Stability results for impulsive functional differential equations with infinite delay

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Abstract

For a family of differential equations with infinitive delay and impulses, we establish conditions for the existence of global solutions and for the global asymptotic and global exponential stabilities of an equilibrium point. The results are used to give stability criteria for a very broad family of impulsive neural network models with both unbounded distributed delays and bounded time-varying discrete delays. Most of the impulsive neural network models with delay recently studied are included in the general framework presented here.

Keywords: infinite delay; impulses; existence of solutions; Cohen-Grossberg neural network; global asymptotic stability; global exponential stability.

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1 Introduction

In nature, many evolutionary systems go through momentary abrupt changes, due to sudden phenomena in the environment. In population dynamics, these short-time phenomena include weather disasters, earthquakes, harvesting, migration of birds. In many different fields, like physics, chemical technology, machinery control, biotechnology or operation research, pulse perturbations have been introduced as a tool to control and stabilize solutions to problems. When considering the entire evolutionary system of such processes, one may consider that these changes are instantaneous, so that they are introduced as impulses in the respective models.

As a result of these multiple applications, the theory of impulsive differential equations has emerged as an important area of investigation [20]. In order to have more realistic models, often the past history of the systems should be taken into account – which has led to the introduction of time-delays in differential equations [14]. In recent years, the stability of an equilibrium for problems involving differential equations with delay and impulses has received a great deal of attention. Namely, natural and artificial neural networks have become an important area of research since the work of Hopfield [18], in view of their multiple applications in pattern recognition, optimization, signal processing, etc. Since the synaptic transmission of information among neurons – or their artificial representation – is not instantaneous, delays have been incorporated in dynamical systems modelling neural networks, as well as impulses occurring at certain fixed times [10]. More recently, functional differential equations (FDEs) with impulses and infinite delay have been considered and used as models for neural netwoks, see [8], [21], [22], [26], [27], [32], and [34]. The stability of equilibria in neural networks is particularly important, since the stationary states represent possible optimal solutions of the system in optimization problems, or stored patterns in associative memories. Establishing the global stability of a unique equilibrium has become one the major goals when implementing an artificial network.

In this paper, we consider a system of impulsive differential equations with infinite delay given in abstract form as

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, x_t),\ 0 \le t \ne t_k, \\ &\Delta(x_i(t_k)) := x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k^-)), \quad i = 1, 2, \dots n,\ k \in \mathbb{N}. \end{aligned} \tag{1.1}$$

Here, $f_i(t, \varphi)$ are real continuous functions for $t \ge 0$ and $\varphi : (-\infty, 0] \to \mathbb{R}^n$ on some space of functions to be defined later, $I_{ik} : \mathbb{R} \to \mathbb{R}$ are continuous, and $\{t_k\}_{k\in\mathbb{N}}$ is a sequence in $(0, \infty)$ such that $t_k \nearrow \infty$ as $k \to \infty$. For $t \ge 0$, we define the history function $x_t : (-\infty, 0] \to \mathbb{R}^n$ by

$$x_t(s) = x(t + s), \quad s \in (-\infty, 0].$$

To give an initial condition for (1.1) at time $t = \sigma$ is to give the past of the system for $s \le \sigma$, i.e., to require that $x(\sigma + s) = \varphi(s)$ for $s \le 0$, for some prescribed function $\varphi : (-\infty, 0] \to \mathbb{R}^n$. With the above notation, we write

$$x_\sigma = \varphi. \tag{1.2}$$

When dealing with continuous FDEs with infinite delay, the choice of an admissible Banach phase space requires special attention, in order to have well-posedness of the initial value problem and standard results on existence, uniqueness, continuation of solutions (see [12], [13], [16]). The situation is far more complicated when impulses are added. For impulsive FDEs with bounded and unbounded delay, existence, uniqueness or global continuation of solutions have been studied by some authors, see e.g. Ballinger and Liu [1, 23], Ouahab [29, 30] or Ye [34]. Most of the literature often refers to a phase space satisfying a certain axiomatic under which these results are achieved, however without providing either an explicit phase space, or proofs of the results. In many papers, a phase space is not even mentioned, much less the problem of such a suitable choice. For this reason, here we spend some time with a correct formulation of a phase space and the proof of existence results.

After the introduction, the present paper is divided into four sections. Section 2 is a preliminary section, where some notation and a concrete phase space for (1.1) are introduced.

Section 3 is dedicated to proving the existence of global solutions to (1.1) in the chosen phase space. In fact, a slightly more general system will be studied, cf. system (2.1).

The main results on stability, concerning both the global asymptotic and the global exponential stabilities of a stationary solution for a class of impulsive FDEs with infinite delay, are presented in Section 4. Note that for the impulsive systems under consideration, from the results in Section 3 it follows that the solutions will be defined for all $t \geq 0$. Most studies use a type of Lyapunov functional to obtain results on global attractivity. Instead, here we use the techniques described in the works of Faria and Oliveira in [5],[6], rather than a Lyapunov functional approach.

We are especially interested in applying these results to neural networks which can be written in the general form (1.1). In Section 5, we establish criteria for stability for several impulsive generalized Cohen-Grossberg models. The advantage of our general formulation is that it applies to a broad family of impulsive FDEs with unbounded delays, which includes as particular cases most of the models analysed in the recent literature (see e.g. [9], [21], [22], [33]).

2 Notation and preliminaries

For a compact interval $[\alpha, \beta]$ of $\mathbb{R}$, let $PC([\alpha, \beta]; \mathbb{R}^n)$ be the space of piecewise continuous functions from $[\alpha, \beta]$ to $\mathbb{R}^n$ and left continuous on $(\alpha, \beta]$, $PC([\alpha, \beta]; \mathbb{R}^n) = \{\phi : [\alpha, \beta] \to \mathbb{R}^n | \ \phi$ is continuous everywhere except for a finite number of points $s \in [\alpha, \beta)$ for which $\phi(s^-), \phi(s^+)$ exist and $\phi(s^-) = \phi(s)\}$, equipped with the supremum norm $\|\phi\|_\infty = \sup_{s\in[\alpha,\beta]} |\phi(s)|$, where $|\cdot|$ is a chosen norm in $\mathbb{R}^n$. Denote by $R([\alpha, \beta]; \mathbb{R}^n)$ the closure of $PC([\alpha, \beta]; \mathbb{R}^n)$ with respect to the supremum norm in the space of all bounded functions from $[\alpha, \beta]$ to $\mathbb{R}^n$. The space $R([\alpha, \beta]; \mathbb{R}^n)$ is the space of normalized (from the left) regulated (or ruled) functions from $[\alpha, \beta]$ to $\mathbb{R}^n$, i.e, the space of functions $f : [\alpha, \beta] \to \mathbb{R}^n$ with only discontinuities of the first kind, and left continuous on $(\alpha, \beta]$; $R([\alpha, \beta]; \mathbb{R}^n)$ is a Banach space and every function in $R([\alpha, \beta]; \mathbb{R}^n)$ has at most countably many discontinuities (see e.g. p. 146 of [3] and Chapter 3 of [17]).

Define the space $PC = PC((-\infty, 0]; \mathbb{R}^n)$ as the space of functions from $(-\infty, 0]$ to $\mathbb{R}^n$ for which the restriction to each compact interval $[\alpha, \beta] \subset (-\infty, 0]$ is in $R([\alpha, \beta]; \mathbb{R}^n)$. Clearly, if $\phi \in PC$ then ϕ is continuous everywhere except at most for a enumerable number of points $s = s_k$, and $\phi(s_k^-), \phi(s_k^+)$ exist with $\phi(s_k) = \phi(s_k^-)$. Denote by BPC the subspace of all bounded functions in PC, $BPC = BPC((-\infty, 0]; \mathbb{R}^n) = \{\phi \in PC : \phi$ is bounded on $(-\infty, 0]\}$, with the supremum norm $\|\phi\|_\infty = \sup_{s\leq 0} |\phi(s)|$. For $\beta \in \mathbb{R}$, in a similar way we define the spaces $PC((-\infty, \beta]; \mathbb{R}^n)$ and $BPC((-\infty, \beta]; \mathbb{R}^n)$.

Fix a function g such that:

(g1) $g : (-\infty, 0] \to [1, \infty)$ is a non-increasing continuous function and $g(0) = 1$;

(g2) $\lim_{u\to 0^-} \frac{g(s + u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

(g3) $g(s) \to \infty$ as $s \to -\infty$.

Note that these conditions are fulfilled for e.g. $g(s) = e^{-\alpha s}$, where $\alpha > 0$. We shall consider the phase space

$$PC_g = \Big\{\phi \in PC : \sup_{s\leq 0} \frac{|\phi(s)|}{g(s)} < \infty\Big\},$$

with the norm

$$\|\phi\|_g = \sup_{s\le 0} \frac{|\phi(s)|}{g(s)}.$$

It is clear that $BPC \subset PC_g$, with $\|\phi\|_g \le \|\phi\|_\infty$ for $\phi \in BPC$. If BPC is considered as a subspace of PC_g, we often write BPC_g. The spaces $(BPC, \|\cdot\|_\infty)$ and $(PC_g, \|\cdot\|_g)$ are Banach spaces (see Section 3).

