\eta(x+2)+\frac{\theta}{2n}\Big], \end{aligned}$$

with respect to a parameter $\theta > 0$. If θ would vanish, particles can jump from site x to $x+1$ exactly when there is at least 1 particle in the vicinity of the bond $(x, x+1)$. However, with $\theta > 0$, the jump from x to $x+1$ may also occur irrespective of the neighboring particle structure with a small rate $\theta/(2n)$.

When $m \geq 2$, the rates generalize to

$$b_x^{R,n}(\eta;\theta) = \eta(x)(1-\eta(x+1))A_n(\eta;\theta) \ \text{ and } \ b_x^{L,n}(\eta;\theta) = \eta(x+1)(1-\eta(x))A_n(\eta;\theta)$$

where $A_n(\eta;\theta)$ equals

$$\prod_{j=-(m-1)}^{-1}\eta(x+j) + \prod_{\substack{j=-(m-2)\\ j\neq 0,1}}^{2}\eta(x+j) + \cdots + \prod_{\substack{j=-1\\ j\neq 0,1}}^{m-1}\eta(x+j) + \prod_{j=2}^{m}\eta(x+j) + \frac{\theta}{2n}.$$

The role of $\theta > 0$ is to make the system 'ergodic'. If $\theta = 0$, there would be an infinite number of invariant measures, such as Dirac measures supported on configurations which cannot evolve under the dynamics. The hydrodynamic limit for this model corresponds to the porous medium equation, $\partial_t\rho_t(t,u) = \Delta\rho^m(t,u)$, and so the model may be thought of as a microscopic porous medium analog.

Now, one may calculate that $b_x^{R,n}(\eta;\theta) - b_x^{L,n}(\eta;\theta) = c_x^n(\eta) - c_{x+1}^n(\eta)$ where, for $m \geq 2$,

$$c^n(\eta;\theta) = \prod_{j=-(m-1)}^{0}\eta(j) + \cdots + \prod_{j=0}^{m-1}\eta(j) - \prod_{\substack{j=-(m-1)\\ j\neq 0}}^{1}\eta(j) - \cdots - \prod_{\substack{j=-1\\ j\neq 0}}^{m-1}\eta(j) + \frac{\theta}{2n}\eta(0).$$

In the case $m = 2$, the last formula reduces to $c^n(\eta;\theta) = \eta(-1)\eta(0) + \eta(0)\eta(1) - \eta(-1)\eta(1) + \frac{\theta}{2n}\eta(0)$.

Of course, uniformly in η, as $n \uparrow \infty$, the terms involving θ vanish,

$$\begin{aligned} &b_x^{R,n}(\eta;\theta) \to b_x^R(\eta) := b_x^{R,1}(\eta), \ \ b_x^{L,n}(\eta;\theta) \to b_x^L(\eta) := b_x^{L,1}(\eta;0) \\ &\qquad \text{and } \ c^n(\eta;\theta) \to c := c^1(\eta;0). \end{aligned}$$

Consider now the Bernoulli product measure on Ω:

$$\nu_\rho = \prod_{x\in\mathbb{Z}}\mu_\rho \quad \text{where } \mu_\rho(1) = 1-\mu_\rho(0) = \rho$$

for $\rho \in [0,1]$. By the construction in [43], it is now standard that L_n is a Markov $L^2(\nu_\rho)$ generator. One may also inspect that condition (R2) holds with respect to ν_ρ. Hence, ν_ρ is invariant for $\rho \in [0,1]$. Condition (IM) also holds as ν_ρ supports two-state configurations. In addition, as ν_ρ is a product measure, $\nu_\rho^{\lambda(z)} = \nu_z$ and (D) holds. Also, by Proposition 5.1, (EE) is satisfied.

We now discuss the spectral gap behavior of the process.

Proposition 2.5. *For kinetically constrained exclusion processes evolving on Λ_ℓ, when $m \geq 2$, there exists a constant C, uniform over ξ and n, such that*

$$W(k,\ell,\xi,n) \ \leq \ C\ell^2\Big(\frac{\ell}{k}\Big)^m 1(k\geq 1).$$

When $m = 2$ and $k \leq \ell/3$, the above spectral gap estimate is already given in [30][Proposition 6.2]. However, a straightforward modification of the proof [30][Proposition 6.2] yields the more general estimate in Proposition 2.5. [Indeed, the difference when $m \geq 2$ is that to bound equation [30][(6.10)] in the general case, one uses that there at most Cj^{m-1} ways to arrange $m-1$ particles in an interval of width j. Now, a similar optimization on j as given in the proof of [30][Proposition 6.2] leads to the desired generalized spectral gap estimate.]

Lemma 2.6. *For the kinetically constrained exclusion model, the spectral gap condition (G) is satisfied.*

Proof. With respect to a constant C, which may change line to line,

$$\begin{aligned}
E_{\nu_\rho}\Big[\Big(W(\sum_{x\in\Lambda_\ell}\eta(x),\ell,\xi,n)\Big)^2\Big] \ &\leq \ C\ell^4 E_{\nu_\rho}\Big[1\Big(\frac{1}{2\ell+1}\leq \eta^{(\ell)}\Big)\big(\eta^\ell\big)^{-2m}\Big]\\
&\leq \ C\ell^4\Big\{\varepsilon^{-2m} + E_{\nu_\rho}\Big[1\Big(\frac{1}{2\ell+1}\leq \eta^{(\ell)}<\varepsilon\Big)\big(\eta^{(\ell)}\big)^{-2m}\Big]\Big\}\\
&\leq \ C\ell^4\Big\{\varepsilon^{-2m} + \ell^{2m}P_{\nu_\rho}\big(\eta^{(\ell)}<\varepsilon\big)\Big\}
\end{aligned}$$

for a fixed $\varepsilon < \rho$. Then, as ν_ρ is a Bernoulli product measure, by Markov's inequality, $E_{\nu_\rho}\Big[W\Big(\sum_{x\in\Lambda_\ell}\eta(x),\ell,\xi,n\Big)^2\Big] \leq C\ell^4$ for all $\ell \geq 1$. □

2.5. **Model 3: Gradient exclusion with speed change.** In this version of exclusion on $\Omega = \{0,1\}^{\mathbb{Z}}$, rates are chosen which correspond to a Hamiltonian with nearest-neighbor interactions,

$$Q_\beta(\eta) \ = \ -\beta\sum_{x\in\mathbb{Z}}(\eta(x)-1/2)(\eta(x+1)-1/2),$$

and which will be reversible with respect to a Markovian measure $\nu_{1/2}$. That is, specify $\nu_{1/2}$ by its finite-dimensional distributions

$$\nu_{1/2}\Big(\eta(x)=e(x): x\in\Lambda_\ell | \eta(y)=\xi(y) \text{ for } y\notin\Lambda_\ell\Big) \ = \ \frac{e^{-Q_{\beta,\ell}(e,\xi)}}{\mathcal{Z}}$$

where

$$\begin{aligned}
Q_{\beta,\ell}(e,\xi) \ &= \ -\beta\sum_{x,x+1\in\Lambda_\ell}(e(x)-1/2)(e(x+1)-1/2)\\
&\quad -\beta(\xi(-\ell-1)-1/2)(e(-\ell)-1/2)-\beta(e(\ell)-1/2)(\xi(\ell+1)-1/2),
\end{aligned}$$

$e,\xi\in\Omega$ and $\mathcal{Z}=\mathcal{Z}(\ell,\xi)$ is the normalization. It is not difficult to see that $\nu_{1/2}$ is Markovian with transition matrix

$$P \ = \ \frac{1}{e^{\beta/4}+e^{-\beta/4}}\left[\begin{array}{cc} e^{\beta/4} & e^{-\beta/4}\\ e^{-\beta/4} & e^{\beta/4}\end{array}\right]$$

and $E_{\nu_{1/2}}[\eta(0)] = 1/2$.

We now introduce a family of reversible measures by use of a 'tilt' or 'chemical potential' λ. Define $\nu^\lambda_{1/2}$, again specified by its finite-dimensional distributions, through the relation

$$\frac{d\nu^\lambda_{1/2}}{d\nu_{1/2}}\big(\eta(x) = e_x : x \in \Lambda_\ell | \eta(y) = \xi(y) \text{ for } y \notin \Lambda_\ell\big) \ = \ \frac{e^{-\lambda \sum_{x\in\Lambda_\ell}(e(x)-1/2)}}{\mathcal{Z}'}$$

where $e, \xi \in \Omega$ and $\mathcal{Z}' = \mathcal{Z}'(\ell, \xi)$ is another normalization. These measures are also Markovian with transition matrix

$$P_\lambda \ = \ \left[\begin{array}{cc} p_1^{-1}e^{\beta/4}e^{-\lambda/2} & p_1^{-1}e^{-\beta/4}e^{\lambda/2} \\ p_2^{-1}e^{-\beta/4}e^{-\lambda/2} & p_2^{-1}e^{\beta/4}e^{\lambda/2}\end{array}\right]$$

where

$$p_1 \ = \ e^{\beta/4-\lambda/2} + e^{-\beta/4+\lambda/2} \quad \text{and} \quad p_2 \ = \ e^{-\beta/4-\lambda/2} + e^{\beta/4+\lambda/2}. \tag{2.15}$$

To parametrize in terms of 'density', recall $\nu^{\lambda(z)}_{1/2} = \nu_z$ where $\lambda(z)$ is chosen so that $E_{\nu_z}[\eta(0)] = z$.

Then, as discussed in [62, Section II.2.4], (R2) is ensured if we take the rates $b^{R,n}_x = b^R_x$ and $b^{L,n}_x = b^L_x$ which don't depend on n as

$$\begin{aligned} b^R_x(\eta) \ &= \ \eta(x)(1-\eta(x+1))\big[\alpha_1\eta(x-1)\eta(x+2) + \alpha_2(1-\eta(x-1))\eta(x+2) \\ &\qquad +\alpha_3\eta(x-1)(1-\eta(x+2)) + \alpha_4(1-\eta(x-1))(1-\eta(x+2))\big] \\ b^L_x(\eta) \ &= \ \eta(x+1)(1-\eta(x))\big[\alpha_1\eta(x-1)\eta(x+2) + \alpha_3(1-\eta(x-1))\eta(x+2) \\ &\qquad +\alpha_2\eta(x-1)(1-\eta(x+2)) + \alpha_4(1-\eta(x-1))(1-\eta(x+2))\big] \end{aligned}$$

where $\alpha_1, \alpha_2 = e^\beta\alpha_3, \alpha_4 > 0$. The condition (R1) also follows if we also assume that $\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0$ so that, as can be checked, $c(\eta)$ takes the form

$$\begin{aligned} c(\eta) \ &= \ \alpha_4\eta(0) + (\alpha_3-\alpha_4)\eta(-1)\eta(0) + (\alpha_3-\alpha_4)\eta(0)\eta(1) \\ &\qquad +(\alpha_4-\alpha_2)\eta(-1)\eta(1) + (\alpha_2-\alpha_3)\eta(-1)\eta(0)\eta(1). \end{aligned}$$

Again, by [43], L_n is a Markov $L^2(\nu_\rho)$ generator for the process. We note when $\beta = 0$ and $\alpha_i = 1$ for $i = 1,2,3,4$, the model is the simple exclusion process and $\nu_{1/2}$ is the Bernoulli product measure with density $1/2$.

The spectral gap for a more general model, including this one, has been bounded as follows [44]: Uniformly over k and ξ (it doesn't depend on n), we have

$$W(k,\ell,\xi,n) \ \leq \ C\ell^2.$$

Hence, (G) holds.

Also, as $\nu_{1/2}$ supports two-state configurations, is FKG, and is exponentially mixing both (IM) and (D) hold. Explicit computations are also possible here to show $E_{\nu^\lambda_{1/2}}[\eta(0)]$ strictly increases in λ. In Proposition 5.2, we show that (EE) holds.

3. Proofs-Outline

The strategies of the proofs for Theorems 2.2 and 2.3 are similar. We consider the stochastic differential of $\mathcal{Y}^{n,\gamma}_t$ and represent it in terms of corrector and martingale terms. Tightness is shown for each term in the decomposition of $\mathcal{Y}^{n,\gamma}_t$. Under the assumption that the initial measure is the invariant state ν_ρ, limit points are identified using a Boltzmann-Gibbs principle, and shown to satisfy (2.10) when $1/2 < \gamma \leq 1$ and to be energy solutions of (2.11) when $\gamma = 1/2$. When the initial measures $\{\mu^n\}$ satisfy (BE), the entropy inequality then allows to characterize the limit points as desired.

In the following Subsections 3.1 - 3.3, associated martingales, Boltzmann-Gibbs principles, and tightness are discussed. In Subsection 3.4, limit points are identified and Theorems 2.2 and 2.3 are proved.

To reduce some of the notation, we will drop the superscript 'n' in the rate functions and write $b_x^{R,n} = b_x^R$, $b_x^{L,n} = b_x^L$, $b_x^n = b_x$, $b^n = b$, $c_x^n = c_x$ and $c^n = c$ until Subsection 3.4.

3.1. **Associated Martingales.** For $H \in \mathbb{S}(\mathbb{R})$, $x \in \mathbb{Z}$ and $n \geq 1$, define

$$\begin{aligned} \triangle_x^n H &= n^2\left\{H\left(\frac{x+1}{n}\right) + H\left(\frac{x-1}{n}\right) - 2H\left(\frac{x}{n}\right)\right\}, \\ \nabla_x^n H &= n\left\{H\left(\frac{x+1}{n}\right) - H\left(\frac{x}{n}\right)\right\}. \end{aligned}$$

Define also, for $\gamma, s \geq 0$, the functions

$$H_{\gamma,s}(\cdot) = H\Big(\cdot - \frac{1}{n}\Big\lfloor \frac{a\varphi_b'(\rho)sn^2}{2n^\gamma}\Big\rfloor\Big) \quad \text{and} \quad \widetilde{H}_{\gamma,s}(\cdot) = H\Big(\cdot - \frac{1}{n}\Big\{\frac{a\varphi_b'(\rho)sn^2}{2n^\gamma}\Big\}\Big). \tag{3.1}$$

We note, in $H_{\gamma,s}$, the process characteristic shift is along $n^{-1}\mathbb{Z}$, which helps make tidy some proofs (in applying a Boltzmann-Gibbs principle (Theorem 3.2) in proofs of Propositions 3.3 and 3.5), instead of along $\mathbb{R}$ as in $\widetilde{H}_{\gamma,s}$.

