Further Geometric Restrictions on Jordan Structure in Matrix Factorization [*]

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October 10, 2011

Abstract: It is known that a nonsingular, nonscalar, n-by-n complex matrix A may be factored as $A = BC$, in which the spectra of B and C are arbitrary, subject to $det(A) = det(B)det(C)$. Furthermore, it is also known that B and C can be taken to be nonderogatory. Additionally it has been shown that when two matrices have eigenvalues of high geometric multiplicity, this restricts the possible Jordan structure of the third. We demonstrate a previously unknown restriction on the Jordan structures of B and C. Furthermore, we show that this generalized geometric multiplicity restriction implies the already known geometric multiplicity restriction, show that the new condition is more restrictive, is not sufficient in general but is sufficient in a situation that we identify.

Keywords: Jordan form; geometric multiplicity; matrix product.

AMS Classifications: 15A23

1   Introduction

In [15] it was shown that any nonsingular nonscalar matrix $A \in M_n(\mathbb{C})$ may be factored $A = BC$, so that $B, C \in M_n(\mathbb{C})$ have arbitrary spectra, subject only to the obvious determinantal condition $detA = \prod_{i=1}^{n} \beta_i \prod_{i=1}^{n} \gamma_i$, in which $\beta_1, \beta_2, ..., \beta_n$ are the eigenvalues of B and $\gamma_1, \gamma_2, ..., \gamma_n$ are the eigenvalues of C (repeats allowed). In [6] Sourour's theorem was extended to the case where exactly $n - rankA$ of the elements $\beta_1, \beta_2, ..., \beta_n$,$\gamma_1, \gamma_2, ..., \gamma_n$ are equal to zero. If B and C have repeated eigenvalues, no indication is given in [15] what sort of

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Jordan structure they may have, and unfortunately, the proof there is not easily adapted to further specify the Jordan structure.

The Jordan structure of B and C cannot generally be taken to be arbitrary. (Suppose that B and/or C have repeated eigenvalues and 1-by-1 Jordan blocks in case $n = 2$, for example.) Thus a natural, deeper, question is to ask for a given A and specified eigenvalues $\beta_1, \beta_2, ..., \beta_n$ for B and $\gamma_1, \gamma_2, ..., \gamma_n$ for C, $\prod_{i=1}^{n} \beta_i \gamma_i = det A$, what the Jordan form for B and C may be?

This is known to be a very difficult problem, in general, but a number of prior works discuss portions of it, some explicitly and some very theoretically. For example, it was shown in [9], [11] that, over the complex field, B and C may be taken to be nonderogatory, for any allowed spectra specified for them. In [1] this result was generized over any field with at least four elements. And [7], [8] showed that the geometric multiplicity restrictions together with the obvious determinantal condition on the eigenvalues, are necessary and sufficient for the problem when possible Jordan forms are given for $n < 4$, but not for $n \geq 4$. Paper [10] considered the possible Jordan canonical forms of A, B, C when they are diagonalizable with two distinct eigenvalues, and [5] gives a criterion for writing a given square matrix as the product of two matrices with prescribed spectra and ranks, etc. A number of variations have been studied in the literatures in the last 20 years; see [13], [14], [17] and references therein.

We must mention that the above factorization problem falls under a more general problem mentioned in the literature in recent years: Given similarity classes $C_1, ... C_k$, determine whether or not the equation $A_1 \cdots A_k = I$ can be solved with $A_i \in C_i$. A weaker version of this problem was solved by Crawley-Boevey [3], [4] over the complex field in which the matrices A_i belong to the closures of the similarity classes C_i. Finally, we adopt the convention that, in a given matrix, the use of a different variable will denote a distinct eigenvalue unless otherwise stated.

2 Observations

We begin with some useful observations. One is the fact that our problem is symmetric in the three matrices A, B and C. Since all three matrices must be invertible, we may also write, for example,

$$B = AC^{-1}.$$

Since the Jordan structure of C^{-1} is determined by that of C, and since AC^{-1} is similar to $C^{-1}A$, it is equivalent to take any of the three matrices as the left hand side and the other two matrices in either order.

Another is that the Jordan structure is similarity invariant, and similarity may be passed through the equation (e.g. $S^{-1}AS = S^{-1}BSS^{-1}CS$). We may take, without loss of generality, any one of the three matrices to actually be in Jordan structure.