Let $D \subset PC_g$, $f : \mathbb{R} \times D \to \mathbb{R}^n, I_k : D \to \mathbb{R}^n$ $(k \in \mathbb{N})$ be continuous functions, and $(t_k)_{k\in\mathbb{N}}$ a given sequence on $(0,\infty)$ such that $t_k \nearrow \infty$ as $k \to \infty$. Consider impulsive FDEs in PC_g in the general form

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \ge 0, t \ne t_k \\ \Delta x(t_k) &= I_k(x_{t_k}), \quad k = 1, 2, \dots, \end{aligned} \tag{2.1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. Here, if $x : (-\infty, a) \to \mathbb{R}^n$ $(a \in \mathbb{R}$ or $a = \infty)$ has at most countably many jump discontinuities and $t \in (-\infty, a)$, as usual x_t denotes the function $x_t : (-\infty, 0] \to \mathbb{R}^n$ defined by $x_t(s) = x(t+s)$, $s \le 0$. For the equation with impulses (2.1), we shall always consider initial conditions in BPC, $x_\sigma = \phi \in BPC$. Alternatively, we may choose BPC as the phase space for (2.1), where now $D \subset BPC$, and $f : \mathbb{R} \times D \to \mathbb{R}^n, I_k : D \to \mathbb{R}^n$ $(k \in \mathbb{R})$ are continuous functions relative to the norm $\|\cdot\|_\infty$ in BPC.

Consider (2.1) and BPC as the set of *admissible* initial conditions.

Definition 2.1. *A function $x(t)$ is called a* **solution** *of system* (2.1) *corresponding to* (σ, ϕ) *if there is $d > \sigma$ such that $x : (-\infty, d] \to \mathbb{R}^n$ is continuous for $t \in [\sigma, d] \setminus \{t_k : k \in \mathbb{N}\}$, $x(t_k^-)$ and $x(t_k^+)$ exist with $x(t_k) = x(t_k^-)$ for $t_k \in [\sigma, d]$ ($k \in \mathbb{N}$), the derivative of $x(t)$ exists for $t \in [\sigma, d] \setminus \{t_k : k \in \mathbb{N}\}$, $x(t)$ satisfies system* (2.1)*, and $x_\sigma = \phi$.*

In order to simplify the notation, in general we shall take f in (2.1) defined in the whole space (i.e, either $D = PC_g$ or $D = BPC$) and initial conditions will be given at $t = 0$:

$$x_0 = \phi, \quad \phi \in BPC. \tag{2.2}$$

For differential equations with impulses and unbounded delay, one should take some care with the choice of a suitable phase space and set of initial conditions. Even for the case without impulses, dealing with FDEs with *unbounded* delay requires a careful abstract formulation of an admissible phase space. For g satisfying (g1)-(g3), the space UC_g defined by

$$UC_g = \left\{ \varphi \in C((-\infty, 0], \mathbb{R}^n) : \sup_{\theta \le 0} \frac{|\varphi(\theta)|}{g(\theta)} < \infty, \frac{\varphi(\theta)}{g(\theta)} \text{ is uniformly continuous on } (-\infty, 0] \right\}$$

with the norm $\|\cdot\|_g$ is an *admissible* phase space in the sense of [13], [16]. This means that the normed space $\mathcal{B} = UC_g$ satisfies the axiomatic conditions for an admissible space given below:

(A) There are a constant $J > 0$ and functions $K, M : [0,\infty) \to [0,\infty)$, with K continuous and M locally bounded, such that the following conditions hold:
If $x : (-\infty, d] \to \mathbb{R}^n$, $d > a$, is such that $x_a \in \mathcal{B}$ and $x|_{[a,d]} \in C([a,d]; \mathbb{R}^n)$, then for all $t \in [a, d]$:

(i) $x_t \in \mathcal{B}$;

(ii) $|x(t)| \leq J\|x_t\|_{\mathcal{B}}$;

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t-a)\sup\{|x(s)| : \ a \leq s \leq t\} + M(t-a)\|x_a\|_{\mathcal{B}}$;

(iv) for $t \in [a,d]$, $t \mapsto x_t \in \mathcal{B}$ is a continuous function.

(B) $\mathcal{B}$ is complete.

Since (A)-(B) are satisfied, for an abstract FDE in the phase space UC_g,

$$\dot{x}(t) = f(t, x_t), \tag{2.3}$$

where $f : D \subset \mathbb{R} \times UC_g \to \mathbb{R}^n$ is continuous, the standard existence and uniqueness type results hold ([13],[16]). Moreover, UC_g is a fading memory space, and therefore precompactness results for bounded positive orbits in UC_g are valid ([12]). Clearly, results for the local existence and uniqueness of solution for the initial value problem (2.1)-(2.2) follow from the corresponding results for DDEs without impulses (2.3).

As regards the existence of global solutions for impulsive FDEs with infinite delay, we refer the reader to [15], [30], [34], also for further references. Nevertheless, the study of continuation of solutions for the impulsive system with unbounded delay (2.1) seems not to have been carried out consistently. Most of the times authors either assume that all solutions with admissible initial conditions are defined for all $t \geq 0$, or introduce axiomatically a phase space $\mathcal{B}$, which should be complete and a 'fading' memory space; this means that $\mathcal{B}$ should satisfy axioms (A)(i)-(iii) and (B), with $C([a,d];\mathbb{R}^n)$ replaced by $PC([a,d];\mathbb{R}^n)$ – note however that hypothesis (A)(iv) fails for FDEs with impulses. Frequently, either an explicit space is not given, or the above axioms are not verified for a given one. In many papers, a phase space is not even mentioned. This situation motivates us to establish sufficient conditions for the existence of *global* solutions to (2.1) in the concrete phase space PC_g. Spaces PC_g will be used throughout Sections 4 and 5, however other concrete suitable phase spaces will be mentioned in Section 3.

In a framework assuring global continuation of all admissible solutions to (2.1), we will be concerned with the stability of an equilibrium solution, if it exists. For $x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$, we also use x^* to denote the constant function $x(t) = x^*$ for t on an interval of $\mathbb{R}$. We say that x^* is an *equilibrium* of (2.1) if $f(t, x^*) = 0$ and $I_k(x^*) = 0$ for all $k \in \mathbb{N}$.

Definition 2.2. *Consider the phase space $\mathcal{D}$ with norm $\|\cdot\|$, where either $(\mathcal{D}, \|\cdot\|) = (PC_g, \|\cdot\|_g)$ for some g satisfying (g1)-(g3), or $(\mathcal{D}, \|\cdot\|) = (BPC, \|\cdot\|_\infty)$, and take BPC as the set of admissible initial conditions. Suppose that x^* is an equilibrium of* (2.1). *The equilibrium x^* is said to be: (i)* **stable** *if for any $\sigma > 0$ and $\varepsilon > 0$ there is $\delta = \delta(\sigma, \varepsilon) > 0$ such that $\|x_t(\sigma, \phi) - x^*\| < \varepsilon$ for all $\phi \in BPC$, with $\|\phi - x^*\| < \delta$ and $t \geq \sigma$; (ii)* **uniformly stable** *if δ in (i) does not depend on σ; (iii)* **globally asymptotically stable** *if x^* is stable and globally attractive in $\mathbb{R}^n$, i.e., $x(t) \to x^*$ as $t \to \infty$, for all solutions $x(t)$ with initial condition in BPC; (iv)* **globally exponentially stable** *if there are positive constants ε, M such that*

$$|x(t, 0, \phi) - x^*| \leq M e^{-\varepsilon t}\|\phi - x^*\|_\infty \quad \text{for} \quad t \geq 0, \ \phi \in BPC.$$

3 Existence of solutions

Although we take BPC as the set of admissible initial conditions for an FDE with impulses, this space is not big enough for the purpose of proving the global attractivity of an equilibrium of (2.1). Rather than BPC, we shall work on a space PC_g where g satisfies (g1)-(g3). It is well known that the spaces $R([\alpha,\beta];\mathbb{R}^n)$ ($\alpha,\beta\in\mathbb{R},\alpha<\beta$) equipped with the norm $\|\cdot\|_\infty$ are Banach spaces, hence BPC is also a Banach space with $\|\cdot\|_\infty$. Now, it is easy to show that:

Lemma 3.1. *PC_g with a Banach space with $\|\cdot\|_g$.*

Proof. Let (φ_n) be a Cauchy sequence in PC_g, and fix any $\varepsilon>0$. There is $n_0\in\mathbb{N}$ such that for all $n>n_0$ and $p\in\mathbb{N}$

$$\|\varphi_{n+p}-\varphi_n\|_g=\sup_{s\le 0}\frac{1}{g(s)}|\varphi_{n+p}(s)-\varphi_n(s)|<\varepsilon.$$

For each $s\le 0$, from the above inequality it follows that the sequence $\varphi_n(s)$ is a Cauchy sequence in $\mathbb{R}^n$; moreover, for its limit $\varphi(s):=\lim\varphi_n(s)$, we get

$$\frac{1}{g(s)}|\varphi_n(s)-\varphi(s)|\le\varepsilon$$

for all $n>n_0$. Consequently, $\sup_{s\le 0}\frac{1}{g(s)}|\varphi_n(s)-\varphi(s)|\to 0$ as $n\to\infty$. On the other hand, for each compact interval $[\alpha,\beta]\subset(-\infty,0]$, ${\varphi_n}_{|_{[\alpha,\beta]}}\to\varphi_{|_{[\alpha,\beta]}}$ uniformly in $[\alpha,\beta]$. Since the spaces $R([\alpha,\beta];\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_\infty$ are complete, $\varphi_{|_{[\alpha,\beta]}}\in R([\alpha,\beta];\mathbb{R}^n)$ for each $[\alpha,\beta]\subset(-\infty,0]$. Therefore $\varphi\in PC_g$ and $\|\varphi_n-\varphi\|_g\to 0$ as $n\to\infty$. □