Let $F(s, \eta_s^n; H, n) = \mathcal{Y}_s^{n,\gamma}(H)$, and $F(\eta; H, n) = n^{-1/2}\sum_{x\in\mathbb{Z}} H(x/n)(\eta(x) - \rho)$. Although, $F(\eta; H, n)$ is an $L^2(\nu_\rho)$ function, in general it's not a local function. However, by approximation by local functions and noting by condition (R1) that $|b(\eta)| \leq C\sum_{|x|\leq R}\eta(x)$, one may conclude $F(\eta; H, n)$ and also $F^2(\eta; H, n)$ belong to the domain of L_n. In particular,

$$L_n F(s, \eta_s^n; H, n) \;=\; \frac{1}{2\sqrt{n}}\sum_{x\in\mathbb{Z}} c_x(\eta_s^n)\triangle_x^n \widetilde{H}_{\gamma,s} + \frac{a}{2n^{\gamma-1/2}}\sum_{x\in\mathbb{Z}} b_x(\eta_s^n)\nabla_x^n \widetilde{H}_{\gamma,s}.$$

Also,

$$\frac{\partial}{\partial_s}F(s, \eta_s^n; H, n) \;=\; \left\{\frac{-a\varphi_b'(\rho)n^2}{2n^\gamma}\right\}\frac{1}{n^{3/2}}\sum_{x\in\mathbb{Z}}\nabla\widetilde{H}_{\gamma,s}\left(\frac{x}{n}\right)(\eta_s^n(x) - \rho).$$

Then,

$$\begin{aligned} \mathcal{M}_t^{n,\gamma}(H) \;:=\; & F(t, \eta_t^n; H, n) - F(0, \eta_0^n; H, n) \\ & - \int_0^t \frac{\partial}{\partial_s}F(s, \eta_s^n; H, n) + L_n F(s, \eta_s^n; H, n)ds \end{aligned}$$

is a martingale. We may decompose

$$\mathcal{M}_t^{n,\gamma}(H) \;=\; \mathcal{Y}_t^{n,\gamma}(H) - \mathcal{Y}_0^{n,\gamma}(H) - \mathcal{I}_t^{n,\gamma}(H) - \mathcal{B}_t^{n,\gamma}(H) - \mathcal{K}_t^{n,\gamma}(H) \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{I}_t^{n,\gamma}(H) &= \frac{1}{2}\int_0^t \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} \big(c_x(\eta_s^n) - \varphi_c(\rho)\big)\triangle_x^n H_{\gamma,s} ds \\
\mathcal{B}_t^{n,\gamma}(H) &= \frac{a}{2n^{\gamma-1/2}}\int_0^t \sum_{x\in\mathbb{Z}} \big(b_x(\eta_s^n) - \varphi_b(\rho) - \varphi_b'(\rho)(\eta_s^n(x)-\rho)\big)\nabla_x^n H_{\gamma,s} ds \\
\mathcal{K}_t^{n,\gamma}(H) &= \int_0^t \Big[\frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} \kappa_x^{n,1}(H,s)\big(c_x(\eta_s^n) - \varphi_c(\rho)\big) \\
&\quad \frac{a}{2n^{\gamma-1/2}}\sum_{x\in\mathbb{Z}} \kappa_x^{n,2}(H,s)\big(b_x(\eta_s^n) - \varphi_b(\rho) - \varphi_b'(\rho)(\eta_s^n(x)-\rho)\big)\Big] ds.
\end{aligned}$$

Here, we introduced the centering constants $\varphi_c(\rho)$ and $\varphi_b(\rho)$ in $\mathcal{I}_t^{n,\gamma}$ and $\mathcal{B}_t^{n,\gamma}$ as $\triangle_x^n H_{\gamma,s}$ and $\nabla_x^n H_{\gamma,s}$ both sum to zero. Also,

$$\begin{aligned}
\kappa_x^{n,1}(H,s) &= \Delta_x^n\big(\widetilde{H}_{\gamma,s} - H_{\gamma,s}\big) = O(n^{-1})\cdot\Delta_x^n H'_{\gamma,s} + O(n^{-2})\cdot H_{\gamma,s}^{(4)}(x'/n) \\
\kappa_x^{n,2}(H,s) &= \nabla_x^n\big(\widetilde{H}_{\gamma,s} - H_{\gamma,s}\big) = O(n^{-1})\cdot\Delta H_{\gamma,s}(x/n) + O(n^{-2})\cdot H'''_{\gamma,s}(x''/n)
\end{aligned}$$

where $|x'-x|, |x''-x| \le 2$.

To capture the quadratic variation $\langle \mathcal{M}_t^{n,\gamma}\rangle$, we compute

$$\begin{aligned}
&L_nF(s,\eta_s^n;H,n)^2 - 2F(s,\eta_s^n;H,n)L_nF(s,\eta_s^n;H,n) \\
&\quad = \frac{1}{2n}\sum_{x\in\mathbb{Z}} b_x(\eta_s^n)(\nabla_x^n\widetilde{H}_{\gamma,s})^2 + \frac{a}{2n^{1+\gamma}}\sum_{x\in\mathbb{Z}}\big(c_x(\eta_s^n) - c_{x+1}(\eta_s^n)\big)(\nabla_x^n\widetilde{H}_{\gamma,s})^2
\end{aligned}$$

so that $(\mathcal{M}_t^{n,\gamma}(H))^2 - \langle \mathcal{M}_t^{n,\gamma}(H)\rangle$ is a martingale with

$$\begin{aligned}
\langle \mathcal{M}_t^{n,\gamma}(H)\rangle &= \int_0^t \frac{1}{2n}\sum_{x\in\mathbb{Z}}(\nabla_x^n\widetilde{H}_{\gamma,s})^2 b_x(\eta_s^n) ds \\
&\qquad + \int_0^t \frac{a}{2n^{1+\gamma}}\sum_{x\in\mathbb{Z}}\big(c_x(\eta_s^n) - c_{x+1}(\eta_s^n)\big)(\nabla_x^n\widetilde{H}_{\gamma,s})^2 ds.
\end{aligned}$$

When starting from the invariant measure ν_ρ, noting the bounds in (R1), we have

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}_t^{n,\gamma}(H) - \mathcal{M}_s^{n,\gamma}(H)\big)^2\big] \\
&\le \Big\{\int_s^t \Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla_x^n\widetilde{H}_{\gamma,s})^2\Big)ds\Big\}\Big[\frac{1}{2}E_{\nu_\rho}[b(\eta)] + \frac{a}{2n^\gamma}E_{\nu_\rho}[|c_0(\eta) - c_1(\eta)|]\Big] \\
&\le C(a)\|b\|_{L^1(\nu_\rho)}\int_s^t \Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla_x^n\widetilde{H}_{\gamma,s})^2\Big)ds. \qquad (3.3)
\end{aligned}$$

To express an exponential martingale, we now observe for $0 \le \lambda \le \lambda(H,n)$ small that $\exp\{\lambda F(\eta;H,n)\}$ is in the domain of L_n. Indeed, if H is a local function, as ν_ρ is assumed in (IM) to have small parameter exponential moments, then $\exp\{\lambda F(\eta;H,n)\} \in L^2(\nu_\rho)$ for all small λ. Again, an approximation argument when $H \in \mathbb{S}(\mathbb{R})$ is not local shows also $\exp\{\lambda F(\eta;H,n)\}$ belongs to the domain of L_n. We calculate

$$\begin{aligned}
&\exp\Big\{-\lambda F(u,\eta^n_u;H,n)\Big\}\Big(\frac{\partial}{\partial_u}+L_n\Big)\exp\Big\{\lambda F(u,\eta^n_u;H,n)\Big\}\\
&= n^2\sum_{x\in\mathbb{Z}}\Big[b^R_x(\eta)p_n\big(\exp\big\{\lambda n^{-3/2}(\nabla^n_x\widetilde{H}_{\gamma,u})\big\}-1\big)\\
&\qquad\qquad +b^L_x(\eta)q_n\big(\exp\big\{-\lambda n^{-3/2}(\nabla^n_x\widetilde{H}_{\gamma,u})\big\}-1\big)\Big]\\
&\quad -\frac{1}{n^{3/2}}\Big\{\frac{a\lambda\varphi_b'(\rho)n^2}{2n^\gamma}\Big\}\sum_{x\in\mathbb{Z}}\nabla\widetilde{H}_{\gamma,u}(x/n)\big(\eta^n_u(x)-\rho\big)
\end{aligned}$$

which, given the assumptions on b in (R1) and on moments of ν_ρ in (IM), belongs to $L^2(\nu_\rho)$.

Hence, by (the proof of) [21][Lemma IV.3.2],

$$\mathcal{Z}_{s,t} = \exp\Big\{\lambda F(t,\eta^n_t)-\lambda F(s,\eta^n_s)-\int_s^t e^{-\lambda F(u,\eta^n_u)}\Big(\frac{\partial}{\partial_u}+L_n\Big)e^{\lambda F(u,\eta^n_u)}du\Big\}$$

is a martingale. We may expand $\mathcal{Z}_{s,t}$ in terms of λ as

$$\begin{aligned}
\mathcal{Z}_{s,t} &= \exp\Big\{\lambda\big(\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)\big)\\
&-\frac{\lambda^2}{2}\langle\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)\rangle+\frac{\lambda^3}{3!}\int_s^t\mathcal{R}_1du+\frac{\lambda^4}{4!}\int_s^t\mathcal{R}_2du+\lambda^5\int_s^t\mathcal{R}_3du\Big\}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_1(u) &= \frac{n^2}{2n^{9/2}}\sum_{x\in\mathbb{Z}}\big(b^R_x(\eta)-b^L_x(\eta)\big)\big(\nabla^n_x\widetilde{H}_{\gamma,u}\big)^3\\
&\quad+\frac{an^2}{2n^{9/2+(1/2+\gamma)}}\sum_{x\in\mathbb{Z}}b_x(\eta)\big(\nabla^n_x\widetilde{H}_{\gamma,u}\big)^3\\
\mathcal{R}_2(u) &= \frac{n^2}{2n^6}\sum_{x\in\mathbb{Z}}b_x(\eta)\big(\nabla^n_x\widetilde{H}_{\gamma,u}\big)^4\\
&\quad+\frac{an^2}{2n^{6+(1/2+\gamma)}}\sum_{x\in\mathbb{Z}}\big(b^R_x(\eta)-b^L_x(\eta)\big)\big(\nabla^n_x\widetilde{H}_{\gamma,u}\big)^4.
\end{aligned}$$

By the gradient condition and the bound on b in assumption (R), one may compute for $i=1,2$ that

$$\|\mathcal{R}_i(u)\|_{L^4(\nu_\rho)} \le \frac{C(a)}{n^{3/2}}\|b(\eta)\|_{L^4(\nu_\rho)}\Big(\frac{1}{n}\sum_x|\nabla^n_x\widetilde{H}_{\gamma,u}|^{2+i}\Big). \tag{3.4}$$

Since $\mathbb{E}_{\nu_\rho}[\mathcal{Z}_{s,t}]=1$, by expanding in powers of λ, using Schwarz inequality, the bound on the quadratic variation (3.3), bounds on $\mathcal{R}_i$ (3.4), and invariance of ν_ρ, we obtain a bound for the fourth moment of $\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)$:

$$\mathbb{E}_{\nu_\rho}\Big[\big(\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)\big)^4\Big] \le C(a,H)\|b\|^4_L(\nu_\rho)\Big(|t-s|^2+n^{-3/2}|t-s|\Big). \tag{3.5}$$

3.2. **Generalized Boltzmann-Gibbs principles.** To treat the stochastic differential of $\mathcal{Y}_t^{n,\gamma}$, we replace the spatial terms of form $\sum_{x\in\mathbb{Z}} h(x)\tau_x f(\eta)$, where h is a function on $\mathbb{Z}$ and f is a local function, in terms of the fluctuation field itself to close the evolution equations. Such replacements fall under the term 'Boltzmann-Gibbs principles' coined by Brox-Rost in [18] which have general validity. For instance, the following result forms the backbone of the argument for Proposition 2.1, when starting from the invariant measure ν_ρ, with respect to the papers cited just before the proposition statement.

Proposition 3.1. *Let f be a local $L^2(\nu_\rho)$ function. For $t \geq 0$ and $h \in \ell^2(\mathbb{Z})$, we have*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \Big(\tau_x f(\eta_s^n) - \varphi_f(\rho) - \varphi_f'(\rho)\big(\eta_s^n(x) - \rho\big)\Big) h(x) ds\Big)^2\Big] = 0.$$

We now state a main result of this paper which provides a sharper estimate, perhaps of independent interest, when starting from ν_ρ. To simplify expressions, we will use the notation

$$\big(\eta_s^n\big)^{(\ell)}(x) \ := \ \frac{1}{2\ell+1} \sum_{y\in\Lambda_\ell} \eta_s^n(x+y).$$

Theorem 3.2 (L^2 generalized Boltzmann-Gibbs principle)**.** *Let f be a local $L^5(\nu_\rho)$ function supported on sites Λ_{ℓ_0} such that $\varphi_f(\rho) = \varphi_f'(\rho) = 0$. There exists a constant $C = C(\rho)$ such that, for $t \geq 0$, $\ell > \ell_0^3$ and $h \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$,*

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \Big(\tau_x f(\eta_s^n) - \frac{\varphi_f''(\rho)}{2}\Big\{\Big(\big(\eta_s^n\big)^{(\ell)}(x) - \rho\Big)^2 - \frac{\sigma_\ell^2(\rho)}{2\ell+1}\Big\}\Big) h(x) ds\Big)^2\Big] \\ &\qquad\qquad \leq \ C\|f\|_{L^5(\nu_\rho)}^2 \Big(\frac{t\ell}{n}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}} h^2(x)\Big) + \frac{t^2 n^2}{\ell^{2+\alpha_0}}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}} |h(x)|\Big)^2\Big). \end{aligned}$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \Big(\tau_x f(\eta_s^n) - \varphi_f'(\rho)\Big\{\big(\eta_s^n\big)^{(\ell)}(x) - \rho\Big\} h(x) ds\Big)^2\Big] \\ &\qquad\qquad \leq \ C\|f\|_{L^5(\nu_\rho)}^2 \Big(\frac{t\ell^2}{n}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}} h^2(x)\Big) + \frac{t^2 n^2}{\ell^{1+\alpha_0}}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}} |h(x)|\Big)^2\Big). \end{aligned}$$

Here, $\alpha_0 > 0$ is the power in assumption (EE).