Throughout all this work, we use $g_T(\lambda)$ denote the geometric multiplicity of the eigenvalue λ in the matrix T. We use $m_T(\lambda)$ to denote the algebraic

multiplicity of λ in the matrix T.

3 The Generalized Geometric Multiplicity Restriction

Here, extending the work of [2] and [16], the geometric multiplicity restriction (GMR) of [8] is properly generalized. The new GGMR is still not sufficient, as we show, but it is quite explicit; and we show that it is sufficient in a restricted setting.

Theorem 1 *(Generalized Geometric Multiplicity Restriction(GGMR)) There exist nonsingular matrices $A, B, C \in M_n(\mathbb{C})$ with prescribed invariant factors $h_1(A)|h_2(A)| \cdots |h_n(A)$, $h_1(B)|h_2(B)| \cdots |h_n(B)$ and $h_1(C)|h_2(C)| \cdots |h_n(C)$, respectively, such that $A = BC$ only if $det(A) = det(B)det(C)$ and the following are satisfied for all $\alpha \in (A), \beta \in (B)$ and $\gamma \in (C)$:*

1. $h_i(\alpha C^{-1})|h_{n-g_A(\alpha)+i}(B)$ *for* $i = 1, ..., g_A(\alpha)$

2. $h_i(\alpha B^{-1})|h_{n-g_A(\alpha)+i}(C)$ *for* $i = 1, ..., g_A(\alpha)$

3. $h_j(\beta C)|h_{n-g_B(\beta)+j}(A)$ *for* $j = 1, ..., g_B(\beta)$

4. $h_j(\frac{1}{\beta}A)|h_{n-g_B(\beta)+j}(C)$ *for* $j = 1, ..., g_B(\beta)$

5. $h_k(\frac{1}{\gamma}A)|h_{n-g_C(\gamma)+k}(B)$ *for* $k = 1, ..., g_C(\gamma)$

6. $h_k(\gamma B)|h_{n-g_C(\gamma)+k}(A)$ *for* $k = 1, ..., g_C(\gamma)$

$\square$

Before we prove the theorem, we first present a previous result of Thompson [16]. The main result of his paper was to determine the possible values of the invariant factors of the matrix A under rank one perturbations.

Theorem 2 *[16] Let $A \in M_n(\mathbb{F})$ have similarity invariants $h_1(A)|h_2(A)| \cdots |h_n(A)$. Then, as column n-tuple x and row n-tuple y range over all vectors with entries in $\mathbb{F}$. The similarity invariants assumed by the matrix*

$$B = A + xy$$

are precisely the monic polynomials $h_1(B)|h_2(B)| \cdots |h_n(B)$ for which

1. $\deg\{h_1(B)h_2(B) \cdots h_n(B)\} = n$

2. $h_1(B)|h_2(A)|h_3(B)|h_4(A) \cdots$
   $h_1(A)|h_2(B)|h_3(A)|h_4(B) \cdots$.

$\square$

Applying this theorem, we have the follow corollary which gives the interlacing relationships for algebraic multiplicity of eigenvalues of A and B.

Corollary 1 *Under the same condition of theorem 2. Suppose that a is an eigenvalue of A, and $(x-a)|h_i(A)$, but $(x-a) \nmid h_{i-1}(A)$, then the algebraic multiplicity of a, satisfy the following interlacing inequality:*

(i) $m_i(A) \leq m_{i+1}(B) \leq m_{i+2}(A) \leq m_{i+3}(B) \cdots$

(ii) $m_{i+1}(A) \leq m_{i+2}(B) \leq m_{i+3}(A) \cdots$.

where $m_i(A)$ (resp. $m_i(B)$) is the multiplicity of a as root of $h_i(A)$ (resp. $h_i(B)$). □

Another corollary that comes directly from theorem 2 is that

Corollary 2 *Assume that $D \in M_n(\mathbb{F})$ is a matrix under rank r perturbations of A, then the invariant factors of D, $h_1(D)|h_2(D)|\cdots|h_n(D)$ and the invariant factors of A, satisfy the following interlacing relations*

(1) $h_1(A)|h_{1+r}(D)|h_{1+2r}(A)|h_{1+3r}(D) \cdots$

(2) $h_1(D)|h_{1+r}(A)|h_{1+2r}(D)|h_{1+3r}(A) \cdots$

Suppose that a is an eigenvalue of A, and $(x-a)|h_i(A)$, but $(x-a) \nmid h_{i-1}(A)$, then the algebraic multiplicity of a, satisfy the following interlacing inequality:

(3) $m_i(A) \leq m_{i+r}(D) \leq m_{i+2r}(A) \leq m_{i+3r}(D) \cdots$

(4) $m_{i+r}(A) \leq m_{i+2r}(D) \leq m_{i+3r}(A) \cdots$. □

In fact, the algebraic multiplicity of a root of $h_i(A)$ is exactly the Jordan block size associated with the eigenvalue of A. So, based on the above analysis, we may give an alternate statement of the theorem to facilitate its understanding. The reader can easily verify the equivalence.

Let nonsingular matrices A, B, $C \in M_n(\mathbb{C})$ have Jordan structures

	eiganvalues	*blocksizes*	
A	$\alpha_1 :$	$a_{11} \leq ... \leq a_{1p_1}$	
	$\alpha_2 :$	$a_{21} \leq ... \leq a_{2p_2}$	
	$\vdots$		
	$\alpha_h :$	$a_{h1} \leq ... \leq a_{hp_h}$	
B	$\beta_i :$	$b_{i1} \leq ... \leq b_{iq_i},$	$i = 1, ..., k$
C	$\gamma_i :$	$c_{i1} \leq ... \leq c_{is_i},$	$i = 1, ..., l$

where $\alpha_i = \alpha_j, \beta_i = \beta_j, \gamma_i = \gamma_j$ only if $i = j$.

Theorem 3 *(GGMR Restated) There exists a triple with the above Jordan structures and $A = BC$ only if $det(A) = det(B)det(C)$ and*

(a) if $q_i + s_j > n$, then some $\alpha_t = \beta_i \gamma_j$, $p_t \geq q_i + s_j - n$, and

$$(a_{t,p_t-(q_i+s_j-n)+1}, ..., a_{t,p_t}) \geq (b_{i1}, ..., b_{i,(q_i+s_j-n)}), (c_{j1}, ..., c_{j,(q_i+s_j-n)})$$

(b) if $q_i + p_j > n$, then some $\gamma_t = \dfrac{\alpha_j}{\beta_i}$, $s_t \geq q_i + p_j - n$, and

$$(c_{t,s_t-(q_i+p_j-n)+1}, ..., c_{t,s_t}) \geq (b_{i1}, ..., b_{i,(q_i+p_j-n)}), (a_{j1}, ..., a_{j,(q_i+p_j-n)})$$

and

(c) if $p_i + s_j > n$, then some $\beta_t = \dfrac{\alpha_i}{\gamma_j}$, $q_t \geq p_i + s_j - n$, and

$$(b_{t,q_t-(p_i+s_j-n)+1}, ..., b_{t,q_t}) \geq (a_{i1}, ..., a_{i,(p_i+s_j-n)}), (c_{j1}, ..., c_{j,(p_i+s_j-n)}).$$

$\square$

Now we prove the theorem 1.

Proof. We will show that condition 3 in theorem 1 is necessary, and the proofs of the other five statements follow along the same lines. Let $\alpha \in (A), \beta \in (B)$ and $\gamma \in (C)$ be given, and let $R = B - \beta I$. We then write $A = BC = (\beta I + R)C$ and we easily obtain

$$A = \beta C + RC = \beta C + \sum_{i=1}^{b} R_i,$$

where $b = rank(RC) = rank(R) = rank(B - \beta I) = n - g_B(\beta)$ and $rank(R_i) = 1$ for $i = 1, 2, ..., b$.

We can consider A as a rank b perturbation of βC. By corollary 2, we have

$$h_t(\beta C) | h_{t+b}(A), \ t = 1, 2, ..., g_B(\beta).$$

Since $b = n - g_B(\beta)$, we have shown 3.

To show 6, let $S = C - \gamma I$ and write $A = B(\gamma I + S) = \gamma B + BS$. Then, proceed along the same lines, substituting variable names as necessary.

For 1 and 5, rewrite as $B = AC^{-1}$; we may then substitute $B, A, C^{-1}, \beta, \alpha$ and $\dfrac{1}{\gamma}$ for A, B, C, α, β and γ, respectively, into the above argument. The proof proceeds in the exact same manner. For 2 and 4, we can rewrite as $C = B^{-1}A$ and a similar substitution will generate a proof along the same lines. $\square$

We will now use the generalized geometric multiplicity restriction to give alternate proofs of the geometric multiplicity restriction from [7], [8].