We return to the space BPC equipped with either the norm of uniform convergence $\|\cdot\|_\infty$ or the norm $\|\cdot\|_g$ where g satisfies (g1)-(g3). The norm in BPC will be denoted simply by $\|\cdot\|$. For a given $x\in BPC((-\infty,b];\mathbb{R}^n)$ where $b>0$, clearly the function $\Phi:[0,b]\to BPC$, $t\mapsto x_t=:\Phi(t)$ is not continuous, unless x is itself continuous: in fact, Φ is discontinuous at any $t\in[0,b]$ such that x has a jump discontinuity for some $t_0\in(-\infty,t)$. In order to have continuity of the map Φ, one has to consider a suitable norm in BPC, see Lemma 3.2 below. Nevertheless, in general the function $t\mapsto f(t,x_t)$ is measurable on intervals $[0,b]$, and therefore summable on $[0,b]$ provided it is dominated by some summable function.

Consider the impulsive FDE in BPC

$$\begin{aligned}\dot{x}(t)&=f(t,x_t),\quad t\ge 0,t\ne t_k\\ \Delta x(t_k)&=I_k(x_{t_k}),\quad k=1,2,\end{aligned}\tag{3.1}$$

where $0<t_1<t_2<\dots$ with $t_k\to\infty$, and the functions $f:[0,\infty)\times BPC\to\mathbb{R}^n, I_k: BPC\to\mathbb{R}^n$ are continuous, $k\in\mathbb{N}$, subject to bounded initial conditions:

$$x_0=\phi,\quad \phi\in BPC.\tag{3.2}$$

Under some conditions, we prove the existence of global solutions for the problem (3.1)-(3.2), following the approach in [29, 30]. The purpose here is not to give optimal sufficient conditions for global solutions to exist (cf. Remark 3.1), but simply to provide a sufficiently general setting which covers the impulsive systems studied in Sections 4 and 5. As usual, a fixed point argument is used, in this case the Leray-Schauder alternative theorem (see [11], p. 124):

Theorem 3.1. *Let C be a convex subset of a normed space E, and assume that $0 \in C$. If $F : C \to C$ is a completely continuous operator, then either the set $\{x \in C : x = \lambda Fx$ for some $\lambda \in (0,1)\}$ is unbounded or F has a fixed point.*

Below, if $b = \infty$, the interval $[0,b]$ means $[0,\infty)$.

Theorem 3.2. *Consider BPC with the norm $\|\cdot\|$, where $\|\cdot\| = \|\cdot\|_\infty$ or $\|\cdot\| = \|\cdot\|_g$. For $b > 0$ or $b = \infty$, assume that:*

(h1) for all $x \in BPC((-\infty,b];\mathbb{R}^n)$, the function $t \mapsto f(t,x_t)$ is measurable on $[0,b]$;

(h2) there are a locally integrable function $p : [0,b] \to [0,\infty)$ and a continuous non-decreasing function $q : [0,\infty) \to [0,\infty)$ with $q(u) > 0$ for $u > 0$ and such that $\int^\infty \frac{du}{q(u)} = \infty$, satisfying

$$|f(t,\psi)| \le p(t)q(\|\psi\|), \quad \text{for} \quad t \in [0,b], \psi \in BPC; \tag{3.3}$$

(h3) for each $k \in \mathbb{N}$, I_k maps bounded sets of BPC into bounded sets of $\mathbb{R}^n$.

Then, for each $\phi \in BPC$, the problem (3.1)-(3.2) has a solution $x(t)$ defined on $[0,b]$.

Proof. We suppose that BPC is equipped with the norm of uniform convergence. For $\|\cdot\| = \|\cdot\|_g$, the proof follows in a similar way. Let $\phi \in BPC$ be given and fix $\bar b > 0$ with $\bar b = b$ if $b \in \mathbb{R}$ and $\bar b < \infty$ if $b = \infty$. We prove that there exists a solution $x(t)$ of the problem (3.1)-(3.2) defined for $t \le \bar b$. For simplicity, in what follows we write b instead of $\bar b$.

Set $t_0 = 0$, and write $b \in (t_{n-1}, t_n]$ for some $n \in \mathbb{N}$. We have $\left(\int_0^t f(s,x_s)\,ds\right)' = f(t,x_t)$ on $[0,b]$ for $x \in BPC((-\infty,b];\mathbb{R}^n)$, and consequently $x(t)$ is a solution of (3.1)-(3.2) on $[0,b]$ if and only if $x(t) = (Nx)(t)$ for $t \le b$, where $N : X \to X$ is the operator defined on $X := \{x : (-\infty,b] \to \mathbb{R}^n |\ x_0 = \phi, x(0^+) = \phi(0)$ and $x_{|[0,b]} \in PC([0,b];\mathbb{R}^n)\}$ by

$$(Nx)(t) = \begin{cases} \phi(t), & t \le 0. \\ \phi(0) + \int_0^t f(s,x_s)\,ds + \sum_{k:0<t_k<t} I_k(x_{t_k}), & 0 \le t \le b, x \in X. \end{cases} \tag{3.4}$$

For $s \le b$, write $x_s = \tilde\phi_s + \bar x_s$, where $\tilde\phi(s) = \phi(s)$ for $s \le 0$ and $\tilde\phi(s) = \phi(0)$ for $0 \le s \le b$. Then, $Nx = x$ if and only if $N_0 \bar x = \bar x$, where, after dropping the bars for simplicity, $N_0 : X_0 \to X_0$ is the operator given by

$$(N_0x)(t) = 0 \text{ for } t \le 0\,,\ (N_0x)(t) = \int_0^t f(s,\tilde\phi_s + x_s)\,ds + \sum_{k:0<t_k<t} I_k(\tilde\phi_{t_k} + x_{t_k}) \text{ for } 0 \le t \le b,$$

and X_0 is the space $X_0 := \{x \in BPC((-\infty,b];\mathbb{R}^n) : x_0 = 0, x(0^+) = 0\}$, which can be identified with the subspace of the functions $x(t)$ in $PC([0,b];\mathbb{R}^n)$ with $x(0^+) = 0$.

Step 1. N_0 is completely continuous.

Let $\{x_m\} \subset X_0$ and $x_m \to x$ uniformly on $[0,b]$. Then,

$$\|N_0x_m - N_0x\|_\infty \le \int_0^b |f(s,\tilde\phi_s + (x_m)_s) - f(s,\tilde\phi_s + x_s)|\,ds + \sum_{1\le k\le n} |I_k(\tilde\phi_{t_k} + (x_m)_{t_k}) - I_k(\tilde\phi_{t_k} + x_{t_k})|.$$

From the continuity of the functions f and I_k, (h2) and Lebesgue's dominated convergence theorem, we get $\|N_0x_m - N_0x\|_\infty \to 0$.

Since $\|N_0x\|_\infty \le \int_0^b p(s)q(\|\tilde\phi_s + x_s\|_\infty)\, ds + \sum_{1\le k\le n} |I_k(\tilde\phi_s + x_{t_k})|$, then N_0 maps bounded sets of X_0 into bounded sets of X_0.

Next, write $N_0 = N_1 + N_2$, where $N_1, N_2 : X_0 \to X_0$ are the operators given by $(N_ix)(t) = 0$ for $t \le 0$ and

$$(N_1x)(t) = \int_0^t f(s, \tilde\phi_s + x_s)\, ds, \quad (N_2x)(t) = \sum_{k:0<t_k<t} I_k(\tilde\phi_s + x_{t_k}) \quad \text{for } 0 \le t \le b.$$

Fix $M > 0$. For any $\tau_0, \tau \in [0, b]$, and $x \in X_0$ such that $\|x\|_\infty \le M$, we have $|(N_1x)(\tau) - (N_1x)(\tau_0)| \le q(\|\phi\|_\infty + M) \int_{\tau_0}^{\tau} p(s)\, ds$, which converges uniformly to zero as $\tau \to \tau_0$, independently of x. This means that N_1 maps a ball $B_M(0)$ of X_0 into an equicontinuous set of the space $\{x : (-\infty, b] \to \mathbb{R}^n : x \text{ is continuous and } x_0 = 0\}$, which can be identified with the subspace of the functions in $C([0, b]; \mathbb{R}^n)$ with $x(0) = 0$. From Ascoli-Arzelà theorem, we conclude that N_1 is completely continuous. On the other hand, hypothesis (h3) means that I_k are completely continuous, and therefore the same happens to the operator N_2.

Step 2. The set $\mathcal{E} = \{x \in X_0 : x = \lambda N_0x \text{ for some } \lambda \in (0, 1)\}$ is bounded.