The proof of Theorem 3.2 in given in Section 4. We note, if the uniform spectral gap holds, $\sup_{k,\xi,n} \ell^{-2} W(k,\ell,\xi,n) < \infty$, then the argument shows one can replace in the right-sides above $\|f\|_{L^5(\nu_\rho)}$ with $\|f\|_{L^3(\nu_\rho)}$.

3.3. **Tightness.** We prove tightness of the fluctuation fields, first starting from the invariant measure ν_ρ, using the L^2 generalized Boltzmann-Gibbs principle. Then by the relative entropy bound (2.6), we deduce tightness when beginning from initial measures $\{\mu^n\}$.

Proposition 3.3. *The sequences $\{\mathcal{Y}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{M}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{I}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{B}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{K}_t^{n,\gamma} : t \in [0,T]\}$ and $\{\langle\mathcal{M}_t^{n,\gamma}\rangle : t \in [0,T]\}_{n\geq 1}$, when starting from invariant measure ν_ρ, are tight in the uniform topology on $D([0,T], \mathbb{S}'(\mathbb{R}))$.*

Proof. By Mitoma's criterion [45], to prove tightness of the sequences with respect to uniform topology on $D([0,T],\mathbb{S}'(\mathbb{R}))$, it is enough to show tightness of $\{\mathcal{Y}^{n,\gamma}_t(H); t\in[0,T]\}_{n\geq1}$, $\{\mathcal{M}^{n,\gamma}_t(H): t\in[0,T]\}_{n\geq1}$, $\{\mathcal{I}^{n,\gamma}_t(H): t\in[0,T]\}_{n\geq1}$, $\{\mathcal{B}^{n,\gamma}_t(H): t\in[0,T]\}_{n\geq1}$, $\{\mathcal{K}^{n,\gamma}_t(H): t\in[0,T]\}$ and $\{\langle\mathcal{M}^{n,\gamma}_t(H)\rangle: t\in[0,T]\}_{n\geq1}$, with respect to the uniform topology for all $H\in\mathbb{S}(\mathbb{R})$. Note that all initial values vanish, except $\mathcal{Y}^{n,\gamma}_0(H)$.

Tightness of $\mathcal{Y}^{n,\gamma}_t(H)$, in view of the decomposition $\mathcal{Y}^{n,\gamma}_t(H) = \mathcal{Y}^{n,\gamma}_0(H) + \mathcal{I}^{n,\gamma}_t(H) + \mathcal{B}^{n,\gamma}_t(H) + \mathcal{K}^{n,\gamma}_t(H) + \mathcal{M}^{n,\gamma}_t(H)$, will follow from tightness of each term. The tightness of $\mathcal{Y}^{n,\gamma}_0(H)$, given that we begin under ν_ρ, follows from assumption (IM).

For the martingale term, we use Doob's inequality and stationarity to obtain

$$\begin{aligned}
&\mathbb{P}_{\nu_\rho}\Big(\sup_{\substack{|t-s|\leq\delta\\0\leq s,t\leq T}}|\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)|>\varepsilon\Big)\\
&\quad\leq\ \varepsilon^{-4}\mathbb{E}_{\nu_\rho}\Big[\sup_{\substack{|t-s|\leq\delta\\0\leq s,t\leq T}}|\mathcal{M}^{n,\gamma}_t(H)-\mathcal{M}^{n,\gamma}_s(H)|^4\Big]\\
&\quad\leq\ C\varepsilon^{-4}\delta^{-1}\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}^{n,\gamma}_\delta(H)\big)^4\big].
\end{aligned}$$

Now, by the fourth moment estimate (3.5), we have

$$\delta^{-1}\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}^n_\delta(H)\big)^4\big]\ \leq\ C\|b\|_{L^4(\nu_\rho)}(\delta+n^{-3/2})$$

which vanishes as $n\uparrow\infty$ and then $\delta\downarrow0$. This is enough to conclude that $\{\mathcal{M}^{n,\gamma}_t(H): t\in[0,T]\}_{n\geq1}$ is tight in the uniform topology.

We now prove tightness for $\mathcal{B}^{n,\gamma}_t(H)$ through the Kolmogorov-Centsov criterion. The argument for $\mathcal{I}^{n,\gamma}_t(H)$ is similar. Also, the proofs for $\langle\mathcal{M}^{n,\gamma}_t(H)\rangle$ and $\mathcal{K}^{n,\gamma}_t(H)$, given their forms, are simpler and can be done using invariance of ν_ρ by squaring all terms. We focus on the case $\gamma=1/2$, given that the estimates are analogous and simpler when $1/2<\gamma\leq1$. Let

$$V_b(\eta)\ =\ b(\eta)-\varphi_b(\rho)-\varphi_b'(\rho)(\eta(0)-\rho).$$

By assumption (R1), V_b has range R. Also, by its form, $\varphi_{V_b}(\rho)=\varphi'_{V_b}(\rho)=0$ and also $\varphi''_{V_b}(\rho)=\varphi''_b(\rho)$.

Then,

$$\mathcal{B}^{n,\gamma}_t(H)\ =\ \frac{a}{2}\int_0^t\sum_{x\in\mathbb{Z}}\big(\nabla^n_x H_{\gamma,s}\big)\tau_x V_b(\eta_s)ds.$$

By invoking Theorem 3.2 and translation-invariance of ν_ρ which allows to replace $\nabla^n_x H_{\gamma,s}$ with $\nabla^n_x H$ (which does not depend on time s), for $\ell\geq\ell_0^3=R^3$, we have

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\Big(\mathcal{B}^{n,\gamma}_t(H)-\frac{a}{4}\int_0^t\sum_{x\in\mathbb{Z}}(\nabla^n_x H_{\gamma,s})\varphi''_b(\rho)\Big\{\Big(\big(\eta^n_s\big)^{(\ell)}(x)-\rho\Big)^2-\frac{\sigma^2_\ell(\rho)}{2\ell+1}\Big\}ds\Big)^2\Big]\\
&\quad\leq\ \mathcal{C}(\rho,a)\|b\|^2_{L^4(\nu_\rho)}\Big\{\frac{t\ell}{n}+\frac{t^2n^2}{\ell^{2+\alpha_0}}\Big\}\Big[\Big(\frac1n\sum_{x\in\mathbb{Z}}(\nabla^n_x H)^2\Big)+\Big(\frac1n\sum_{x\in\mathbb{Z}}|\nabla^n_x H|\Big)^2\Big].
\end{aligned}\tag{3.6}$$

On the other hand, given $\sup_{\ell\geq R}E_{\nu_\rho}[(\sqrt{\ell}(\eta^\ell-\rho))^4]<\infty$ by assumption (IM) and $|\varphi''_b(\rho)|\leq C\|b\|_{L^2(\nu_\rho)}$ by assumption (D), and the Schwarz inequality

$(\sum_x h(x)r(x))^2 \leq (\sum_x |h(x)|) \sum_x |h(x)| r^2(x)$, we have for $\ell > R^3$ that

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} (\nabla_x^n H_{\gamma,s}) \frac{\varphi_b''(\rho)}{2}\Big\{\Big(\big(\eta_s^n\big)^{(\ell)}(x) - \rho\Big)^2 - \frac{\sigma_\ell^2(\rho)}{2\ell+1}\Big\}ds\Big)^2\Big] \\ \leq C(\rho)\|b\|^2_{L^2(\nu_\rho)} \frac{t^2n^2}{\ell^2}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}|\nabla_x^n H|\Big)^2.\end{aligned}$$

Hence, for $\ell > R^3$, we have $\mathbb{E}_{\nu_\rho}\big[(\mathcal{B}_t^{n,\gamma}(H))^2\big] \leq C(a,\rho,H)\|b\|^2_{L^4(\nu_\rho)}[t\ell/n + t^2n^2/\ell^2]$, noting the domination $n^2/\ell^{2+\alpha_0} \leq n^2/\ell^2$. Then, if ℓ is taken as $\ell = t^{1/3}n > R^3$, we conclude $\mathbb{E}_{\nu_\rho}\big[(\mathcal{B}_t^{n,\gamma}(H))^2\big] \leq C(a,\rho,H)\|b\|^2_{L^4(\nu_\rho)} t^{4/3}$.

However, when $t^{1/3}n \leq R^3$, we have by the same Schwarz bound that

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[(\mathcal{B}_t^{n,\gamma}(H))^2\Big] &\leq C(\rho,a)\|b\|^2_{L^2(\nu_\rho)} t^2n^2\Big(\frac{1}{n}\sum_x |\nabla_x^n H|\Big)^2 \\ &\leq C(\rho,a,H,R)\|b\|^2_{L^2(\nu_\rho)} t^{4/3}.\end{aligned}$$

This shows tightness of $\mathcal{B}_t^{n,\gamma}(H)$.

Combining these estimates, we conclude the proof of the proposition. □

We now update to when the process begins from the measures $\{\mu^n\}$.

Proposition 3.4. *The fluctuation field sequences* $\{\mathcal{Y}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{M}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{I}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{B}_t^{n,\gamma} : t \in [0,T]\}_{n\geq 1}$, $\{\mathcal{K}_t^{n,\gamma} : t \in [0,T]\}$ *and* $\{\langle\mathcal{M}_t^{n,\gamma}\rangle : t \in [0,T]\}_{n\geq 1}$ *are tight in the uniform topology on* $D([0,T], \mathbb{S}'(\mathbb{R}))$ *when starting from* $\{\mu^n\}$ *satisfying assumption (BE).*

Proof. As before, all initial values vanish except $\mathcal{Y}_0^{n,\gamma}$ which however is tight by (CLT). Next, by Proposition 3.3, we have $\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\mathbb{P}_{\nu_\rho}(O^n_{\delta,\varepsilon}) = 0$ where

$$O^n_{\delta,\varepsilon} = \Big\{\sup_{\substack{|t-s|\leq\delta \\ s,t\in[0,T]}} \|X_t^n - X_s^n\| > \varepsilon\Big\},$$

and X_t^n may be equal to $\mathcal{Y}_t^{n,\gamma}$, $\mathcal{M}_t^{n,\gamma}$, $\mathcal{I}_t^{n,\gamma}$, $\mathcal{B}_t^{n,\gamma}$, $\mathcal{K}_t^{n,\gamma}$ or $\langle\mathcal{M}_t^{n,\gamma}\rangle$. Then, we have by the entropy inequality (2.6) that also $\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\mathbb{P}_{\mu^n}(O_{\delta,\varepsilon}) = 0$ which allows to conclude. □

3.4. **Identification of limit points: Proofs of Theorems 2.2 and 2.3.** With tightness (Proposition 3.4) in hand, we now identify the limit points of $\{\mathcal{Y}_t^{n,\gamma} : t \in [0,T]\}$ and its parts in decomposition (3.2). Let Q^n be the distribution of

$$\Big(\mathcal{Y}_t^{n',\gamma}, \mathcal{M}_t^{n',\gamma}, \mathcal{I}_t^{n',\gamma}, \mathcal{B}_t^{n',\gamma}, \mathcal{K}_t^{n,\gamma}, \langle\mathcal{M}_t^{n',\gamma}\rangle\Big),$$

and let n' be a subsequence where $Q^{n'}$ converges to a limit point Q. Let also $\mathcal{Y}_t$, $\mathcal{M}_t$, $\mathcal{I}_t$, $\mathcal{B}_t$, $\mathcal{K}_t$ and $\mathcal{D}_t$ be the respective limits in distribution of the components. Since tightness is shown in the uniform topology on $D([0,T], \mathbb{S}'(\mathbb{R}))$, we have that $\mathcal{Y}_t$, $\mathcal{M}_t$, $\mathcal{I}_t$, $\mathcal{B}_t$, $\mathcal{K}_t$ and $\mathcal{D}_t$ have a.s. continuous paths.

Let now $G_\varepsilon : \mathbb{R} \to [0,\infty)$ be a smooth compactly supported function for $0 < \varepsilon \leq 1$ which approximates $\iota_\varepsilon(z) = \varepsilon^{-1}1_{[-1,1]}(z\varepsilon^{-1})$ as in the definition of energy solution before Theorem 2.3. That is, $\|G_\varepsilon\|^2_{L^2(\mathbb{R})} \leq 2\|\iota_\varepsilon\|^2_{L^2(\mathbb{R})} = \varepsilon^{-1}$ and

$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1/2}\|G_\varepsilon - \iota_\varepsilon\|_{L^2(\mathbb{R})} = 0$. Define

$$\mathcal{A}^{n,\gamma,\varepsilon}_{s,t}(H) \ := \ \int_s^t \frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla^n_x H)\big[\tau_x\mathcal{Y}^{n,\gamma}_u(G_\varepsilon)\big]^2 du.$$

Since for fixed $0 < \varepsilon \le 1$ the map $\pi_\cdot \mapsto \int_s^t du\int dx\big(\nabla H(x)\big)\big\{\pi_u(\tau_{-x}G_\varepsilon)\big\}^2$ is continuous in the uniform topology on $D([0,T];\mathbb{S}'(\mathbb{R}))$, we have subsequentially in distribution that

$$\lim_{n'\uparrow\infty}\mathcal{A}^{n',\gamma,\varepsilon}_{s,t}(H) \ = \ \int_s^t du\int dx\big(\nabla H(x)\big)\big\{\mathcal{Y}_u(\tau_{-x}G_\varepsilon)\big\}^2 \ =: \ \mathcal{A}^\varepsilon_{s,t}(H).$$

Proposition 3.5. *Suppose the initial distribution is the invariant measure ν_ρ and $t\in[0,T]$.*

When $\gamma = 1/2$, there is a constant $C = C(a,\rho)$ such that

$$\begin{aligned}&\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big|\mathcal{B}^{n,\gamma}_t(H) - \frac{a\varphi_b''(\rho)}{4}\mathcal{A}^{n,\gamma,\varepsilon}_{0,t}(H)\Big|^2\Big]\\ &\qquad\le \ Ct\Big(\varepsilon + \varepsilon^{-1}\|G_\varepsilon - \iota_\varepsilon\|^2_{L^2(\mathbb{R})}\Big)\|b\|^2_{L^4(\nu_\rho)}\Big[\|\nabla H\|^2_{L^2(\mathbb{R})} + \|\nabla H\|^2_{L^1(\mathbb{R})}\Big].\end{aligned}$$

Then, in $L^2(\mathbb{P}_{\nu_\rho})$, $A^\varepsilon_{0,t}(H)$ is a Cauchy ε-sequence. Hence,

$$\frac{a\varphi_b''(\rho)}{4}\mathcal{A}_{0,t}(H) \ := \ \lim_{\varepsilon\downarrow 0}\frac{a\varphi_b''(\rho)}{4}\mathcal{A}^\varepsilon_{0,t}(H) \ = \ \mathcal{B}_t(H).$$

In particular, we conclude $\mathcal{A}_{s,t}(H) \overset{d}{=} \mathcal{A}_{0,t-s}(H)$ does not depend on the specific smoothing family $\{G_\varepsilon\}$. Moreover, when $1/2 < \gamma \le 1$, we have $\mathcal{B}_t(H) = 0$.