Corollary 3 *If $A, B, C \in M_n(\mathbb{C})$, A is nonsingular and $A = BC$, then*

(i) $g_A(\beta\gamma) \geq g_B(\beta) + g_C(\gamma) - n$ *and*

(ii) $g_B(\frac{\alpha}{\gamma}) \geq g_A(\alpha) + g_C(\gamma) - n.$

Proof. Here we show only (i), (ii) can be shown similarly. If $g_B(\beta) + g_C(\gamma) - n \leq 0$, obviously, $g_A(\beta\gamma) \geq 0$. Thus we assume $g_B(\beta) + g_C(\gamma) - n > 0$. We know that $(x - \gamma)|h_{n-g_c(\gamma)+1}(C)$, so $(x - \beta\gamma)|h_{n-g_c(\gamma)+1}(\beta C)$. Now, since $n - g_C(\gamma) < g_B(\beta)$, so $n - g_C(\gamma) + 1 \leq g_B(\beta)$, by theorem 1 part 3, we have that

$$(x - \beta\gamma)|h_{n-g_C(\gamma)+1}(\beta C)|h_{(n-g_B(\beta))+(n-g_C(\gamma)+1)}(A) = h_{n-(g_B(\beta)+g_C(\gamma)-n)+1}(A).$$

Thus, we have $g_A(\beta\gamma) \geq g_B(\beta) + g_C(\gamma) - n$. $\qquad\square$

The following example shows the generalized geometric multiplicity restriction(GGMR) is more general than geometric multiplicity restriction(GMR).

Example The follow triple

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

satisfy the geometric multiplicity restriction and determinant restriction, but fail the generalized geometric multiplicity restriction.

Observation: In case that all A, B, C are all diagonalizable, the generalized geometric multiplicity restriction and geometric multiplicity restriction is equivalent since in this case the algebraic and geometric multiplicities are equal. And we can use the same example which we showed in [8] as an example to say that theorem 1 is not sufficient.

4   A Particular Sufficient Result

Here we consider a special case and begin with an example. Suppose that $n = 6$, the invariant factors of B are $1, 1, 1, (x - \beta)^2, (x - \beta)^2, (x - \beta)^2$ and the invariant factors of C are $1, (x-\gamma), (x-\gamma), (x-\gamma), (x-\gamma), (x-\gamma)(x-\delta), \gamma \neq \delta$, respectively. So $g_B(\beta) = 3$, $g_C(\gamma) = 5$. Necessary that there is an A such that $A = BC$, we have that $g_A(\beta\gamma) \geq g_B(\beta) + g_C(\gamma) - n = 3 + 5 - 6 = 2$ and the 6 interlacing conditions of theorem 1 must be verified. By item 6 of theorem 1, $h_5(\gamma B)|h_6(A)$ and $h_4(\gamma B)|h_5(A)$, that is, $(x - \beta\gamma)^2$ must be a factor of $h_5(A)$ and of $h_6(A)$. Then, by condition 5 of theorem 1, $h_5(\frac{1}{\gamma}A)|h_6(B)$, so $h_5(A)$ is exactly $(x - \beta\gamma)^2$. And since $h_2(\frac{1}{\gamma}A)|h_3(B)$, and $h_3(B) = 1$, so $h_2(A) = 1$, and $g_A(\beta\gamma)$ is at most 4.

The possible invariant factors of A are

(1) $1, 1, (x - \beta\gamma), (x - \beta\gamma), (x - \beta\gamma)^2, (x - \beta\gamma)^2$.

(2) $1, 1, 1, (x - \beta\gamma)^2, (x - \beta\gamma)^2, (x - \beta\gamma)^2$.

(3) $1, 1, 1, (x - \beta\gamma), (x - \beta\gamma)^2, (x - \beta\gamma)^2(x - \beta\delta)$.

(4) $1, 1, 1, 1, (x - \beta\gamma)^2, (x - \beta\gamma)^3(x - \beta\delta)$.