Let $\lambda \in (0, 1)$ and $x = \lambda N_0x$. For $t \in [0, \bar t_1]$ where $\bar t_1 = min\{t_1, b\}$, we have

$$\|x_t\|_\infty \le \lambda \int_0^t p(s)q(\|\tilde\phi_s + x_s\|_\infty)\, ds \le \int_0^t p(s)q(\|\tilde\phi_s + x_s\|_\infty)\, ds =: \beta(t).$$

The function $\beta(t)$ is non-decreasing and $\beta(0) = 0$; since q is non-decreasing, it also satisfies the differential inequality $\beta'(t) \le p(t)q(\|\phi\|_\infty + \beta(t))$. We now use the well-known Osgood's test argument: if the solution $u(t)$ of the initial value problem $u(0) = 0$, $u'(t) = p(t)q(\|\phi\|_\infty + u(t))$ is not bounded on $[0, \tau)$ with $\tau \le \bar t_1$, then $u(t) \to \infty$ as $t \to \tau^-$ and

$$\int_0^\tau p(t)\, dt = \int_0^\tau \frac{u'(t)}{q(\|\phi\|_\infty + u(t))}\, dt = \int_{\|\phi\|_\infty}^\infty \frac{du}{q(u)} = \infty,$$

a contradiction. This implies that there is $C_1 > 0$ such that $\beta(t) \le C_1$ on $[0, \bar t_1]$, where C_1 may dependent on ϕ but not on x. Hence, $\|x_t\|_\infty \le C_1$ for all $t \in [0, \bar t_1]$.

Now, suppose that $b > t_1$. From (h3), consider $c_1 > 0$ such that $|I_1(y)| \le c_1$ for all $y \in BCP$ with $\|y\|_\infty \le C_1 + \|\phi\|_\infty$. For $t \in (t_1, \bar t_2]$ where $\bar t_2 = \min\{b, t_2\}$, we obtain

$$\|x_t\|_\infty \le \int_0^t p(s)q(\|\tilde\phi_s + x_s\|_\infty)\, ds + c_1.$$

Proceeding as above, where now $\beta(t) = c_1 + \int_0^t p(s)q(\|\tilde\phi_s + x_s\|_\infty)\, ds$ for $t \in [0, \bar t_2]$, in a similar way we deduce that there is a constant C_2 such that $\|x_t\|_\infty \le \beta(t) \le C_2$ for all $t \in [0, \bar t_2]$. By induction, we conclude that there is $C > 0$ such that $\|x_t\|_\infty \le C$ for all $t \in [0, b]$, hence the set $\mathcal{E}$ is bounded.

Thus, from Theorem 3.1 there is a fixed point of N, and the proof is complete. □

Clearly (3.3) holds if $|f(t, \psi)|$ has a sublinear-type growth as described in the following criterion:

Corollary 3.1. *Let BPC be equipped with either the norm $\|\cdot\| = \|\cdot\|_\infty$ or $\|\cdot\| = \|\cdot\|_g$. Assume (h1), (h3), and that for each $b > 0$ there are integrable functions $p, q : [0, b], \to [0, \infty)$ such that*

$$|f(t, \psi)| \leq p(t) + q(t)\|\psi\|, \quad \text{for all} \quad t \in [0, b], \psi \in BPC.$$

Then, for each $\phi \in BPC$, the problem (3.1)-(3.2) has a solution $x(t)$ defined for $t \geq 0$.

Remark 3.1. As for FDEs without impulses, in Theorem 3.2 it is actually sufficient to require that $f(t, \psi)$ satisfies the Carathéodory conditions, instead of the continuity of the function $f(t, \psi)$ on both variables (see e.g. [14] p. 58, and [29, 30]). On the other hand, if the impulse functions I_k are uniformly bounded by constants $c_k > 0$, for each given initial condition $\phi \in BCP$ we may replace condition $\int^\infty \frac{du}{q(u)} = \infty$ in (h2) by

$$\int_0^{t_n+1} p(s)\, ds < \int_{M_n + \|\phi\|_\infty}^\infty \frac{du}{q(u)} \text{ for all } n \in \mathbb{N}, \quad \text{where } M_n = \sum_{k=1}^n c_k,$$

and deduce the existence of a global solution of (3.1)-(3.2). This is proven by applying in Step 2 of the above proof the argument used for the Osgood's test in each interval $[0, t_1], [t_1, t_2]$, etc. However, the setting in Theorem 3.2 is general enough for our purposes, since the aim is to study impulsive neural network in the form (4.1).

As mentioned previously, the impulsive nature of system (3.1) brings difficulties when working with *delayed* differential equations, even for the case of finite discrete delays, due to the fact that the map $t \mapsto x_t$ is not continuous if x has at least a simple jump discontinuity. Together with the subtle problem of working with infinite delay, this makes the choice of a suitable phase space, for which the standard results on existence, uniqueness, continuation of solutions should be valid, a difficult task. As we shall see in Sections 4 and 5, in terms of applications it is very useful to use the norm $\|\cdot\|_g$ (or $\|\cdot\|_\infty$) in BPC, but in fact a more efficient norm is defined below.

For g satisfying (g1)-(g3), define the space $L_g^1 = L_g^1((-\infty, 0]; \mathbb{R}^n)$ as

$$L_g^1 = \{\varphi : (-\infty, 0] \to \mathbb{R}^n \mid \frac{\varphi(s)}{g(s)} \in L^1((-\infty, 0]; \mathbb{R}^n)\}$$

with the norm

$$\|\varphi\|_{L_g^1} = \int_{-\infty}^0 \frac{|\varphi(s)|}{g(s)}\, ds.$$

The space $L_g^1((-\infty, b]; \mathbb{R}^n)$ is defined in a similar way. We further assume

(g4) $g^{-1} \in L^1((-\infty, 0]; \mathbb{R})$, where $g^{-1}(s) = 1/g(s)$.

Clearly, if $\varphi \in BPC$ then $\varphi \in L_g^1$, with $\|\varphi\|_{L_g^1} \leq \|g^{-1}\|_{L^1}\|\varphi\|_\infty$; and $L_g^1 \cap PC \subset PC_g$, however this latter inclusion is not continuous for $L_g^1 \cap PC$ endowed with $\|\cdot\|_{L_g^1}$.

Lemma 3.2. *Let g satisfy (g1)-(g4). For a given $x \in BPC((-\infty, b]; \mathbb{R}^n)$ where $b > 0$, the function $\Phi : [0, b] \to L_g^1$, $t \mapsto x_t$, is continuous.*

Proof. Let $\{s_k\}$ be the set of discontinuity points for x. For $t_n, t_0 \in [0, b]$ with $t_n \to t_0$, $\|x_{t_n} - x_{t_0}\|_{L_g^1} = \int_{-\infty}^0 h_n(s)\, ds$, where $h_n(s) := |x(t_n + s) - x(t_0 + s)|/g(s) \to 0$ as $n \to \infty$ for every $s \in (-\infty, 0], s \neq s_k - t_0$, and $|h_n(s)| \leq 2\|x\|_\infty/g(s) \in L^1$. Hence $\|x_{t_n} - x_{t_0}\|_{L_g^1} \to 0$. □

If $f:[0,\infty)\times(BPC,\|\cdot\|_{L^1_g})\to\mathbb{R}^n$ is continuous, then the map from $[0,b]$ to $\mathbb{R}^n, t\mapsto f(t,x_t)$, is continuous. An existence result similar to Theorem 3.2 is valid for the norm $\|\cdot\|_{L^1_g}$, as follows:

Theorem 3.3. *Consider BPC with a norm $\|\cdot\|_{L^1_g}$, where g satisfies (g1)-(g4). For $b>0$ or $b=\infty$, assume that:*

(i) there are a locally integrable function $p:[0,b]\to[0,\infty)$ and a continuous non-decreasing function $q:[0,\infty)\to[0,\infty)$ with $q(u)>0$ for $u>0$ and such that $\int^\infty \frac{du}{q(u)}=\infty$, satisfying

$$|f(t,\psi)|\le p(t)q(\|\psi\|_{L^1_g}),\quad \text{for all}\quad t\in[0,b],\psi\in BPC;$$

(ii) for each $k\in\mathbb{N}$, I_k maps bounded sets of BPC into bounded sets of $\mathbb{R}^n$.

Then, for each $\phi\in BPC$, the problem (3.1)-(3.2) has a solution $x(t)$ defined on $[0,b]$.

Proof. The aim is to use Theorem 3.1, to guarantee the existence of a fixed point to the operator $N:X\to X$ given by (3.4), where now $X:=\{x:(-\infty,b]\to\mathbb{R}^n|\ x_0=\phi, x(0^+)=\phi(0)$ and $x_{|[0,b]}\in L^1([0,b];\mathbb{R}^n)\}$ is considered as a subset of $L^1_g((-\infty,b];\mathbb{R}^n)$. The proof follows along the lines of the proof of Theorem 3.2, with the necessary adaptations for $\|\cdot\|_{L^1_g}$ (instead of $\|\cdot\|_\infty$), and where instead of the Ascoli-Arzelà theorem we use the following characterization of compactness in $L^1[0,b]$ (cf. [4]): a set $K\subset L^1[0,b]$ is relatively compact if and only if $\int_0^b|y(t+u)-y(t)|\,dt\to 0$ uniformly for $y\in K$. Details are omitted. □

Remark 3.2. Due to the continuity of the map $t\mapsto x_t\in L^1_g$, in fact other nice properties can be derived in the phase space L^1_g, such as pre-compactness of bounded positive orbits, however such study is behond the purposes of the present paper.