In addition, when $1/2 \le \gamma \le 1$,

$$\begin{aligned}\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big|\mathcal{I}^{n,\gamma}_t(H) - \frac{\varphi_c'(\rho)}{2}\int_0^t\mathcal{Y}^{n,\gamma}_s(\Delta H)ds\Big|^2\Big] \ &= \ 0\\ \lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big|\langle\mathcal{M}^{n,\gamma}_t(H)\rangle - \frac{\varphi_b(\rho)}{2}t\|\nabla H\|^2_{L^2(\mathbb{R})}\Big|^2\Big] \ &= \ 0\\ \lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big|\mathcal{K}^{n,\gamma}_t(H)\Big|^2\Big] \ &= \ 0.\end{aligned}$$

Then, in $L^2(\mathbb{P}_{\nu_\rho})$, $\mathcal{K}_t(H) = 0$ and

$$\mathcal{I}_t(H) \ = \ \frac{\varphi_c'(\rho)}{2}\int_0^t\mathcal{Y}_s(\Delta H)ds \ \text{ and } \ D_t(H) \ = \ \frac{\varphi_b(\rho)}{2}t\|\nabla H\|^2_{L^2(\mathbb{R})}.$$

Proof. We prove the limit display for $\mathcal{B}_t(H)$ when $\gamma = 1/2$ which shows, by a Fatou's lemma, that $\mathbb{E}_{\nu_\rho}\big[\big|\mathcal{B}_t(H) - (a\varphi_b''(\rho)/4)\mathcal{A}^\varepsilon_{0,t}(H)\big|^2\big] \le C(a,\rho,H)t\varepsilon$. Therefore, $\mathcal{A}^\varepsilon_{0,t}(H)$, as a sequence in ε, is Cauchy in $L^2(\mathbb{P}_{\nu_\rho})$. The arguments for $\mathcal{I}_t(H)$, $\mathcal{D}_t(H)$, and $\mathcal{K}_t(H)$, noting their forms, are similar; for $\mathcal{D}_t(H)$ and $\mathcal{K}_t(H)$ one might also use spatial mixing assumed in (IM). To simplify notation, we will call $n = n'$.

Note, for $\ell = \varepsilon n$, that

$$\begin{aligned}\sum_{x\in\mathbb{Z}}(\nabla^n_x H_{\gamma,s})\Big(\big(\eta^n_s\big)^{(\ell)}(x) - \rho\Big)^2 \ &= \ \sum_{x\in\mathbb{Z}}(\nabla^n_x H_{\gamma,s})\Big(\frac{1}{2n\varepsilon+1}\sum_{|z|\le n\varepsilon}(\eta^n_s(z+x)-\rho)\Big)^2\\ &= \ \frac{1+O(n^{-1})}{n}\sum_{x\in\mathbb{Z}}(\nabla^n_x H)\big[\tau_x\mathcal{Y}^{n,\gamma}_s(\iota_\varepsilon)\big]^2.\end{aligned}$$

Here, the shift by $n^{-1}\lfloor a\varphi_b'(\rho)sn^2/(2n^\gamma)\rfloor$ in $\nabla_x^n H_{\gamma,s}$ (cf. (3.1)) was transferred to $\tau_x\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)$.

Then, with $\ell=\varepsilon n$, by Theorem 3.2, as in the bound (3.6), we have

$$\begin{aligned}
&\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big(\mathcal{B}_t^{n,\gamma}(H)-\frac{a\varphi_{b^n}''(\rho)}{4}\int_0^t\frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla_x^nH)\tau_x\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)^2ds\Big)^2\Big]\\
&= \lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big(\mathcal{B}_t^{n,\gamma}(H)\\
&\qquad -\frac{a\varphi_{b^n}''(\rho)}{4}\int_0^t\frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla_x^nH)\tau_x\Big\{\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)^2-\frac{\sigma_\ell^2(\rho)}{2\varepsilon}\Big\}ds\Big)^2\Big]\\
&\le \lim_{n\uparrow\infty}C(a,\rho)\|b^n\|^2_{L^4(\nu_\rho)}t\Big(\varepsilon+\frac{1}{\varepsilon^{2+\alpha_0}n^{\alpha_0}}\Big)\\
&\qquad\times\Big[\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}\big(\nabla_x^nH\big)^2\Big)+\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}\big|\nabla_x^nH\big|\Big)^2\Big].
\end{aligned}$$

Here, as the sum of $\nabla_x^nH_{\gamma,s}$ on x vanishes, we introduced the centering constant $(2\varepsilon)^{-1}\sigma_\ell^2(\rho)$ in the second line.

Now,

$$\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)^2-\mathcal{Y}_x^{n,\gamma}(G_\varepsilon)^2 \;=\; \big[\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)-\mathcal{Y}_s^{n,\gamma}(G_\varepsilon)\big]\cdot\big[\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)+\mathcal{Y}_s^{n,\gamma}(G_\varepsilon)\big]$$

and by (IM2)

$$\mathcal{C}_{\nu_\rho}\big(\iota_\varepsilon-G_\varepsilon,\iota_\varepsilon-G_\varepsilon\big)^{1/2}\cdot\mathcal{C}_{\nu_\rho}\big(\iota_\varepsilon+G_\varepsilon,\iota_\varepsilon+G_\varepsilon\big)^{1/2}\;\le\;C(\rho)\varepsilon^{-1/2}\|G_\varepsilon-\iota_\varepsilon\|_{L^2(\mathbb{R})}.$$

Hence, by Schwarz inequality,

$$\begin{aligned}
&\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t\frac{1}{n}\sum_{x\in\mathbb{Z}}(\nabla_x^nH)\tau_x\mathcal{Y}_s^{n,\gamma}(\iota_\varepsilon)^2ds-\mathcal{A}_{0,t}^{n,\gamma,\varepsilon}(H)\Big)^2\Big]\\
&\le\;C(\rho)\varepsilon^{-1}\|G_\varepsilon-\iota_\varepsilon\|^2_{L^2(\mathbb{R})}t^2\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}\big|\nabla_x^nH\big|\Big)^2.
\end{aligned}$$

Finally, combining these estimates with the inequality $(a+b)^2\le 2a^2+2b^2$, and by assumption (D) that $\lim_{n\uparrow\infty}\varphi_{b^n}''(\rho)=\varphi_b''(\rho)$, we finish the proof. □

Proposition 3.6. *Suppose the initial measures $\{\mu^n\}$ satisfy assumption (BE), and $t\in[0,T]$.*

When $\gamma=1/2$, we have $\mathcal{A}_{0,t}^\varepsilon(H)$ is a Cauchy ε-sequence in probability with respect to the limit measure Q, and hence

$$\frac{a\varphi_b''(\rho)}{4}\mathcal{A}_{0,t}(H)\;:=\;\lim_{\varepsilon\downarrow 0}\frac{a\varphi_b''(\rho)}{4}\mathcal{A}_{0,t}^\varepsilon(H)\;=\;\mathcal{B}_t(H).$$

On the other hand, when $1/2<\gamma\le 1$, we have $\mathcal{B}_t(H)\equiv 0$.

When $1/2\le\gamma\le 1$, we have $\mathcal{K}_t^n(H)\equiv 0$,

$$\mathcal{I}_t(H)\;=\;\frac{\varphi_c'(\rho)}{2}\int_0^t\mathcal{Y}_s(\Delta H)ds\;\;\text{and}\;\;\mathcal{D}_t(H)\;=\;\frac{\varphi_b(\rho)}{2}t\|\nabla H\|^2_{L^2(\mathbb{R})}.$$

Proof. By assumption (BE), and lower semi-continuity of entropy, the limit measure Q also has bounded entropy with respect to $\mathbb{P}_{\nu_\rho}$, $H(Q;\mathbb{P}_{\nu_\rho})<\infty$. When $\gamma=1/2$, by the $L^2(\mathbb{P}_{\nu_\rho})$ statements in Proposition 3.5, we have for $\delta>0$ that

$\lim_{\varepsilon\downarrow 0} Q\big(\big|\mathcal{B}_t(H) - (a\varphi_b''(\rho)/4)\mathcal{A}_t^\varepsilon(H)\big| > \delta\big) = 0$ and so $\mathcal{A}_t^\varepsilon(H)$ is Cauchy in probability with respect to Q. Therefore, $\lim_{\varepsilon\downarrow 0}(a\varphi_b''(\rho)/4)\mathcal{A}_t^\varepsilon(H) = \mathcal{B}_t(H)$.

The other claims follow similarly. □

Proof of Theorems 2.2 and 2.3. Let $H \in \mathbb{S}(\mathbb{R})$, $t \in [0,T]$, and suppose the initial measures are $\{\mu^n\}$. When $\gamma = 1/2$, by the decomposition (3.2), Proposition 3.6, and tightness of the constituent processes $\mathcal{M}_t^{n,\gamma}$, $\mathcal{Y}_t^{n,\gamma}$, $\mathcal{Y}_0^{n,\gamma}$, $\mathcal{I}_t^{n,\gamma}$, $\mathcal{B}_t^{n,\gamma}$ and $\mathcal{K}_t^{n,\gamma}$ in the uniform topology, any limit point of

$$(\mathcal{M}_t^{n,\gamma}, \mathcal{Y}_t^{n,\gamma}, \mathcal{Y}_0^{n,\gamma}, \mathcal{I}_t^{n,\gamma}, \mathcal{B}_t^{n,\gamma}, \mathcal{K}_t^{n,\gamma})$$

satisfies

$$\mathcal{M}_t(H) \;=\; \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{\varphi_c'(\rho)}{2}\int_0^t \mathcal{Y}_s(\triangle H)ds - (a\varphi_b''(\rho)/2)\mathcal{A}_t(H).$$

However, when $1/2 < \gamma \leq 1$,

$$\mathcal{M}_t(H) \;=\; \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \frac{\varphi_c'(\rho)}{2}\int_0^t \mathcal{Y}_s(\triangle H)ds. \tag{3.7}$$

Also, in both cases, $\mathcal{Y}_0(H) = \bar{\mathcal{Y}}_0(H)$ by assumption (CLT).

We also claim in both cases that $\mathcal{M}_t(H)$ is a continuous martingale with a quadratic variation

$$\langle \mathcal{M}_t(H)\rangle \;=\; \frac{\varphi_b(\rho)}{2} t\|\nabla H\|^2_{L^2(\mathbb{R})}.$$

Indeed, by Proposition 3.6, any limit point of the quadratic variation sequence equals $\mathcal{D}_t(H) = (\varphi_b(\rho)/2)t\|\nabla H\|^2_{L^2(\mathbb{R})}$. Next, $\mathcal{M}_t(H)$ as the limit of martingales with respect to the uniform topology is a continous martingale. Also, by the triangle inequality, Doob's inequality and the quadratic variation bound (3.3),

$$\begin{aligned}\sup_n \mathbb{E}_{\nu_\rho}\Big[\sup_{0\leq s\leq t} |\mathcal{M}_s^{n,\gamma}(H) - \mathcal{M}_{s^-}^{n,\gamma}(H)|\Big] \;&\leq\; 2\sup_n \mathbb{E}_{\nu_\rho}\Big[\sup_{u\in[0,t]} |\mathcal{M}_u^{n,\gamma}(H)|^2\Big]^{1/2}\\ &\leq\; 2\sup_n \mathbb{E}_{\nu_\rho}\Big[\langle M_t^{n,\gamma}(H)\rangle\Big]^{1/2} \;\leq\; C(a,T)\|b\|_{L^1(\nu_\rho)}\|\nabla H\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Then, by [34][Corollary VI.6.30], $(\mathcal{M}_t^{n,\gamma}(H), \langle \mathcal{M}_t^{n,\gamma}(H)\rangle)$ converges on a subsequence in distribution to $(\mathcal{M}_t(H), \langle \mathcal{M}_t(H)\rangle)$. Since, also $\langle \mathcal{M}_t^{n,\gamma}(H)\rangle$ converges on a subsequence in distribution to $\mathcal{D}_t(H) = (\varphi_b(\rho)/2)t\|\nabla H\|^2_{L^2(\mathbb{R})}$, we have $\langle \mathcal{M}_t(H)\rangle = (\varphi_b(\rho)/2)t\|\nabla H\|^2_{L^2(\mathbb{R})}$.

By Proposition 3.6, when $\gamma = 1/2$, $\mathcal{Y}_t$ is a 'Cauchy energy solution' corresponding to the stochastic Burgers equation (2.11). But, if initially $\mu^n \equiv \nu_\rho$, by Proposition 3.5, $\mathcal{Y}_t$ is an 'L^2 energy solution'. This completes the proof of Theorem 2.3.

However, when $1/2 < \gamma \leq 1$, by the form of $\mathcal{M}_t(H)$ in (3.7), we conclude $\mathcal{Y}_t(H)$ solves the Ornstein-Uhlenbeck equation (2.10). By uniqueness, all subsequences converge to the same limit, and we obtain Theorem 2.2. □

4. Proof of the generalized Boltzmann-Gibbs Principle

We start by recalling the notion of $H_{1,n}$ and $H_{-1,n}$ spaces. For $n \geq n_0$, recall $S_n = (L_n + L_n^*)/2$ (cf. near (2.2)), and define the $H_{1,n}$ semi-norm $\|\cdot\|_{1,n}$ on $L^2(\nu_\rho)$ functions by

$$\|f\|^2_{1,n} \;:=\; E_{\nu_\rho}\big[f(-S_n)f\big] \;=\; n^2 D_{\nu_\rho}(f).$$

The Hilbert space $H_{1,n}$ is then the completion of functions with finite $H_{1,n}$ norm modulo norm-zero functions. In particular, local bounded functions are dense in $H_{1,n}$.

Correspondingly, one can define the dual semi-norm $\|\cdot\|_{-1,n}$ with respect to the $L^2(\nu_\rho)$ inner-product by

$$\|f\|_{-1,n} \ := \ \sup\Big\{\frac{E_{\nu_\rho}[f\phi]}{\|\phi\|_{1,n}} : \phi \neq 0 \text{ local, bounded}\Big\},$$

and the Hilbert space $H_{-1,n}$ which is the completion over those functions with finite $\|\cdot\|_{-1,n}$ norm modulo norm-zero functions.