(5) $1, 1, 1, 1, (x - \beta\gamma)^2, (x - \beta\gamma)^2 (x - d)^2, d^2 = \beta^2 \gamma\delta.$

(6) $1, 1, 1, 1, (x - \beta\gamma)^2, (x - \beta\gamma)^2 (x - a)(x - b), \quad a \neq b, \quad ab = \beta^2 \gamma\delta.$

But, by determinant restriction, the first and the second are ruled out, or $\gamma = \delta$, so only the other four remain. When (3) happens, let

$$B = \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix}, C = \gamma I_5 \oplus \delta. \text{ Then, } BC \text{ gives directly}$$
A.

In case (4), it is easy to verify that there are

$$B = \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \quad \text{and} \quad C' = \begin{bmatrix} \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 1 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \simeq C,$$

such that BC' has (4) as invariant factor.

In case (5) and (6), we write $B = \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix}, C = \gamma I_4 \oplus \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$ and $A = \begin{bmatrix} \beta\gamma & 1 \\ 0 & \beta\gamma \end{bmatrix} \oplus \begin{bmatrix} \beta\gamma & 1 \\ 0 & \beta\gamma \end{bmatrix} \oplus A'$, where $A' = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, since the last block of B and of C are nonderogatory, and A' is nonscalar, so by [9], there are B' similar to B and C' similar to C, such that $B'C'$ with (5) or (6) as invariant factors.

In general, for this particular case, we have the following corollary which shows that in a particular situation the theorem 1 is necessary and sufficient.

Corollary 4 *Suppose that* $B = \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix}, C = \gamma I_{n-1} \oplus \delta.$ *Then the necessary conditions if theorem 1 are also sufficient for there exist B' similar to B and C' similar to C, such that $B'C'$ lies in the similarity class of a given A.*

Proof. Here we consider only the sufficiency. In this case, $g_B(\beta) = \frac{n}{2}$, $g_C(\gamma) = n - 1$. So, $g_A(\beta\gamma) \geq g_B(\beta) + g_C(\gamma) - n = \frac{n}{2} + n - 1 - n = \frac{n}{2} - 1$ must be satisfied. Assume that A satisfies the 6 interlacing conditions in theorem 1. By 6,

$$h_i(\gamma B) | h_{1+i}(A), \quad i = 1, ..., n - 1$$

which means the invariant factors $h_i(A), i = \frac{n}{2} + 2, ..., n$ must have $(x - \beta\gamma)^2$ as factors. By 5,

$$h_i(\frac{1}{\gamma} A) | h_{1+i}(B), i = 1, ..., n - 1.$$

So the invariant factors $h_i(A), i = \frac{n}{2} + 2, ..., n - 1$ are exactly $(x - \beta\gamma)^2$. Then the possible invariant factors of A are the follows

(1) $\overbrace{1,...,1}^{\frac{n}{2}-1}, (x-\beta\gamma), (x-\beta\gamma), (x-\beta\gamma)^2, ..., (x-\beta\gamma)^2.$

(2) $\overbrace{1,...,1}^{\frac{n}{2}}, (x-\beta\gamma)^2, (x-\beta\gamma)^2, (x-\beta\gamma)^2.$

(3) $\overbrace{1,...,1}^{\frac{n}{2}-1}, (x-\beta\gamma), (x-\beta\gamma)^2, ..., (x-\beta\gamma)^2, (x-\beta\gamma)^2(x-\beta\delta).$

(4) $\overbrace{1,...,1}^{\frac{n}{2}+1}, (x-\beta\gamma)^2, ..., (x-\beta\gamma)^2, (x-\beta\gamma)^3(x-\beta\delta).$

(5) $\overbrace{1,...,1}^{\frac{n}{2}+1}, (x-\beta\gamma)^2, ..., (x-\beta\gamma)^2, (x-\beta\gamma)^2(x-d)^2, d^2=\beta^2\gamma\delta.$

(6) $\overbrace{1,...,1}^{\frac{n}{2}+1}, (x-\beta\gamma)^2, ..., (x-\beta\gamma)^2, (x-\beta\gamma)^2(x-a)(x-b), \ \ a\neq b, \ \ ab=\beta^2\gamma\delta.$

Now, taking into account the determinant restriction, the cases (1) and (2) can not happen. In the other four cases, we can use the same arguments as we used in the 6×6 example to get B' similar to B and C' similar to C, such that $B'C'$ lies in the similarity class of a given A.

$\square$

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