4 Main Results on Stability

In this section, sufficient conditions for the existence, global asymptotic stability and global exponential stability of an equilibrium point for (1.1) (???ou será (4.1)??) will be established.

We start with an auxiliary lemma.

Lemma 4.1. *Suppose that $\mathbb{R}^n$ is equipped with the maximum norm, and assume the following hypotheses:*

(H1) for all $t\ge 0$ and $\varphi\in PC_g$ such that $\frac{1}{g(\theta)}|\varphi(\theta)|<|\varphi(0)|$, for all $\theta<0$, then $f_i(t,\varphi)\varphi_i(0)<0$ for some $i\in\{1,...,n\}$ such that $|\varphi(0)|=|\varphi_i(0)|$;

(H2) for each function $\widehat{I}_k(u):=I_k(u)+u$ ($u\in\mathbb{R}^n$), there is $\xi_k>0$ such that $|\widehat{I}_k(u)|\le\xi_k|u|$ and $\prod_{k=1}^\infty\max\{1,\xi_k\}<\infty$.

If $x(t,0,\varphi)$ is a solution of (1.1)-(1.2) defined on $\mathbb{R}$, then $x(t,0,\varphi)$ is bounded and

$$|x(t,0,\varphi)|\le\|\varphi\|_g\prod_{k=1}^\infty\max\{1,\xi_k\}.$$

Proof. If $x(t,0,\varphi)$ is the solution of (1.1)-(1.2), then $x(t,0,\varphi)$ is a continuous function for $t \in (0,t_1)$. From Lemma 2.1 in [6], we obtain

$$|x(t,0,\varphi)| \leq \|\varphi\|_g, \quad \forall t \in (0,t_1].$$

By using (H2), for $t = t_1$

$$|x(t_1^+)| = |\widehat{I}_1(x(t_1))| \leq \xi_1 |x(t_1)| \leq \xi_1 \|\varphi\|_g,$$

and we get

$$|x(t^+,0,\varphi)| \leq \|\varphi\|_g \max\{1,\xi_1\}, \quad t \in (0,t_1].$$

For $t \in (t_1,t_2]$ and using the same arguments, we have

$$|x(t,0,\varphi)| \leq \|\varphi\|_g \max\{1,\xi_1\}, \quad t \in (t_1,t_2].$$

Again by using (H2), for $t = t_2$ we obtain

$$|x(t_2^+)| = |\widehat{I}_2(x(t_2))| \leq \xi_2 |x(t_2)| \leq \xi_2 \|\varphi\|_g \max\{1,\xi_1\}.$$

Therefore,

$$|x(t^+,0,\varphi)| \leq \|\varphi\|_g \max\{1,\xi_1\} \max\{1,\xi_2\}, \quad t \in (0,t_2].$$

Repeating the above procedure, for $t \in (0,t_l]$, $l \in \mathbb{N}$, we conclude that

$$|x(t^+,0,\varphi)| \leq \|\varphi\|_g \max\{1,\xi_1\} \dots \max\{1,\xi_l\}, \quad t \in (0,t_l].$$

Consequently,

$$|x(t,0,\varphi)| \leq \|\varphi\|_g \prod_{k=1}^{\infty} \max\{1,\xi_k\}, \quad t \geq 0.$$

□

Remark 4.1. A similar estimate is obtained using the norm $\|\cdot\|_\infty$ in BPC (cf. Lemma 2.2 in [6]). In fact, if (H1) is replaced by

(H1') for all $t \geq 0$ and $\varphi \in BPC$ such that $|\varphi(\theta)| < |\varphi(0)|$, for all $\theta < 0$, then $f_i(t,\varphi)\varphi_i(0) < 0$ for some $i \in \{1,...,n\}$ such that $|\varphi(0)| = |\varphi_i(0)|$,

then in the above lemma we obtain the estimate $|x(t,0,\varphi)| \leq \|\varphi\|_\infty \prod_{k=1}^{\infty} \max\{1,\xi_k\}$.

We now consider the following non-autonomous impulsive system:

$$\begin{aligned} &\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(t,x_t)], \ 0 \leq t \neq t_k, \ i = 1,2,\dots n, \\ &\Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)), \end{aligned} \tag{4.1}$$

where $\Delta(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-)$, $a_i : \mathbb{R} \to (0,\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$, $f_i : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ and $I_{ik} : \mathbb{R} \to \mathbb{R}$ are continuous functions for all $k \in \mathbb{N}, 1 \leq i \leq n$, and either $\mathcal{D} = PC_g$ or $\mathcal{D} = BPC$. We shall also consider the non-impulsive version of (4.1),

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(t,x_t)], \quad t \geq 0, \ i = 1,2,\dots n. \tag{4.2}$$

For $i = 1,\dots,n$ and $k = 1,2,\dots,$ we designate

$$\widehat{I}_{ik}(u) := I_{ik}(u) + u, \quad u \in \mathbb{R}. \tag{4.3}$$

In the sequel, we fix the maximum norm in $\mathbb{R}^n$, $|x| = \max_{1\leq i\leq n} |x_i|$ for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, and the following hypotheses will be referred to:

(A1) there exist constants $\beta_i > 0$ such that $\dfrac{b_i(u) - b_i(v)}{u - v} \geq \beta_i > 0$, for all $u, v \in \mathbb{R}, u \neq v$, $i = 1, 2, \ldots, n$;

(A2) $f_i : \mathbb{R} \times PC_g \to \mathbb{R}$ are uniformly Lipschitz continuous with respect to $\varphi \in PC_g$, with $|f_i(t, \varphi) - f_i(t, \psi)| \leq l_i \|\varphi - \psi\|_g$, for $t \in \mathbb{R}$ and $\varphi, \psi \in PC_g$, $i = 1, 2, \ldots, n$;

(A3) $\beta_i > l_i$, $i = 1, 2, \ldots, n$;

(A4) $\widehat{I}_{ik}$ are Lipschitz continuous, with $|\widehat{I}_{ik}(u) - \widehat{I}_{ik}(v)| \leq \widehat{\gamma}_{ik}|u - v|$ for $u, v \in \mathbb{R}$, $i = 1, ..., n$, $k \in \mathbb{N}$, where $\widehat{I}_{ik}(u) = u + I_{ik}(u), u \in \mathbb{R}$;

(A5) for $\widehat{\gamma}_k := \max_{1 \leq i \leq n} \widehat{\gamma}_{ik}$, $\prod_{k=1}^{\infty} \max\{1, \widehat{\gamma}_k\} < \infty$.

For the impulsive system (4.1), the application of Theorem 3.2 yields the following existence result:

Proposition 4.1. *Assume (A1)–(A4). Then the initial value problem (4.1)-(2.2) has a solution $x(t)$ defined on $[0, \infty)$.*

Proof. Let $\phi \in BPC$ be given. For each $b > 0$ fixed, if there is a solution $x(t)$ of (4.1)-(2.2) defined on $[0, b]$, then it follows from the proof of Lemma 4.1 that it must be uniformly bounded by a constant $M = M(b, \phi)$. The functions $a_i(u), b_i(u)$ are bounded for $u \in [-M, M], 1 \leq i \leq n$, and $f_i(t, \phi)$ are Lipschitz continuous with respect to ϕ, hence (3.3) is satisfied on $[0, b]$. Thus, (4.1)-(2.2) has a solution defined on $[0, b]$, for all $b > 0$. □

Lemma 4.2. *Assume (A1), (A2) and (A3), and that for each $x \in \mathbb{R}^n$ the functions $t \mapsto f_i(t, x)$ are constant on $\mathbb{R}$, $1 \leq i \leq n$. Then there exists a unique equilibrium point $x^* = (x_1^*, \ldots, x_n^*)$ of (4.2)*

Proof. The result is an immediate consequence of Lemma 3.1 in [6]. □

Lemma 4.3. *Suppose that conditions (A1)–(A5) are satisfied. Assume also that there exists a unique equilibrium point $x^* = (x_1^*, \ldots, x_n^*)$ of (4.2), with $\widehat{I}_{ik}(x_i^*) = x_i^*$ for all $i = 1, \ldots, n, k \geq 1$. Then, any solution $x(t) = x(t, 0, \phi)$ of (4.1) with initial condition $\phi \in BPC$ satisfies*

$$|x(t) - x^*| \leq \|\phi - x^*\|_g \prod_{k=1}^{\infty} \max\{1, \widehat{\gamma}_k\}, \quad t \geq 0. \tag{4.4}$$

Proof. Let $x^* = (x_1^*, ..., x_n^*)$ be the equilibrium of the continuous FDE (4.2). Clearly x^* is also an equilibrium of the impulsive system (4.1) if $\widehat{I}_{ik}(x^*) = x^*$ for all $k \geq 1$ and $i = 1, \ldots, n$.