We now state a helping lemma for the results in this section. Define the restricted Dirichlet form on local, bounded functions with respect to the grand canonical measure ν_ρ as

$$D_{\nu_\rho,\ell}(\phi) \ = \sum_{x,x+1\in\Lambda_\ell} E_{\nu_\rho}\Big[b_x^{R,n}(\eta)\big(\nabla_{x,x+1}\phi(\eta)\big)^2\Big].$$

Recall the collection $\eta_r^c := \{\eta(x) : x \notin \Lambda_r\}$.

Proposition 4.1. *Let $r : \Omega \to \mathbb{R}$ be an $L^4(\nu_\rho)$ function and $\ell_0 \geq 2$. Suppose that $E_{\nu_\rho}[r|\eta^{(\ell_0)}, \eta_{\ell_0}^c] = 0$ a.s. Then, for local, bounded functions ϕ, we have*

$$\big|E_{\nu_\rho}[r(\eta)\phi(\eta)]\big| \ \leq \ E_{\nu_\rho}\Big[W\Big(\sum_{x\in\Lambda_{\ell_0}}\eta(x), \ell_0, \eta_{\ell_0}^c, n\Big)^2\Big]^{1/4}\|r\|_{L^4(\nu_\rho)} D_{\nu_\rho,\ell_0}^{1/2}(\phi).$$

Proof. Recall, from Subsection 2.1, for $k \geq 0$, $\ell_0 \geq 2$ and $\xi \in \Omega$, the space

$$\mathcal{G}_{k,\ell_0,\xi} \ = \ \Big\{\eta : \sum_{x\in\Lambda_{\ell_0}}\eta(x) = k, \eta(y) = \xi(y) \text{ for } y \notin \Lambda_{\ell_0}\Big\}$$

and generator $S_{n,\mathcal{G}_{k,\ell_0,\xi}}$ which governs the evolution of the symmetrized process on $\mathcal{G}_{k,\ell_0,\xi}$. Suppose $W(k,\ell_0,\xi,n) < \infty$ and the measure $\nu_{k,\ell_0,\xi}$ is the unique invariant measure for the process.

Given $E_{\nu_\rho}[r|\sum_{|x|\leq\ell_0}\eta(x) = k, \eta(y) = \xi(y) \text{ for } y \notin \Lambda_{\ell_0}] = E_{\nu_{k,\ell_0,\xi}}[r] = 0$, we have r restricted to $\mathcal{G}_{k,\ell_0,\xi}$ is orthogonal to constant functions and therefore belongs to the range of $-S_{n,\mathcal{G}_{k,\ell_0,\xi}}$, that is the equation $r = -S_{n,\mathcal{G}_{k,\ell_0,\xi}}u$ can be solved for some function $u : \mathcal{G}_{k,\ell_0,\xi} \to \mathbb{R}$.

Now, with $k = \sum_{x\in\Lambda_{\ell_0}}\eta(x)$, $W(k,\ell_0,\eta_{\ell_0}^c,n) < \infty$ a.s. by assumption (G). Hence,

$$\begin{aligned}
\big|E_{\nu_\rho}[r\phi]\big| \ &= \ \big|E_{\nu_\rho}\big[E_{\nu_\rho}[r\phi|\eta^{(\ell_0)}, \eta_{\ell_0}^c]\big]\big| \\
&= \ \big|E_{\nu_\rho}\big[E_{\nu_\rho}[(-S_{n,\mathcal{G}_{k,\ell_0,\eta_{\ell_0}^c}}u)\phi|\eta^{(\ell_0)}, \eta_{\ell_0}^c]\big]\big| \\
&\leq \ E_{\nu_\rho}\Big[E_{\nu_\rho}[u(-S_{n,\mathcal{G}_{k,\ell_0,\eta_{\ell_0}^c}}u)|\eta^{(\ell_0)}, \eta_{\ell_0}^c]^{1/2} \\
&\qquad\qquad \times E_{\nu_\rho}[\phi(-S_{n,\mathcal{G}_{k,\ell_0,\eta_{\ell_0}^c}}\phi)|\eta^{(\ell_0)}, \eta_{\ell_0}^c]^{1/2}\Big].
\end{aligned}$$

The last line follows as $-S_{n,\mathcal{G}_{k,\ell_0,\xi}}$ is a nonnegative symmetric operator, and therefore has a square root.

Further, since $W(k,\ell_0,\xi,n)$ is the reciprocal of the spectral gap for $-S_{n,\mathcal{G}_{k,\ell_0,\xi}}$, we have

$$E_{\nu_\rho}\big[ru|\eta^{(\ell_0)}, \eta_{\ell_0}^c\big] \ \leq \ W(k,\ell_0,\eta_{\ell_0}^c,n)E_{\nu_\rho}\big[r^2|\eta^{(\ell_0)}, \eta_{\ell_0}^c\big].$$

Therefore, we conclude

$$\big|E_{\nu_\rho}[r\phi]\big| \;\le\; E_{\nu_\rho}\Big[W\Big(\sum_{x\in\Lambda_{\ell_0}}\eta(x),\ell_0,\eta^c_{\ell_0},n\Big)E_{\nu_\rho}[r^2|\eta^{(\ell_0)},\eta^c_{\ell_0}]\Big]^{1/2}D^{1/2}_{\nu_\rho,\ell_0}(\phi).$$

The desired bound now follows from Schwarz inequality. □

The following bound on the variance of additive functionals is the main way we control the fluctuations of several quantities in the sequel. A proof of Proposition 4.2 can be found in [36][Appendix 1.6].

To simplify notation, for the rest of the section, we will drop the superscript 'n' and write $\eta^n = \eta$.

Proposition 4.2. *Let $r:\Omega\to\mathbb{R}$ be a mean-zero $L^2(\nu_\rho)$ function, $\varphi_r(\rho)=0$. Then,*

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t r(\eta_s)ds\Big)^2\Big] \;\le\; 20t\|r\|^2_{-1,n}.$$

The proof of Theorem 3.2, given at the end of the section, is made through a succession of steps, labeled 'one-block', 'renormalization step', 'two-blocks' and 'equivalence of ensembles' estimates.

Lemma 4.3 (One-block estimate)**.** *Let $f:\Omega\to\mathbb{R}$ be a local $L^4(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\varphi_f(\rho)=0$. Then, there exists a constant $C=C(\rho)$ such that for $\ell\ge\ell_0$, $t\ge 0$ and $h\in\ell^1(\mathbb{Z})\cap\ell^2(\mathbb{Z})$:*

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t\sum_{x\in\mathbb{Z}}h(x)\tau_x\Big\{f(\eta_s)-E_{\nu_\rho}[f(\eta_s)|\eta_s^{(\ell)},(\eta_s)^c_\ell]\Big\}ds\Big)^2\Big]$$
$$\le\; Ct\frac{\ell^3}{n^2}\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x).$$

Proof. By Proposition 4.2, we need only to estimate the $H_{-1,n}$ norm of the integrand (which is in $L^2(\nu_\rho)$ since $h\in\ell^1(\mathbb{Z})$). Bound the $H_{-1,n}$ norm multiplied by n, using Proposition 4.1, as follows:

$$\begin{aligned}&\sup_\phi\Big\{D^{-\frac12}_{\nu_\rho}(\phi)E_{\nu_\rho}\big[\sum_{x\in\mathbb{Z}}h(x)\tau_x\Big\{f-E_{\nu_\rho}[f|\eta^{(\ell)},\eta^c_\ell]\Big\}\phi\big]\Big\}\\ &=\;\sup_\phi\sum_{x\in\mathbb{Z}}D^{-\frac12}_{\nu_\rho}(\phi)E_{\nu_\rho}\Big[h(x)\tau_x\big(f-E_{\nu_\rho}[f|\eta^{(\ell)},\eta^c_\ell]\big)\phi\Big] \qquad (4.1)\\ &\le\;\sup_\phi D^{-\frac12}_{\nu_\rho}(\phi)\sum_{x\in\mathbb{Z}}|h(x)|E_{\nu_\rho}\big[W\big(\sum_{x\in\Lambda_\ell}\eta(x),\ell,\eta^c_\ell,n\big)^2\big]^{\frac14}\|f\|_{L^4(\nu_\rho)}D^{\frac12}_{\nu_\rho,\ell}(\tau_{-x}\phi).\end{aligned}$$

Observe now, by translation-invariance of ν_ρ, that

$$\sum_{x\in\mathbb{Z}}D_{\nu_\rho,\ell}(\tau_{-x}\phi)\;\le\;(2\ell+1)D_{\nu_\rho}(\phi).$$

Then, noting the spectral gap assumption (G), and using the relation $2ab=\inf_{\kappa>0}[a^2\kappa+\kappa^{-1}b^2]$, we bound (4.1) by

$$\sup_\phi D^{-\frac12}_{\nu_\rho}(\phi)\inf_{\kappa>0}\Big\{\kappa C\ell^2\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x)+\kappa^{-1}C\ell D_{\nu_\rho}(\phi)\Big\}$$
$$\le\Big(C\ell^3\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x)\Big)^{\frac12}$$

where $C = C(\rho)$ is a constant. This finishes the proof. □

Now we double the size of the box in the conditional expectation.

Lemma 4.4 (Renormalization step)**.** *Let $f : \Omega \to \mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\varphi_f(\rho) = \varphi'_f(\rho) = 0$. There exists a constant $C = C(\rho)$ such that for $\ell \geq \ell_0$, $t \geq 0$ and $h \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$:*

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta_s)|\eta_s^{(\ell)}, (\eta_s)_\ell^c] - E_{\nu_\rho}[f(\eta_s)|\eta_s^{(2\ell)}, (\eta_s)_{2\ell}^c]\Big\}h(x)ds\Big)^2\Big]$$
$$\leq\ C\|f\|^2_{L^5(\nu_\rho)} t\frac{\ell}{n^2}\sum_{x\in\mathbb{Z}} h^2(x).$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta_s)|\eta_s^{(\ell)}, (\eta_s)_\ell^c] - E_{\nu_\rho}[f(\eta_s)|\eta_s^{(2\ell)}, (\eta_s)_{2\ell}^c]\Big\}h(x)ds\Big)^2\Big]$$
$$\leq\ C\|f\|^2_{L^5(\nu_\rho)} t\frac{\ell^2}{n^2}\sum_{x\in\mathbb{Z}} h^2(x).$$

Proof. We prove the first statement as the second is similar. Since

$$E_{\nu_\rho}\Big[E_{\nu_\rho}\big[f(\eta)\big|\eta^{(\ell)}, \eta_\ell^c\big]\big|\eta^{(2\ell)}, \eta_{2\ell}^c\Big] \ =\ E_{\nu_\rho}\big[f(\eta)\big|\eta^{(2\ell)}, \eta_{2\ell}^c\big],$$

we follow now the same steps as in the proof of Lemma 4.3 to the last line. At this point, we need a sufficient bound on the variance

$$\begin{aligned}
&\|E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}, \eta_\ell^c] - E_{\nu_\rho}[f(\eta)|\eta^{(2\ell)}, \eta_{2\ell}^c]\|^2_{L^4(\nu_\rho)} \\
&=\ \Big\|E_{\nu_\rho}\Big[f(\eta) - \frac{\varphi''_f(\rho)}{2}\Big\{(\eta^{(\ell)}-\rho)^2 - \frac{\sigma_\ell^2(\rho)}{2\ell+1}\Big\}\Big|\eta^{(\ell)}\Big] \\
&\quad -E_{\nu_\rho}\Big[f(\eta) - \frac{\varphi''_f(\rho)}{2}\Big\{(\eta^{(2\ell)}-\rho)^2 - \frac{\sigma_{2\ell}^2(\rho)}{2(2\ell+1)}\Big\}\Big|\eta^{(2\ell)}\Big]\Big\|^2_{L^4(\nu_\rho)} \\
&\quad +O(\|f\|^2_{L^2(\nu_\rho)}\ell^{-2}).
\end{aligned}$$

The last equality follows from bounding the fourth moment of $(\eta^{(\ell)} - \rho)$ using assumptions (IM) and (D). We now apply the equivalence of ensembles assumption (EE), obtaining a further bound on the right-hand side of $O(\ell^{-2})$. □

Lemma 4.5 (Two-blocks estimate)**.** *Let $f : \Omega \to \mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\varphi_f(\rho) = \varphi'_f(\rho) = 0$. Then, there exists a constant $C = C(\rho)$ such that for $\ell \geq \ell_0$, $t \geq 0$ and $h \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$:*

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta)|\eta^{(\ell_0)}, \eta_{\ell_0}^c] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}, \eta_\ell^c]\Big\}h(x)ds\Big)^2\Big]$$
$$\leq C\|f\|^2_{L^5(\nu_\rho)} t\frac{\ell}{n^2}\sum_{x\in\mathbb{Z}} h^2(x).$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta)|\eta^{(\ell_0)},\eta^c_{\ell_0}] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)},\eta^c_\ell]\Big\}h(x)ds\Big)^2\Big]$$
$$\leq C\|f\|^2_{L^5(\nu_\rho)} t\frac{\ell^2}{n^2}\sum_{x\in\mathbb{Z}} h^2(x).$$

Proof. We prove the first display as the second is analogous. Again, we invoke Proposition 4.2 and bound the square of the $H_{-1,n}$ norm. To this end, write $\ell = 2^{m+1}\ell_0 + r$ where $0 \leq r \leq 2^{m+1}\ell_0 - 1$. Then,

$$\begin{aligned}
&E_{\nu_\rho}[f(\eta)|\eta^{(\ell_0)},\eta^c_{\ell_0}] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)},\eta^c_\ell]\\
&= E_{\nu_\rho}[f(\eta)|\eta^{(2^{m+1}\ell_0)},\eta^c_{2^{m+1}\ell_0}] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)},\eta^c_\ell]\\
&\quad + \sum_{i=0}^m \big\{E_{\nu_\rho}[f(\eta)|\eta^{(2^i\ell_0)},\eta^c_{2^i\ell_0}] - E_{\nu_\rho}[f(\eta)|\eta^{(2^{i+1}\ell_0)},\eta^c_{2^{i+1}\ell_0}]\big\}.
\end{aligned}$$

Now, by Minkowski's inequality, with respect to the $H_{-1,n}$ norm, over the $m+2$ terms, and Lemma 4.4, we obtain that the left-side of the display in the lemma statement is bounded by

$$\Big\{\big(\frac{Ct2^{m+2}\ell_0}{n^2}\big)^{1/2} + \sum_{i=0}^m \big(\frac{Ct2^{i+1}\ell_0}{n^2}\big)^{1/2}\Big\}^2 \sum_{x\in\mathbb{Z}} h^2(x) \;\leq\; \frac{C\|f\|^2_{L^5(\nu_\rho)}t\ell}{n^2}\sum_{x\in\mathbb{Z}} h^2(x)$$

to finish the proof. □

Lemma 4.6 (Equivalence of ensembles estimate)**.** *Let $f:\Omega\to\mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\varphi_f(\rho) = \varphi'_f(\rho) = 0$. Then, there exists a constant $C = C(\rho)$ such that for $\ell \geq \ell_0$, $t \geq 0$ and $h \in \ell^1(\mathbb{Z})$:*

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta_s)|\eta_s^{(\ell)},(\eta_s)^c_\ell]\\
&\qquad\qquad - \frac{\varphi''_f(\rho)}{2}\Big((\eta_s^{(\ell)}-\rho)^2 - \frac{\sigma^2_\ell(\rho)}{2\ell+1}\Big)\Big\}h(x)ds\Big)^2\Big]\\
&\leq\; C\|f\|^2_{L^5(\nu_\rho)} t^2\frac{n^2}{\ell^{2+\alpha_0}}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2
\end{aligned}$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} \tau_x\Big\{E_{\nu_\rho}[f(\eta_s)|\eta_s^{(\ell)},(\eta_s)^c_\ell] - \varphi'_f(\rho)\big(\eta_s^{(\ell)}-\rho\big)\Big)^2\Big]$$
$$\leq\; C\|f\|^2_{L^5(\nu_\rho)} t^2\frac{n^2}{\ell^{1+\alpha_0}}\Big(\frac{1}{n}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2.$$

Here, $\alpha_0 > 0$ is the power mentioned in assumption (EE).