Translating x^* to the origin by the change $\tilde{x}(t) = x(t) - x^*$, system (4.1) becomes

$$\begin{aligned} &\dot{\tilde{x}}_i(t) = -\tilde{a}_i(\tilde{x}_i(t))[\tilde{b}_i(\tilde{x}_i(t)) - \tilde{f}_i(t, \tilde{x}_t)], \quad 0 \leq t \neq t_k,\ i = 1, 2, \ldots n \\ &\Delta(\tilde{x}_i(t_k)) = \tilde{I}_{ik}(\tilde{x}_i(t_k^-)), \end{aligned} \tag{4.5}$$

where $\tilde{a}_i(u) = a_i(u + x_i^*)$, $\tilde{b}_i(u) = b_i(u + x_i^*)$, $\tilde{f}_i(t, \varphi) = f_i(t, \varphi + x^*)$ and $\tilde{I}_{ik}(u) = I_{ik}(u + x_i^*)$ with zero as the unique equilibrium, i.e., $\tilde{b}_i(0) + \tilde{f}_i(t, 0) = 0$ for $i = 1, ...n$, $t \in \mathbb{R}$. Clearly b_i,

f_i and I_{ik} satisfy (A1)-(A5) if only if $\tilde{b}_i$, $\tilde{f}_i$ and $\tilde{I}_{ik}$ satisfy (A1)-(A5). For simplicity, instead of (4.5) we consider (4.1) subject to the conditions $b_i(0) + f_i(t, 0) = 0$ and $\widehat{I}_{ik}(0) = 0$ for all $t \in \mathbb{R}$, $i = 1, \dots, n$, and $k \in \mathbb{N}$.

Define $\widehat{I}_k(u) = (\widehat{I}_{1k}(u_1), \dots, \widehat{I}_{nk}(u_n))$ for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$, where as before $\widehat{I}_{ik}(x) = I_{ik}(x) + x,\ x \in \mathbb{R}$. We prove that hypotheses (H1) and (H2) in Lemma 4.1 are satisfied.

Consider $\psi \in PC_g$ with $\frac{1}{g(\theta)}|\psi(\theta)| < |\psi(0)|$ for all $\theta < 0$, and let $|\psi(0)| = \psi_i(0) > 0$ (analogous if $\psi_i(0) < 0$). From (A1)-(A3), we derive

$$\begin{aligned}[b_i(\psi_i(0)) + f_i(t, \psi)] &= [b_i(\psi_i(0)) - b_i(0)] + [f_i(t, \psi) - f_i(t, 0)] \\ &\geq \beta_i \psi_i(0) - l_i \|\psi\|_g = (\beta_i - l_i)\|\psi\|_g > 0.\end{aligned}$$

Therefore, $F_i(t, \psi) := -a_i(\psi_i(0))[b_i(\psi_i(0)) + f_i(t, \psi)] < 0$, and (H1) holds. Since

$$|\widehat{I}_k(u)| = |\widehat{I}_k(u) - \widehat{I}_k(0)| \leq \widehat{\gamma}_k |u|,$$

for $u \in \mathbb{R}^n$ and $\widehat{\gamma}_k = \max_{1 \leq i \leq n} \widehat{\gamma}_{ik}$, from (A5) we conclude that (H2) holds. From Lemma 4.1, each solution $x(t) = x(t, 0, \phi)$ with $\phi \in PC_g$ is bounded on $\mathbb{R}$ and satisfies (4.4). □

Remark 4.2. Instead of (A2), suppose that the following condition is satisfied:

(A2') $f_i : \mathbb{R} \times BPC \to \mathbb{R}$ are uniformly Lipschitz continuous with respect to $\varphi \in BPC$, with $|f_i(t, \varphi) - f_i(t, \psi)| \leq l_i \|\varphi - \psi\|_\infty$, for $t \in \mathbb{R}$ and $\varphi,\ \psi \in BPC$, $i = 1, 2, \dots, n$.

Then, one can verify that (4.5) satisfies (H1') instead of (H1), and from Remark 4.1 we conclude that the estimate (4.4) is replaced by $|x(t) - x^*| \leq \|\phi - x^*\|_\infty \prod_{k=1}^\infty \max\{1, \widehat{\gamma}_k\}, t \geq 0$.

Theorem 4.1. *Assume that conditions (A1)–(A6) are satisfied, where*

(A6) $\inf_{k \geq 1}(t_{k+1} - t_k) > 0$.

Suppose also that

$$I_{ik}(x_i^*) = 0, \quad i = 1, \dots, n,\ k = 1, 2, \dots, \tag{4.6}$$

where x^* is the unique equilibrium of the continuous system (4.2). Then x^* is the unique equilibrium of the impulsive system (4.1), and is globally asymptotically stable.

Proof. From the lemma above, all solutions are bounded on $\mathbb{R}$ and the equilibrium x^* is uniformly stable. After translating x^* to the origin by the change $\tilde{x}(t) = x(t) - x^*$, (4.1) is transformed into (4.5), where for simplicity we drop the tildes and consider (4.1) subject to the conditions $b_i(0) + f_i(t, 0) = 0$ and $\widehat{I}_{ik}(0) = 0$ for all $t \in \mathbb{R}$, $i = 1, \dots, n$, and $k \in \mathbb{N}$.

It remains to prove that $x = 0$ is globally attractive. Let $x(t) = x(t, 0, \varphi)$ be the solution of (4.1) with an initial condition $x_0 = \varphi \in BPC$, $x(t) = (x_1(t), ..., x_n(t))$ for $t \in \mathbb{R}$.

Define $-v_i = \liminf_{t \to \infty} x_i(t)$ and $u_i = \limsup_{t \to \infty} x_i(t)$ for $i = 1, ..., n$, and $v = \max_i\{v_i\}$ and $u = \max_i\{u_i\}$. Note that $-\infty < -v \leq u < \infty$. Suppose that $|v| \leq u$, then $\max(u, v) = u$ (the case $|u| \leq v$ is analogous). It is sufficient to prove that $u = 0$, which is done in several steps.

Claim 1. For each $\varepsilon > 0$, there exists $T > 0$ such that $\|x_t\|_g \leq u + \varepsilon$ for all $t \geq T$.

Since $x(t)$ is uniformly bounded, take $M > 0$ such that $|x(t)| \leq M$ for all $t \in \mathbb{R}$. Now, fix any $\varepsilon > 0$. There is $T_0 > 0$ such that $|x(t)| \leq u + \varepsilon$ for all $t \geq T_0$. Let $T_1 > T_0$ be sufficiently large so that $M/g(-T_1) < u + \varepsilon$, and take $T = 2T_1$.

Consider $t \geq T$. If $s \leq -T_1$, then

$$\frac{|x(t+s)|}{g(s)} \leq \frac{M}{g(-T_1)} < u + \varepsilon; \tag{4.7}$$

if $-T_1 \leq s \leq 0$, then $t + s \geq T_1$ and therefore

$$\frac{|x(t+s)|}{g(s)} \leq |x(t+s)| \leq u + \varepsilon. \tag{4.8}$$

From (4.7) and (4.8), we obtain $\|x_t\|_g \leq u + \varepsilon$ for $t \geq T$. This proves Claim 1.

Next, since $a_i(x) > 0$ for all $x \in \mathbb{R}$ and $x_i(t)$ is bounded, we observe that there is $m > 0$ such that $a_i(x_i(t)) \geq m$ for $t \geq 0$ and $i = 1, \ldots, n$. If $u > 0$, then $\eta > 0$, where

$$\eta := m(\beta_i - l_i)u/2.$$

Claim 2. If $u > 0$ and $s_j \to \infty, x_i(s_j) \to u$ (respectively $x_i(s_j) \to -u$) as $j \to \infty$ for some $i = 1, \ldots, n$, with $s_j \neq t_k$ for all $j, k \in \mathbb{N}$, then there exists $j_1 \in \mathbb{N}$ such that $\dot{x}_i(s_j) \leq -\eta$ (respectively $\dot{x}_i(s_j) \geq \eta$) for all $j \geq j_1$.

Suppose that $u > 0$ and that $x_i(s_j) \to u$ for some sequence (s_j) with $s_j \to \infty$. Using (A1), (A2) and (A3) and for $\varepsilon \in \left(0, \frac{(\beta_i - l_i)u}{2(\beta_i + l_i)}\right)$, from Claim 1 we have

$$\begin{aligned} b_i(x_i(s_j)) + f_i(s_j, x_{s_j}) &= (b_i(x_i(s_j)) - b_i(0)) + (f_i(s_j, x_{s_j}) - f_i(s_j, 0)) \\ &\geq \beta_i x_i(s_j) - l_i \|x_{s_j}\|_g \geq \beta_i(u - \varepsilon) - l_i(u + \varepsilon) \\ &= (\beta_i - l_i)u - (\beta_i + l_i)\varepsilon \geq (\beta_i - l_i)u/2 > 0 \end{aligned}$$

for j sufficiently large. Thus,

$$\dot{x}_i(s_j) \leq -a_i(x_i(s_j))(\beta_i - l_i)u/2 \leq -m(\beta_i - l_i)u/2 = -\eta \tag{4.9}$$

for j sufficiently large. If $x(s_j) \to -u$, in a similar way we conclude that $\dot{x}_i(s_j) \geq \eta$, and claim 2 is proven.

Now fix $i \in \{1, ..., n\}$ such that $u_i = u$. Then, there is a sequence $(s_j)_j$ such that $s_j \nearrow \infty$ and

$$\max\{x_i(s_j), x_i(s_j^+)\} \to u \quad \text{as} \quad j \to \infty.$$

Since $x_i(t)$ is continuous on each interval $(t_{k-1}, t_k]$, we may assume that $x_i(s_j) \to u$ as $j \to \infty$ with

$$s_j \in (t_{k(j)-1}, t_{k(j)}). \tag{4.10}$$

for some subsequence $(t_{k(j)})_j$ of $(t_k)_k$.

Claim 3. If $u > 0$, then $\limsup_{j\to\infty} \max\{x_i(t_{k(j)}), |x_i(t_{k(j)-1}|)\} = u$.

We first prove that $\limsup_{j\to\infty} \max\{x_i(t_{k(j)}), x_i(t^+_{k(j)-1})\} = u$. Otherwise, consider a subsequence of s_j, still denoted by s_j, such that $\max\Big(\{x_i(t) : t \in (t_{k(j)-1}, t_{k(j)}]\} \cup \{x(t^+_{k(j)-1})\}\Big) = x_i(\bar{s}_j)$, for some $\bar{s}_j \in (t_{k(j)-1}, t_{k(j)})$. So we have $x_i(\bar{s}_j) \geq x_i(s_j) \to u$ as

$j \to \infty$ and $\dot{x}_i(\bar{s}_j) = 0$. Clearly, we can replace s_j with $\bar{s}_j$ in (4.9), which leads to a contradiction.

Thus $\limsup_{j\to\infty} \max\{x_i(t_{k(j)}), x_i(t^+_{k(j)-1})\} = u$. On the other hand, from (A5) it is known that $\log(\max\{1, \widehat{\gamma}_k\}) \to 0$ as $k \to \infty$, hence $\limsup \widehat{\gamma}_k \le 1$. For any $\varepsilon > 0$ small, we get

$$|x_i(t^+_{k(j)-1})| \le \widehat{\gamma}_{k(j)-1}|x_i(t_{k(j)-1})| \le u + \varepsilon$$

for j large, we conclude that $\limsup_{j\to\infty} \max\{x_i(t_{k(j)}), |x_i(t_{k(j)-1}|)\} = u$.

We finally prove:

Claim 4. $u = 0$.

By way of contradiction, suppose again that $u > 0$.

Claim 2 and the above arguments also show that, if a subsequence $x_i(t_{k(j)})$ of $x_i(t_k)$ converges to u (respectively $-u$), then, for j large, there are no local maximum (respectively minimum) points of $x_i(t)$ in the intervals $\big(t_{k(j)-1}, t_{k(j)}\big)$.

For some subsequence of $t_{k(j)}$, still designated by $t_{k(j)}$, from Claim 3, we obtain one of the following cases: (i) $x_i(t_{k(j)}) \to u$; (ii) $x_i(t_{k(j)-1}) \to u$; or (iii) $x_i(t_{k(j)-1}) \to -u$. We consider separately these three situations.

In Case (i), $x_i(t)$ is strictly decreasing on $(t_{k(j)-1}, t_{k(j)}]$. Now, define the functions

$$\varphi_j : [0, \sigma] \to \mathbb{R},\ \varphi_j(s) = x_i(t_{k(j)} - \sigma + s),$$

where by (A6) we choose σ such that $0 < \sigma < \inf_{k\ge1}(t_{k+1} - t_k)$. Thus, $\varphi_j \in C^1([0, \sigma], \mathbb{R})$, $\varphi_j(\sigma) \to u$, and $\varphi_j(s)$ decreasing in $[0, \sigma]$, for each j. Thus $\varphi_j(s) \to u$ for each $s \in [0, \sigma]$. As in the proof of Claim 2, for j sufficiently large and $t \in (t_{k(j)-1}, t_{k(j)})$ we get

$$b_i(x_i(t)) + f_i(t, x_t) \ge \beta_i x_i(t) - l_i\|x_t\|_g \ge \beta_i x_i(t_{k(j)}) - l_i\|x_t\|_g \ge (\beta_i - l_i)u/2,$$

hence $\dot{\varphi}_j(s) \le -\eta < 0$ for all $s \in [0, \sigma)$. Clearly, this is not possible, since it leads to $\varphi_j(0) \ge \varphi_j(\sigma) + \eta\sigma \to u + \eta\sigma > u$, a contradiction.

Case (ii) is treated as Case (i), by replacing the intervals $(t_{k(j)-1}, t_{k(j)}]$ with $(t_{k(j)-2}, t_{k(j)-1}]$.

In Case (iii), we derive that $x_i(t)$ is strictly increasing on $(t_{k(j)-2}, t_{k(j)-1}]$. We proceed as in Case (i), with $(t_{k(j)-2}, t_{k(j)-1}]$ instead of $(t_{k(j)-1}, t_{k(j)}]$, and where now we obtain $\varphi_j(s) \to -u$, $\dot{\varphi}_j(s) \ge \eta > 0$ for all $s \in [0, \sigma)$, yielding a contradiction in a similar way. □

Remark 4.3. In Claim 2, in fact one can consider $s_j = t_k$ for some $k, j \in \mathbb{N}$, in which case $\dot{x}_i(s_j)$ means the left derivative of $x_i(t)$ at $t = t_k$. For f regular enough, note that a solution $x(t)$ of (4.1) necessarily has left and right derivatives at the points $t = t_k$, since the function $y(t)$ with components $y_i(t) = x_i(t) - \sum_{k:0<t_k<t} I_{ik}(x_i(t_k))$ is absolutely continuous on each interval $[t_{k-1}, t_k]$.

Remark 4.4. If the functions $I_{ik} : \mathbb{R} \to \mathbb{R}$ are Lispchitz continuous,

(A4') $|I_{ik}(u) - I_{ik}(v)| \le \gamma_{ik}|u - v|$ for $u, v \in \mathbb{R}$, $i = 1, ..., n$, $k \in \mathbb{N}$,

then (A4) holds with $\widehat{\gamma}_{ik} \le (1 + \gamma_{ik})$. If in addition

(A5') $\sum_k \gamma_k < \infty$ for $\gamma_k := \max_{1\le i\le n} \gamma_{ik}$,

then the series $\sum_k \log(1 + \gamma_k)$ is also convergent, and consequently (A5) is satisfied. Therefore, Theorem 4.1 is still valid if we replace assumptions (A4), (A5) by (A4'), (A5').

We shall now study the global exponential stability of an equilibrium point of the impulsive system (4.1). For this purpose, in the remaining part of this section we shall consider the phase space PC_g, with

$$g(s) = e^{-\alpha s}, \quad s \in (-\infty, 0], \tag{4.11}$$

for some $\alpha > 0$, and denote PC_g by PC_α and $\|\cdot\|_g$ by $\|\cdot\|_\alpha$

Lemma 4.4. *Assume (A1), (A2), (A3), and*

(A7) $a_i(u) \geq \underline{a_i} > 0$, *for all* $u \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Suppose also that x^* *is the unique equilibrium point of* (4.2). *Consider a solution* $x : (-\infty, b] \to \mathbb{R}^n$ *of the non-impulsive system* (4.2) *on* $[a, b]$ *with* $x_a \in PC_\alpha$ $(-\infty < a < b \leq \infty)$. *If there exist constants* $q > 0$ *and* $\varepsilon \in (0, \alpha]$ *with* $\varepsilon < \min_i\{\underline{a_i}(\beta_i - l_i)\}$, *such that*

$$|x(t) - x^*| \leq q e^{-\varepsilon(t-a)} \quad \text{for} \quad t \in (-\infty, a]$$

then

$$|x(t) - x^*| \leq q e^{-\varepsilon(t-a)} \quad \text{for} \quad t \in (-\infty, b]. \tag{4.12}$$

Proof. After the change $y(t) = x(t) - x^*$, we may assume that the equilibrium point is zero, i.e., we consider (4.2) subject to the constraints $b_i(0) + f_i(t, 0) = 0,\ t \in \mathbb{R}, 1 \leq i \leq n$.