Proof. By squaring and using invariance of ν_ρ, the left-hand side of the display is bounded by

$$2t^2\mathbb{E}_{\nu_\rho}\Big[\big(\sum_{x\in\mathbb{Z}}|h(x)||r(x)|\big)^2\Big]$$

where $r(x)$ is the expression in curly braces in the display of Lemma 4.6. Now, by Schwarz inequality,

$$\Big(\sum_{x\in\mathbb{Z}}|h(x)|r(x)\Big)^2 \;\leq\; \Big(\sum_{x\in\mathbb{Z}}|h(x)|\Big)\sum_{x\in\mathbb{Z}}|h(x)|r^2(x).$$

Since ν_ρ is translation-invariant, the desired bound is now obtained by noting the form of $r(x)$ and the equivalence of ensembles assumption (EE). □

Proof of Theorem 3.2. By combining Lemma 4.3 with $\ell = \ell_0$, and Lemmas 4.5 and Lemma 4.6, we straightforwardly obtain the result. □

5. Equivalence of ensembles

We prove in Proposition 5.1 that condition (EE) holds for a large class of systems with product invariant measures. In this case, $\nu_{k,\ell,\xi}$ does not depend on ξ, which simplifies the conditional expectation in the statement of (EE).

Next, we show in Proposition 5.2 that (EE) also holds for the Markov chain measure $\nu_{1/2}$ defined in Subsection 2.5. Some parts of the proofs of these statements are similar those in [61].

Define $\Lambda_n^+ = \{x : 1 \leq x \leq n\}$.

Proposition 5.1. *Let ν_ρ be a product measure on Ω such that (IM) holds, and $0 < \nu_\rho(\eta(0) = 1) < 1$. Let also f be a local $L^5(\nu_\rho)$ function, supported on sites Λ_ℓ^+, such that $\varphi_f(\rho) = \varphi'_f(\rho) = 0$. Then, there exists a constant $C = C(\rho)$ such that*

$$\Big\|E_{\nu_\rho}[f(\eta)|y] - \Big\{y^2 - \frac{\sigma^2(\rho)}{n}\Big\}\frac{\varphi''_f(\rho)}{2}\Big\|_{L^4(\nu_\rho)} \;\leq\; \frac{C\|f\|_{L^5(\nu_\rho)}}{n^{3/2}}.$$

On the other hand, when only $\varphi_f(\rho) = 0$ is known,

$$\Big\|E_{\nu_\rho}[f(\eta)|y] - y\varphi'_f(\rho)\Big\|_{L^4(\nu_\rho)} \;\leq\; \frac{C\|f\|_{L^5(\nu_\rho)}}{n}.$$

Here, $y := \frac{1}{n}\sum_{x\in\Lambda_n^+}\eta(x) - \rho$.

Proof. We prove the first display as the second statement, following the same scheme, has a simpler argument. Recall the tilted measures $\{\nu_z : \rho_* < z < \rho^*\}$ given after assumption (D1) which are well defined as ν_ρ is a product measure. Let $\sigma^2(z) = E_{\nu_z}[(\eta(0) - z)^2]$. Note also the canonical expectation $E_{\nu_z}[f|y]$ does not depend on the specific value z, and that we are free to choose it as desired.

Then,

$$\begin{aligned} E_{\nu_\rho}[f(\eta)|y] &= E_{\nu_{y+\rho}}\big[f(\eta)\big|\frac{1}{n}\sum_{x\in\Lambda_n^+}\eta(x) - \rho = y\big] \\ &= \frac{E_{\nu_{y+\rho}}\big[f(\eta)1\big(\frac{1}{n}\sum_{x\in\Lambda_n^+}\eta(x) - \rho = y\big)\big]}{\nu_{y+\rho}\big(\frac{1}{n}\sum_{x\in\Lambda_n^+}\eta(x) - \rho = y\big)}. \end{aligned}$$

Define $\theta_m(z) = \sqrt{m}\nu_{y+\rho}(\sum_{x\in\Lambda_m^+}\eta(x) - \rho - y = z)$, and write the last expression as

$$E_{\nu_{y+\rho}}\left[f(\eta)\frac{\sqrt{n}\theta_{n-\ell}(-\sum_{x\in\Lambda_\ell^+}(\eta(x) - y - \rho))}{\sqrt{n-\ell}\theta_n(0)}\right].$$

Let $\psi_y(t) = E_{\nu_{y+\rho}}[e^{it(\eta(x)-\rho-y)}]$ be the characteristic function. Then, one can write

$$\begin{aligned}\theta_m(x) &= \frac{\sqrt{m}}{2\pi}\int_{-\pi}^{\pi} e^{itx}\psi_y^m(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi\sqrt{m}}^{\pi\sqrt{m}} e^{itx/\sqrt{m}}\psi_y^m(t/\sqrt{m})dt.\end{aligned}$$

By Taylor expansion,

$$\begin{aligned}2\pi\theta_m(x) &= \int_{-\pi\sqrt{m}}^{\pi\sqrt{m}} \psi_y^m(t/\sqrt{m})dt - \int_{-\pi\sqrt{m}}^{\pi\sqrt{m}} \frac{ixt}{\sqrt{m}}\psi_y^m(t/\sqrt{m})dt \qquad (5.1)\\ &-\frac{1}{2}\int_{\sqrt{-\pi\sqrt{m}}}^{\pi\sqrt{m}} \frac{x^2t^2}{m}\psi_y^m(t/\sqrt{m})dt + O\Big(\frac{|x|^3}{m^{3/2}}\Big)\int_{\sqrt{-\pi\sqrt{m}}}^{\pi\sqrt{m}} |t|^3|\psi_y^m(t/\sqrt{m})|dt.\end{aligned}$$

Let now $\delta > 0$ be a small number. We observe that the integral, when $|y| \le \delta$, in the last term in (5.1) is uniformly bounded in m.

Indeed, given ν_ρ governs i.i.d. $\mathbb{N}_0$-valued random variables with moment generating function and which are nondegenerate in that $0 < \nu_\rho(\eta(0) = 1) < 1$, for $\varepsilon < |t/\sqrt{m}| \le \pi$, one computes $|\psi_y^m(t/\sqrt{m})| < C_0(y)^m$ where $\sup_{|y|\le\delta} C_0(y) < 1$; also, for $0 \le |t/\sqrt{m}| < \varepsilon$, $\psi_y^m(t/\sqrt{m}) = (1 - (t^2\sigma^2(y+\rho)/(2m)) + O(m^{-3/2}))^m$ and $|\psi_y^m(t/\sqrt{m})| \le e^{-C_1(y)t^2}$. Here, $\sup_{|y|\le\delta}\sigma^2(y+\rho) < \infty$ and $\inf_{|y|\le\delta}\min\{\sigma^2(y+\rho), C_1(y)\} > 0$.

Also, the second integral in (5.1), when $|y| \le \delta$, given

$$\psi_y^m(t/\sqrt{m}) = \Big(1 - \frac{t^2\sigma^2(y+\rho)}{2m}\Big)^m\Big[1 + O(m^{-1/2})\Big],$$

is almost the integral of an odd function and is of order $O(m^{-1/2})$.

On the other hand, given ν_ρ is a translation-invariant product measure whose marginal has moment generating function, by the classical local limit theorem, $\lim_{m\uparrow\infty}\theta_m(0) = (2\pi\sigma^2(y+\rho))^{-1/2}$. Then, for $|y| \le \delta$, we have

$$\begin{aligned}E_{\nu_\rho}[f(\eta)|y] &= \kappa_0 E_{\nu_{y+\rho}}[f(\eta)] + \frac{\kappa_1}{\sqrt{n}}E_{\nu_{y+\rho}}\big[f(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x) - \rho - y\big)\big] \\ &\quad + \frac{\kappa_2}{n}E_{\nu_{y+\rho}}\big[f(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x) - \rho - y\big)^2\big] + \varepsilon_f(n)\end{aligned}$$

where $|\varepsilon_f(n)| \le C(\rho)\|f\|_{L^2(\nu_\rho)}n^{-3/2}$ and $\kappa_i = \kappa_i(n)$ for $i = 0, 1, 2$ are explicit expressions. Indeed, one observes

$$\begin{aligned}\kappa_0(n) &= \frac{\sqrt{n}}{\sqrt{n-\ell}}\frac{\theta_{n-\ell}(0)}{\theta_n(0)} = 1 + O(n^{-1/2}) \\ \kappa_1(n) &= \frac{\sqrt{n}}{\theta_n(0)\sqrt{n-\ell}}\frac{1}{2\pi}\int_{-\pi\sqrt{n-\ell}}^{\pi\sqrt{n-\ell}} t\psi_y^{n-\ell}\Big(\frac{t}{\sqrt{n-\ell}}\Big)dt = O(n^{-1/2}) \\ \kappa_2(n) &= \frac{-\sqrt{n}}{2\theta_n(0)\sqrt{n-\ell}}\frac{1}{2\pi}\int_{-\pi\sqrt{n-\ell}}^{\pi\sqrt{n-\ell}} t^2\psi_y^{n-\ell}\Big(\frac{t}{\sqrt{n-\ell}}\Big)dt \\ &= \frac{-1}{2\sigma^2(y+\rho)} + O(n^{-1/2}).\end{aligned}$$

Now, for a local $L^2(\nu_\rho)$ function h supported on coordinates in Λ_ℓ^+, we have

$$E_{\nu_{y+\rho}}[h] \;=\; E_{\nu_\rho}\left[h(\eta)\frac{e^{\lambda(y+\rho)\sum_{x\in\Lambda_\ell^+}(\eta(x)-\rho)}}{M^\ell(\lambda(y+\rho))}\right]$$

where $\lambda(y+\rho)$ is the 'tilt' chosen to change the density to $y+\rho$ and $M(\lambda) = E_{\nu_\rho}[e^{\lambda(\eta(x)-\rho)}]$. Note that $z = M'(\lambda(z))/M(\lambda(z))$ and

$$\lambda'(z) \;=\; \Big[\frac{M''(\lambda(z))}{M(\lambda(z))} - \Big(\frac{M'(\lambda(z))}{M(\lambda(z))}\Big)^2\Big]^{-1} \;=\; \frac{1}{\sigma^2(z)}.$$

For $|y|\leq\delta$, after a straightforward calculation, one obtains

$$E_{\nu_{y+\rho}}[h(\eta)] \quad=\quad (\lambda'(\rho))^2\frac{y^2}{2}E_{\nu_\rho}[h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-\rho\big)^2] + |y|^3 r(\rho,\delta,h)$$

when $\varphi_h(\rho) = \varphi_h'(\rho) = 0$. When, only $\varphi_h(\rho)=0$ is known,

$$E_{\nu_{y+\rho}}[h(\eta)] \quad=\quad \lambda'(\rho)yE_{\nu_\rho}\big[h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-\rho\big)\big] + |y|^2 r(\rho,\delta,h).$$

Here, the remainders are bounded $|r(\rho,\delta,h)| \leq C(\rho,\delta)\|h\|_{L^2(\nu_\rho)}$. Indeed, the second remainder estimate holds by noting the second derivative given in (2.4) is bounded for $|y|\leq\delta$; the first remainder bound also holds given that the third derivative is in form

$$\begin{aligned}
&\frac{d^3}{dy^3}E_{\nu_{y+\rho}}[h(\eta)] \;=\; \lambda'''(y+\rho)E_{\nu_{y+\rho}}\big[\bar h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-y-\rho\big)\big]\\
&+3\lambda'(y+\rho)\lambda''(y+\rho)E_{\nu_{y+\rho}}\big[\bar h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-y-\rho\big)^2\big]\\
&+(\lambda'(y+\rho))^3E_{\nu_{y+\rho}}\big[\bar h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-y-\rho\big)^3\big]\\
&-3(\lambda'(y+\rho))^3E_{\nu_{y+\rho}}\big[\bar h(\eta)\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-y-\rho\big)\big]E_{\nu_{y+\rho}}\big[\big(\sum_{x\in\Lambda_\ell^+}\eta(x)-y-\rho\big)^2\big]
\end{aligned}$$

where $\bar h(\eta) = h(\eta) - E_{\nu_{y+\rho}}[h]$.

Now, from (2.4), we have

$$\varphi_h^{(k)}(\rho) \;=\; (\lambda'(\rho))^k E_{\nu_\rho}\Big[h(\eta)\Big(\sum_{x\in\Lambda_\ell^+}(\eta(x)-\rho)\Big)^k\Big]$$

when $\varphi_h^{(k-1)}(\rho) = \varphi_h(\rho) = 0$, for $k=1,2$. Also, given the assumptions on f and $E_{\nu_\rho}[y^{2p}] = O(n^{-p})$ (which means each y factor is of order $n^{-1/2}$), we can group the dominant terms to arrive at

$$\begin{aligned}
&E_{\nu_\rho}\Big[1(|y|\leq\delta)\Big(E_{\nu_\rho}[f(\eta)|y] - \Big\{\frac{\kappa_0 y^2}{2} + \frac{1}{\lambda'(\rho)}\frac{\kappa_1 y}{\sqrt{n}} + \frac{1}{(\lambda'(\rho))^2}\frac{\kappa_2}{n}\Big\}\varphi_f''(\rho)\Big)^4\Big]\\
&\qquad\leq\; C(\rho,\delta)\|f\|^4_{L^4(\nu_\rho)}n^{-6}
\end{aligned}$$

and hence, as $\kappa_0(n) = 1 + O(n^{-1/2}), \kappa_1(n) = O(n^{-1/2})$ and, by expanding $\sigma^2(y+\rho)$, $\kappa_2(n) = -2^{-1}\sigma^{-2}(\rho) + O(y)$,

$$E_{\nu_\rho}\Big[1(|y|\le\delta)\Big(E_{\nu_\rho}[f(\eta)|y] - \Big\{y^2 - \frac{\sigma^2(\rho)}{n}\Big\}\frac{\varphi_f''(\rho)}{2}\Big)^4\Big] \ \le\ C(\rho,\delta)\|f\|^4_{L^4(\nu_\rho)}n^{-6}.$$

On the other hand, we bound, noting $|\varphi_f''(\rho)| \le C\|f\|_{L^2(\nu_\rho)}$ by assumption (D), and simple estimates (one can use large deviations bounds), that also

$$E_{\nu_\rho}\Big[1(|y|>\delta)\Big(E_{\nu_\rho}[f(\eta)|y] - \Big\{y^2 - \frac{\sigma^2(\rho)}{n}\Big\}\frac{\varphi_f''(\rho)}{2}\Big)^4\Big] \ \le\ C(\rho,\delta)\|f\|^4_{L^5(\nu_\rho)}O(n^{-6})$$

to finish the proof. □

We now prove the equivalence ensembles estimate (EE) with respect to a Markovian measure. Recall the Gibbs measures $\nu_{1/2}$ and $\nu_x = \nu_{1/2}^{\lambda(x)}$ defined in Subsection 2.5. To see how the next proposition can be used to satisfy assumption (EE), we note (1) the estimate is uniform in the 'outside variables' η_ℓ^c, and (2) since the transition matrix P_β is positive, the L^∞ norm of any local function supported on sites Λ_{ℓ_0} can be bounded $\|f\|_{L^\infty} \le C(\ell_0,\beta)\|f\|_{L^p(\nu_\rho)}$ for $p>0$. Recall also the definitions of $\varphi_f(\rho)$ and its derivatives in (2.4).

Proposition 5.2. *Let f be a local function, supported on sites indexed by Λ_ℓ, such that $\varphi_f(1/2) = \varphi_f'(1/2) = 0$. Then, for each $0<\varepsilon<1$, there is a constant $C = C(\varepsilon)$ such that for every fixed $a,b\in\{0,1\}$,*

$$\Big\|E_{\nu_{1/2}}[f|y,\eta(-n-1)=a,\eta(n+1)=b] - \frac{\varphi_f''(1/2)}{2}\Big[y^2 - \frac{\sigma_n^2(1/2)}{2n+1}\Big]\Big\|_{L^4(\nu_{1/2})} \le \frac{C\|f\|_{L^\infty}}{n^{3/2-\varepsilon}}$$

On the other hand, when only $\varphi_f(1/2)=0$ is known,

$$\Big\|E_{\nu_{1/2}}[f|y,\eta(-n-1)=a,\eta(n+1)=b] - y\varphi_f'(1/2)\Big\|_{L^4(\nu_{1/2})} \le \frac{C\|f\|_{L^\infty}}{n^{1-\varepsilon}}.$$

Here, $y = (2n+1)^{-1}\sum_{x\in\Lambda_n}\eta(x) - 1/2$.

Proof. The argument has the same structure as for Proposition 5.1. We will concentrate on the first display as the second statement has a similar and easier argument. Since $\nu_{1/2}$ corresponds to an ergodic finite-state Markov chain with uniform invariant measure, it is exponentially mixing and allows standard block approximations, which are used in many steps.

Let $0<\chi<1/6$. Let also $n' = n - n^\chi$. Develop

$$\begin{aligned} &E_{\nu_{1/2}}[f|y,\eta(-n-1)=a,\eta(n+1)=b]\\ &= E_{\nu_{y+1/2}}[f(\eta)|y,\eta(-n-1)=a,\eta(n+1)=b]\\ &= E_{\nu_{y+1/2}}\Big[f(\eta)\frac{\sqrt{2n+1}\theta^\chi_{n,y,a,b}(-\sum_{x\in\Lambda_{n^\chi}}(\eta(x)-y-1/2))}{\sqrt{2n'}\theta_{n,y,a,b}(0)}\\ &\qquad\Big|\eta(-n-1)=a,\eta(n+1)=b\Big] \end{aligned}$$

where

$$\theta^\chi_{n,y,a,b}(z) \;=\; \sqrt{2n'}\nu_{y+1/2}\Big(\sum_{n^\chi<|x|\le n}\eta(x)-y-1/2=z \\ \Big|\eta(n^\chi),\eta(-n^\chi),\eta(-n-1)=a,\eta(n+1)=b\Big)$$

and

$$\theta_{n,y,a,b}(z) \;=\; \sqrt{2n+1}\nu_{y+1/2}\Big(\sum_{x\in\Lambda_n}\eta(x)-y-1/2=z\Big|\eta(-n-1)=a,\eta(n+1)=b\Big).$$

By a local central limit theorem for ergodic Markov chains [38], we have

$$\lim_{n\uparrow\infty}\theta_{n,y,a,b}(0) \;=\; \frac{1}{\sqrt{2\pi\sigma^2(y+1/2)}}$$

where we recall $\sigma^2(z)=\lim_{n\uparrow\infty}n^{-1}E_{\nu_z}[(\sum_{x\in\Lambda_n}\eta(x)-z)^2]$. Here, $\inf_{|y|\le\delta}\sigma^2(y+1/2)>0$.

Let the characteristic function $\psi_{n,y,\chi,a,b}(t)$ for $|t|\le\pi$ be defined by

$$E_{\nu_{y+1/2}}\Big[e^{it\sum_{n^\chi<|x|\le n}(\eta(x)-y-1/2)}\big|\eta(n^\chi),\eta(-n^\chi),\eta(-n-1)=a,\eta(n+1)=b\Big].$$

Recall formulas p_1 and p_2 in (2.15). By diagonalizing the transfer matrix,

$$\begin{bmatrix} p_1^{-1}e^{\beta/4}e^{-\lambda/2}e^{-it(1/2+y)} & p_1^{-1}e^{-\beta/4}e^{\lambda/2}e^{it(1/2-y)} \\ p_2^{-1}e^{-\beta/4}e^{-\lambda/2}e^{-it(1/2+y)} & p_2^{-1}e^{\beta/4}e^{\lambda/2}e^{it(1/2-y)} \end{bmatrix},$$

one can show for $|y|,|t|\le\delta$ that

$$\psi_{n,y,\chi,a,b}\left(\frac{t}{\sqrt{2n'}}\right) \;=\; \Big(1-\frac{t^2\sigma^2(y+1/2)}{2n'}\Big)^{2n+1}\big[1+O(n'^{-1/2})\big].$$

In particular, for $|y|,|t|\le\delta$, $|\psi_{n,y,\chi,a,b}(t)|<\exp\{-Ct^2\}$ for some $C=C(\delta)>0$. Also, one can obtain for $|y|\le\delta$ and $\delta<|t|\le\pi$ that $|\psi_{n,y,\chi,a,b}(t)|<A^n$ where $A=A(\delta)<1$.

Now, write as before

$$\begin{aligned}\theta^\chi_{n,y,a,b}(x) \;&=\; \frac{\sqrt{2n'}}{2\pi}\int_{-\pi}^{\pi}e^{itx}\psi_{n,y,\chi,a,b}(t)dt\\ &=\; \frac{1}{2\pi}\int_{-\pi\sqrt{2n'}}^{\pi\sqrt{2n'}}e^{itx/\sqrt{n'}}\psi_{n,y,\chi,a,b}(t/\sqrt{2n'})dt.\end{aligned}$$

One can rewrite the last expression as

$$\frac{1}{2\pi}\int_{-\pi\sqrt{2n'}}^{\pi\sqrt{2n'}}\psi_{n,y,\chi,a,b}(t/\sqrt{2n'})dt+\frac{ix}{2\pi\sqrt{2n'}}\int_{-\pi\sqrt{2n'}}^{\pi\sqrt{2n'}}t\psi_{n,y,\chi,a,b}(t/\sqrt{2n'})dt \\ -\frac{x^2}{4\pi n'}\int_{-\pi\sqrt{2n'}}^{\pi\sqrt{2n'}}t^2\psi_{n,y,\chi,a,b}(t/\sqrt{2n'})dt+r_0(x)n^{-3/2}$$

in terms of error $r_0(x)$ which is bounded $|r_0(x)|=O(|x|^3)$.

Then, as in the proof of Proposition 5.1, we have, for $|y| \leq \delta$, that

$$
\begin{aligned}
&E_{\nu_{1/2}}\big[f|y, \eta(-n-1)=a, \eta(n+1)=b\big] \\
&= \kappa_0 E_{\nu_{y+1/2}}\big[f(\eta)|\eta(-n-1)=a, \eta(n+1)=b\big] \\
&+\frac{\kappa_1}{\sqrt{2n'}} E_{\nu_{y+1/2}}\big[f(\eta)\big(\sum_{|x|\leq n^\chi} \eta(x)-1/2-y\big)\big|\eta(-n-1)=a, \eta(n+1)=b\big] \\
&+\frac{\kappa_2}{2n'} E_{\nu_{y+1/2}}\big[f(\eta)\big(\sum_{|x|\leq n^\chi} \eta(x)-1/2-y\big)^2\big|\eta(-n-1)=a, \eta(n+1)=b\big] \\
&+\varepsilon_f(n)
\end{aligned}
$$

where $|\varepsilon_f(n)| \leq C\|f\|_{L^2(\nu_\rho)} n^{-3/2}$ and $\kappa_i = \kappa_i(n)$ for $i = 0,1,2$ have the same asymptotics as before.

Now, for $|y| \leq \delta$, h supported on sites in Λ_{n^χ} and $i = 0,1,2$, using the exponentially mixing property of the measures $\{\nu_{y+1/2} : |y| \leq \delta\}$ and $\varphi_f(1/2) = \varphi'_f(1/2) = 0$, we can expand

$$
\begin{aligned}
&E_{\nu_{y+1/2}}\Big[f(\eta)\Big(\sum_{|x|\leq n^\chi} \eta(x)-1/2-y\Big)^i\big|\eta(-n-1)=a, \eta(n+1)=b\Big] \qquad (5.2)\\
&= \frac{\lambda'(1/2)^{2-i}y^{2-i}}{(2-i)!} E_{\nu_{1/2}}\Big[f(\eta)\Big(\sum_{|x|\leq n^{2\chi}} \eta(x)-1/2\Big)^2\Big] + |y|^{3-i} r_1(f,n) + r_2(f,n)
\end{aligned}
$$

Here, the error $r_1(f,n)$ stands for the error made first in Taylor approximation with respect to the conditioned measure. Using that $\nu_{y+1/2}$ is exponentially mixing, one can bound the first, second and third derivatives uniformly in a, b and $|y| \leq \delta$ after a straightforward but tedious calculation so that $|r_1(f,n)| \leq C(\delta) n^{3\chi}\|f\|_{L^\infty}$. The error $r_2(f,n)$ represents other errors made by exponential approximations and $|r_2(h,n)| \leq C\|f\|_{L^2(\nu_\rho)} n^{-3/2}$.

Here, for h supported on sites in Λ_{n^χ}, and notation

$$
\bar h(\eta) = h(\eta) - E_{\nu_{y+1/2}}[h|\eta(-n-1)=a, \eta(n+1)=b] \ \text{ and } \ \bar\eta(x) = \eta(x) - y - 1/2,
$$

the first derivative is

$$
\begin{aligned}
&\frac{d}{dy} E_{\nu_{y+1/2}}[h(\eta)|\eta(-n-1)=a, \eta(n+1)=b] \\
&= \lambda'(y+1/2) E_{\nu_{y+1/2}}\Big[\bar h(\eta)\Big(\sum_{|x|\leq n} \bar\eta(x)\Big)\big|\eta(-n-1)=a, \eta(n+1)=b\Big].
\end{aligned}
$$

The second derivative is

$$
\begin{aligned}
&\frac{d^2}{dy^2} E_{\nu_{y+1/2}}\big[\bar h(\eta)|\eta(-n-1)=a, \eta(n+1)=b\big] \\
&= \lambda''(y+1/2) E_{\nu_{y+1/2}}\Big[\bar h(\eta)\Big(\sum_{|x|\leq n} \bar\eta(x)\Big)\big|\eta(-n-1)=a, \eta(n+1)=b\Big] \\
&\quad + (\lambda'(y+1/2))^2 E_{\nu_{y+1/2}}\Big[\bar h(\eta)\Big(\sum_{|x|\leq n} \bar\eta(x)\Big)^2\big|\eta(-n-1)=a, \eta(n+1)=b\Big].
\end{aligned}
$$

The third derivative is

$$\begin{aligned}
&\frac{d^3}{dy^3}E_{\nu_{y+1/2}}\big[\bar{h}(\eta)|\eta(-n-1)=a,\eta(n+1)=b\big]\\
&=\lambda'''(y+1/2)E_{\nu_{y+1/2}}\Big[\bar{h}(\eta)\Big(\sum_{|x|\le n}\bar{\eta}(x)\Big)\Big|\eta(-n-1)=a,\eta(n+1)=b\Big]\\
&+3\lambda'(y+1/2)\lambda''(y+1/2)\\
&\qquad\times E_{\nu_{y+1/2}}\Big[\bar{h}(\eta)\Big(\sum_{|x|\le n}\bar{\eta}(x)\Big)^2\Big|\eta(-n-1)=a,\eta(n+1)=b\Big]\\
&+(\lambda'(y+1/2))^3E_{\nu_{y+1/2}}\Big[\bar{h}(\eta)\Big(\sum_{|x|\le n}\bar{\eta}(x)\Big)^3\Big|\eta(-n-1)=a,\eta(n+1)=b\Big]\\
&-3(\lambda'(y+1/2))^3E_{\nu_{y+1/2}}\Big[\bar{h}(\eta)\Big(\sum_{|x|\le n}\bar{\eta}(x)\Big)\Big|\eta(-n-1)=a,\eta(n+1)=b\Big]\\
&\qquad\times E_{\nu_{y+1/2}}\Big[\Big(\sum_{|x|\le n}\bar{\eta}(x)\Big)^2\Big|\eta(-n-1)=a,\eta(n+1)=b\Big].
\end{aligned}$$

We have applied these expressions with $h(\eta)=f(\eta)\big(\sum_{|x|\le n^\chi}(\eta(x)-1/2)\big)^i$ for $i=0,1,2$ to bound (5.2).

The rest of the proof navigates a virtually similar route as for Proposition 5.1, noting that a factor $n^{3\chi}y$ is of order $O(n^{-(1/2-3\chi)})$. The parameter χ may be chosen small enough to fit with the desired estimate. □

6. Tightness in Hermite space

We prove that $\{\mathcal{Y}^{n,\gamma}_t : t\in[0,T]\}$, when $a\neq 0$ and $\gamma=1/2$, is tight in a dual Hermite Hilbert space $\mathcal{H}_{-k}\subset\mathbb{S}'(\mathbb{R})$ with $k=4$. When $\gamma\in(1/2,1]$, the proof is the same with similar estimates.

To define $\mathcal{H}_k$, let $\{h_z : z\ge 0\}$ be the Hermite functions on $L^2(\mathbb{R})$, that is $h_0(u)=\pi^{-1/2}e^{-u^2/2}$ and, for $z\ge 1$,

$$h_z(u) \;=\; (2^z z!)^{-1/2}(-1)^z\pi^{-1/2}e^{u^2/2}\frac{d^z}{du^z}e^{-u^2}.$$

It is standard that $\{h_z\}$ are orthonormal and complete on $L^2(\mathbb{R})$. Each Hermite function is an eigenfunction with respect to operator $\mathcal{U}=|u|^2-\triangle$, that is $|u|^2h_z-\triangle h_z=\lambda_z h_z$ with eigenvalue $\lambda_z=2z+1$. Also, the recursion holds: $h'_z(u)=(z/2)^{1/2}h_{z-1}(u)-((z+1)/2)^{1/2}h_{z+1}(u)$ for $z\ge 1$. See [54][Chapter V] for more discussion.

We will need the following L^p estimates on h_z which follow from the above properties: Namely, $\|h^{(i)}_z\|^2_{L^2(\mathbb{R})}\le C(1+z)^i$ for $i=1,2$. Also, $\|h_z\|_{L^1(\mathbb{R})}\le C|1+z|^{1/4}$ from Lemma 6.3, and therefore $\|h^{(i)}_z\|_{L^1(\mathbb{R})}\le C|1+z|^{(2i+1)/4}$ for $1\le i\le 4$. Here, C is a universal constant.

Now, any function $f\in L^2(\mathbb{R})$ can be expressed as

$$f \;=\; \sum_{z\ge 0}\langle f,h_z\rangle h_z$$

where

$$\langle f, h_z\rangle = \int f(u)h_z(u)du.$$

Define, for $k \geq 1$, the Hilbert spaces $\mathcal{H}_k$ which are the completions of smooth compactly supported functions with inner product

$$\langle f, g\rangle_k \;=\; \langle f, \mathcal{U}^k g\rangle.$$

In particular, $L^2(\mathbb{R}) = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_k \supset \mathbb{S}(\mathbb{R})$ are those functions such that

$$\sum_{z\geq 0}\langle f, h_z\rangle^2\lambda_z^k \;<\; \infty.$$

The duals of $\mathcal{H}_k$ are $\mathcal{H}_{-k}$, relative to completion with respect to the inner-product, can be identified as those functions such that

$$\sum_{z\geq 0}\langle f, h_z\rangle^2\lambda_z^{-k} \;<\; \infty.$$

We have the ordering $\mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-k}$. Also, $\mathcal{H}_{-k} \subset \mathbb{S}'(\mathbb{R})$ for $k \geq 0$. We will endow $\mathcal{H}_{-k}$ with the uniform weak-* topology.

Now, to prove that $\{\mathcal{Y}_t^{n,\gamma} : t \in [0,T]\}$ is tight in the uniform topology on $D([0,T];\mathcal{H}_{-k})$ with $k \geq 3+d = 4$, starting from the invariant measure ν_ρ, it is sufficient to show the following result–see [36][Chapter 11].

Then, tightness when starting under a pertubed state $\{\mu^n\}$ would follow the same argument given in Subsection 3.3 with respect to $\mathbb{S}'(\mathbb{R})$.

Proposition 6.1. *For each $z \geq 0$, there is a constant $C = C(\rho, T)$ such that*

$$\limsup_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}|\mathcal{Y}_t^{n,\gamma}(h_z)|^2\Big] \;\leq\; C|1+z|^{5/2}$$

and for $\varepsilon > 0$ that

$$\lim_{\delta\downarrow 0}\limsup_{n\uparrow\infty}\mathbb{P}_{\nu_\rho}\Big(\sup_{\substack{|s-t|<\delta\\ s,t\in[0,T]}}|\mathcal{Y}_t^{n,\gamma}(h_z) - \mathcal{Y}_s^{n,\gamma}(h_z)| > \varepsilon\Big) \;=\; 0.$$

Proof. The second line in the display follows already from the tightness bounds given in Subsection 3.3 with h_z substituted for H. We now argue the first estimate. As in Subsection 3.3, we write

$$\mathcal{Y}_t^{n,\gamma}(h_z) \;=\; \mathcal{Y}_0^{n,\gamma} + \mathcal{I}_t^{n,\gamma}(h_z) + \mathcal{B}_t^{n,\gamma}(h_z) + \mathcal{K}_t^{n,\gamma}(h_z) + \mathcal{M}_t^{n,\gamma}(h_z).$$

The term $\mathcal{Y}_0^{n,\gamma}(h_z)$, noting assumption (IM2), is bounded

$$\limsup_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{Y}_0^{n,\gamma}(h_z)\big)^2\big] \;\leq\; C(\rho)\|h_z\|^2_{L^2(\mathbb{R})} \;=\; C(\rho).$$

The martingale term is bounded, by Doob's inequality and the quadratic variation bound (3.3), noting bounds in (R1), as

$$\limsup_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\big[\sup_{t\in[0,T]}\big(\mathcal{M}_t^{n,\gamma}(h_z)\big)^2\big] \;\leq\; C(a,\rho,T)\|\nabla h_z\|^2_{L^2(\mathbb{R})} \;\leq\; C(a,\rho,T)|1+z|^2.$$

The most involved term is $\mathcal{B}_t^{n,\gamma}(h_z)$ which we now treat. Recall the notation of Subsection 3.3, in particular $\tau_x V_b(\eta) = \tau_x b(\eta) - \varphi_b(\rho) - \varphi_b'(\rho)(\eta(x) - \rho)$.

We will also, to simplify notation for the rest of the proof, drop the superscript 'n' and write $\eta^n = \eta$.

We may bound

$$\begin{aligned}
&(a/2)^{-2}\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}\big(\mathcal{B}^n_t(h_z)\big)^2\Big]\\
&\leq\ 3\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}\Big(\int_0^t W_1(\eta_s)ds\Big)^2\Big]+3\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}\Big(\int_0^t W_2(\eta_s)ds\Big)^2\Big]\\
&\quad +3\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}\Big(\int_0^t W_3(\eta_s)ds\Big)^2\Big]\\
&=\ Q_1+Q_2+Q_3
\end{aligned}$$

where

$$\begin{aligned}
W_1 &= \sum_{x\in\mathbb{Z}}(\nabla^n_x h_z)\Big\{\tau_x V_b(\eta)-E_{\nu_\rho}\big[V_b(\eta)|\eta^{(\ell)}_s,\eta^c_\ell\big]\Big\}\\
W_2 &= \sum_{x\in\mathbb{Z}}(\nabla^n_x h_z)\Big\{E_{\nu_\rho}\big[V_b(\eta)|\eta^{(\ell)}_s,\eta^c_\ell\big]-\frac{\varphi''_b(\rho)}{2}\big\{\big(\eta^{(\ell)}(x)-\rho\big)^2-\frac{\sigma^2_\ell(\rho)}{2\ell+1}\big\}\Big\} \text{ and}\\
W_3 &= \sum_{x\in\mathbb{Z}}(\nabla^n_x h_z)\frac{\varphi''_b(\rho)}{2}\big\{\big(\eta^{(\ell)}(x)-\rho\big)^2-\frac{\sigma^2_\ell(\rho)}{2\ell+1}\big\}.
\end{aligned}$$

In the following we will take $\ell = n$ and use the bounds on b, c, $\varphi''_b(\rho)$ and $\varphi'_c(\rho)$ afforded by (R1), (IM) and (D). From the proofs of Lemma 4.3 and Lemma 4.5, which together bound the H_{-1} norm of the integrand W_1, and Lemma 6.2 below, we have

$$Q_1\ \leq\ C(b,\rho)T\frac{\ell}{n}\|\nabla h_z\|^2_{L^2(\mathbb{R})}\ \leq C(b,\rho)T|1+z|.$$

On the other hand, Q_2 can be bounded by use of Schwarz inequality,

$$Q_2\ \leq\ T\mathbb{E}_{\nu_\rho}\Big[\int_0^T\big(W_2(\eta_s)\big)^2ds\Big].$$

Then, by the proof of Lemma 4.6, which makes use of the equivalence of ensembles assumption (EE), and the bound on $\|\nabla h_z\|_{L^1(\mathbb{R})}$ mentioned at the beginning of the section,

$$Q_2\ \leq\ C(b,\rho)T^2\frac{n^2}{\ell^{2+\alpha_0}}\|\nabla h_z\|^2_{L^1(\mathbb{R})}\ \leq\ C(b,\rho)T^2|z|^{3/2}n^{-\alpha_0}.$$

Similarly, by the assumption in (IM) that the fourth moment of $\eta^{(\ell)}-\rho$ is bounded as $O(\ell^{-2})$, we have

$$Q_3\ \leq\ C(b,\rho)T^2\frac{n^2}{\ell^2}\|\nabla h_z\|^2_{L^1(\mathbb{R})}\ \leq\ C(b,\rho)T^2|z|^{3/2}.$$

Hence, we have

$$\mathbb{E}_{\nu_\rho}[\sup_{t\in[0,T]}(\mathcal{B}^{n,\gamma}_t(h_z))^2]\ =\ O(|z|^{3/2}).$$

The term $\mathcal{I}^{n,\gamma}_t(h_z)$ is handled similarly. Noting $\|\Delta h_z\|^2_{L^2(\mathbb{R})}, \|\Delta h_z\|^2_{L^1(\mathbb{R})}\ \leq C|z|^{5/2}$, we have the bound

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]}|\mathcal{I}^{n,\gamma}_t(h_z)|^2\Big]\ \leq\ C(c,\rho,T)|1+z|^{5/2}.$$

Also, the term $\mathcal{K}_t^{n,\gamma}(h_z)$, noting its form and the $L^1(\mathbb{R})$ bounds on $h_z^{(i)}$ for $1 \leq i \leq 4$, can be analyzed as for Q_3 above:

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]} |\mathcal{K}_t^{n,\gamma}(h_z)|^2\Big] \;\leq\; C(\rho)\Big[|1+z|^{5/2} + \frac{1}{n}|1+z|^{7/2} + \frac{1}{n^3}|1+z|^{9/2}\Big]$$

which tends to $C(\rho)|1+z|^{5/2}$ as $n \uparrow \infty$. This completes the proof. $\square$

We now state a case of [60][Theorem 2.2], valid for our processes, that we used above.

Lemma 6.2. *For $f \in L^2(\nu_\rho) \cap H_{-1,n}$, we have*

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{t\in[0,T]} \Big|\int_0^t f(\eta_s^n)ds\Big|^2\Big] \;\leq\; 8T\|f\|_{-1,n}^2.$$

We could not find a reference for a bound on the L^1 norm of $h_n(u)$. Although a sharp bound using [41][Theorem 2.1] can be made, the following cruder bound is sufficient for our purposes.

Lemma 6.3. *There is a universal constant C such that $\|h_z\|_{L^1(\mathbb{Z})} \leq C(1+z)^{1/4}$.*

Proof. Consider the related Hermite polynomials, for $z \geq 1$,

$$\kappa_z(u) \;=\; (-1)^z(z!)^{-1/2}e^{u^2/2}\frac{d^z}{du^z}e^{-u^2/2}$$

and $\kappa_0(u) = 1$. One can relate $\{\kappa_z(u)\}$ to the Hermite functions $\{h_z(u)\}$ defined at the beginning of the section: For $z \geq 0$,

$$h_z(u) \;=\; e^{-u^2/2}\pi^{-1/2}\kappa_z(2^{1/2}u).$$

L^p estimates for $\{\kappa_z(u)\}$ have been proved in [33][Theorem 5.19] with respect to weight function $w(u) = (2\pi)^{-1/2}\exp\{-u^2/2\}$. Namely, for $2 < p < \infty$,

$$\Big[\int_{\mathbb{R}} w(u)|\kappa_z(u)|^p w(u)du\Big]^{1/p} \;\leq\; (p-1)^{z/2}.$$

Then,

$$\begin{aligned}\|h_z\|_{L^1(\mathbb{R})} &= \pi^{-1/2}\int_{\mathbb{R}} e^{-u^2/2}|\kappa_z(2^{1/2}u)|du \\ &= 2^{-1/2}\pi^{-1/2}\int_{\mathbb{R}} e^{-y^2/4}|\kappa_z(y)|dy \\ &= \int_{\mathbb{R}} w(y)e^{y^2/4}|\kappa_z(y)|dy \\ &\leq \Big[\int_{\mathbb{R}} w(x)e^{qy^2/4}dy\Big]^{1/q}\Big[\int_{\mathbb{R}} w(y)|\kappa_z(y)|^p dy\Big]^{1/p}\end{aligned}$$

where we choose $p = 2 + 1/(1+z)$ and so $q \sim 2 - 1/(1+z)$. Now, by the L^p estimate above, for some universal constant C,

$$\Big[\int_{\mathbb{R}} w(y)|\kappa_z(y)|^p dy\Big]^{1/p} \;\leq\; (2-p)^{z/2} \;=\; (1+1/(1+z))^{z/2} \;\leq\; Ce^{1/2}$$

and $\Big[\int_{\mathbb{R}} w(x)e^{qy^2/4}dy\Big]^{1/q} \sim \Big[\int_{\mathbb{R}} e^{-y^2/4(1+z)}dy\Big]^{(2-1/(1+z))^{-1}} \leq C(1+z)^{1/4}$. $\square$

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