By contradiction, suppose that (4.12) does not hold. Consequently there are $\delta > 0$, $m \in \{1, \ldots, n\}$, and $t^* \in (a, b]$ such that

$$|y_m(t^*)| = (q + \delta)e^{-\varepsilon(t^*-a)} \text{ and } |y_i(t)| < (q + \delta)e^{-\varepsilon(t-a)}, \quad t < t^*, i = 1, \ldots, n.$$

Assume that $y_m(t^*) > 0$ (the situation $y_m(t^*) < 0$ is analogous). Denoting $z(t) := (q + \delta)e^{-\varepsilon(t-a)}$, $t \in [a, b]$, we have

$$y'_m(t^*) \geq z'(t^*).$$

On the other hand, using the hypotheses (A1)–(A3) and (A7), we have

$$\begin{aligned}
y'_m(t^*) &= -a_m(y_m(t^*))[b_m(y_m(t^*)) + f_m(t^*, y_{t^*})] \\
&= -a_m(y_m(t^*))[b_m(y_m(t^*)) - b_m(0) + f_m(t^*, y_{t^*}) - f_m(t^*, 0)] \\
&\leq -\underline{a_m}[\beta_m y_m(t^*) - l_m \|y_{t^*}\|_\alpha] \\
&\leq -\underline{a_m}[\beta_m z(t^*) - l_m \sup_{s \leq 0} \frac{(q + \delta)e^{-\varepsilon(t^*+s-a)}}{e^{-\alpha s}}] \\
&\leq -\underline{a_m} \left[\beta_m z(t^*) - l_m (q + \delta)e^{-\varepsilon(t^*-a)}\right] \\
&= -\underline{a_m}(\beta_m - l_m) z(t^*) < -\varepsilon z(t^*) = z'(t^*),
\end{aligned}$$

which is a contradiction. □

Theorem 4.2. *Assume that there is an equilibrium* x^* *of* (4.1). *Assume also (A1)–(A4), (A7) and*

(A8) *for* $\widehat{\gamma}_k := \max_{1 \leq i \leq n} \widehat{\gamma}_{ik}$ *and* $\widehat{\gamma}_{ik}$ *as in (A4) and some* $k_0 \in \mathbb{N}$,

$$\eta := \sup_{k \geq k_0} \left(\frac{\log(\max\{1, \widehat{\gamma}_k\})}{t_k - t_{k-1}} \right) < \min_i\{\underline{a_i}(\beta_i - l_i)\}, \tag{4.13}$$

where the space PC_g *in (A2) is* $PC_g = PC_\varepsilon$ *for some* $\varepsilon > \eta$. *Then the equilibrium* x^* *of (4.1) is globally exponentially stable.*

Proof. For η as in (4.13), choose $\alpha \in (0, \varepsilon]$ such that

$$\eta < \alpha < \min_i \left\{ \underline{a_i}(\beta_i - l_i) \right\}. \tag{4.14}$$

Clearly, (A2) still holds with PC_ε replaced by PC_α, with the same Lipschitz constants l_i. Write $\eta_k = \max\{1, \widehat{\gamma}_k\},\ k = 1, 2, \ldots$. As above, we may assume that $x^* = 0$ is the unique equilibrium point of (4.1). Let $y(t)$ be a solution of (4.1) defined on $\mathbb{R}$, with $y_0 \in PC_\alpha$. Then,

$$|y(t)|_\infty \leq \|y_0\|_\alpha \, e^{-\alpha t}, \quad \text{for} \quad t \in (-\infty, 0].$$

From Lemma 4.4, we conclude that

$$|y(t)|_\infty \leq \|y_0\|_\alpha \, e^{-\alpha t} \quad \text{for} \quad t \in (-\infty, t_1].$$

Next, we observe that $|y(t_1^+)|_\infty = |y_i(t_1^+)|$, for some $i = 1, \ldots, n$, and

$$\begin{aligned} |y(t_1^+)|_\infty &= |y_i(t_1^+)| = |I_{i1}(y_i(t_1)) + y_i(t_1)| = |\widehat{I}_{i1}(y_i(t_1))| \\ &\leq \widehat{\gamma}_{i1}|y_i(t_1)| \leq \widehat{\gamma}_{i1}\|y_0\|_\alpha \, e^{-\alpha t_1} \leq \eta_1 \|y_0\|_\alpha \, e^{-\alpha t_1}. \end{aligned}$$

Consequently,

$$|y(t)|_\infty \leq \eta_1 \|y_0\|_\alpha \, e^{-\alpha t_1} e^{-\alpha(t - t_1)}, \quad t \in (-\infty, t_1^+]$$

and, again from Lemma 4.4, we conclude that

$$|y(t)|_\infty \leq \eta_1 \|y_0\|_\alpha \, e^{-\alpha t} \quad \text{for} \quad t \in (-\infty, t_2].$$

Noting that $|y(t_2^+)|_\infty = |y_i(t_2^+)|$, for some $i = 1, \ldots, n$, then

$$|y(t_2^+)|_\infty = |\widehat{I}_{i2}(y_i(t_2))| \leq \widehat{\gamma}_{i2}|y_i(t_2)| \leq \eta_2 \eta_1 \|y_0\|_\alpha \, e^{-\alpha t_2}.$$

Iterating the above procedure, we obtain

$$|y(t)|_\infty \leq \eta_1 \eta_2 \cdots \eta_{k-1} \|y_0\|_\alpha \, e^{-\alpha t},\ t \in (t_{k-1}, t_k], k = 1, 2, \ldots,$$

where $t_0 = 0$ and $\eta_0 = 1$. The definition of η in (4.13) yields $\log \eta_k \leq \eta(t_k - t_{k-1})$, hence

$$\eta_k \leq e^{\eta(t_k - t_{k-1})}, \quad k \geq k_0.$$

Thus, for $t \in (t_{k-1}, t_k]$ and $k > k_0$,

$$|y(t)|_\infty \leq \eta_1 \cdots \eta_{k_0 - 1} \|y_0\|_\alpha \, e^{\eta t_{k-1}} e^{-\alpha t} \leq \eta_1 \cdots \eta_{k_0 - 1} \|y_0\|_\alpha \, e^{-(\alpha - \eta)t},$$

and, consequently

$$|y(t)|_\infty \leq \eta_1 \cdots \eta_{k_0 - 1} \|y_0\|_\alpha \, e^{-(\alpha - \eta)t}, \quad t \geq 0.$$

□

Corollary 4.1. *In the space PC_ε for some $\varepsilon > 0$, assume that there is an equilibrium point x^* of (4.1) and that (A1)–(A7) hold. Then x^* is globally exponentially stable.*

Proof. Observe that (A5) and (A6) imply (A8). □

Remark 4.5. Consider system (4.1) in some space PC_ε with $\varepsilon > 0$, and suppose that the functions $a_i(x)$ are bounded from below by some positive constant (i.e., (A7) holds). Then, the above corollary shows that the sufficient conditions in Theorem 4.1 for the global *asymptotic* stability of an equlibrium x^* are stronger then the requirements in Theorem 4.2 for its global *exponential* stability, which seems to indicate that Theorem 4.1 does not provide a good criterion. However, in the applications given in the next section it will be apparent that it is much more restrictive to set (4.1) in the framework of a space $PC_\varepsilon = PC_g$ with $g(s) = e^{-\varepsilon s}$ ($\varepsilon > 0$), then in a generic space PC_g with g satisfying (g1)-(g3).

5 Applications

Here, we apply the previous results to the following impulsive generalized Cohen-Grossberg neural network model with both time-varing delays and distributed delays:

$$\dot{x}_i(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(h_{ij}^{(p)}(x_j(t-\tau_{ij}^{(p)}(t)))+ \\ + f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s))\, d\eta_{ij}^{(p)}(s)\right)\Bigg)\Bigg],\ 0 \le t \ne t_k, \qquad (5.1)$$

$$\Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)),\ i = 1, \dots n,\ k \in \mathbb{N}, \qquad (5.2)$$

where $t_k \nearrow \infty$ as $k \to \infty$, $a_i : \mathbb{R} \to (0, \infty)$, $b_i, h_{ij}^{(p)}, f_{ij}^{(p)}, g_{ij}^{(p)}, I_{ik} : \mathbb{R} \to \mathbb{R}$, $\tau_{ij}^{(p)} : [0, \infty) \to [0, \infty)$ are continuous functions with $\tau_{ij}^{(p)}(t) \le \tau_{ij}^{(p)} \le \tau$ for some $\tau > 0$, and $\eta_{ij}^{(p)} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded functions, normalized so that $\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1$, for all $i, j \in \{1, \dots, n\}$, $p \in \{1, \dots, P\}$. For (5.1), in the sequel we further assume that the functions b_i satisfy (A1), I_{ik} satisfy (A4), and that $h_{ij}^{(p)}, f_{ij}^{(p)}, g_{ij}^{(p)}$ are Lipschitzian with Lipschitz constants $\zeta_{ij}^{(p)}, \mu_{ij}^{(p)}, \sigma_{ij}^{(p)}$, respectively.

Model (5.1) was introduced in [28] and is particulary relevant in terms of applications, since it includes different types of neural network models as subclasses, as we shall illustrate with several examples.

We define the square real matrices,

$$B = diag(\beta_1, \dots, \beta_n),\ \ L = [l_{ij}] \ \text{ and } \ N = B - L, \qquad (5.3)$$

where $\beta_1, \dots, \beta_n$ are as in (A1) and $l_{ij} = \sum_{p=1}^{P} \zeta_{ij}^{(p)} + \mu_{ij}^{(p)}\sigma_{ij}^{(p)}$.

Proposition 5.1. *Consider (5.1) under the hypotheses assumed above. If N is a non-singular M-matrix, then the model (5.1) has a unique equilibrium point $x^* = (x_1^*, \dots, x_n^*)$.*

Proof. See [28]. □

In addition, in what follows we always assume that if x^* is an equilibrium point of (5.1), then it satisfies

$$I_{ik}(x_i^*) = 0, \quad i = 1, \dots, n, k \in \mathbb{N}. \qquad (5.4)$$

This means that x^* is also an equilibrium point of the impulsive model (5.1)-(5.2).

Theorem 5.1. *Consider (5.1)-(5.2) under the hypotheses assumed above. In addition, assume the hypotheses (A5), (A6), and that the matrix N, defined in (5.3), is a non-singular M-matrix.*

Then there is a unique equilibrium point $x^* \in \mathbb{R}^n$ of (5.1)-(5.2) which is globally asymptotically stable.

Proof. The existence and uniqueness of the equilibrium point of (5.1)-(5.2) comes from Proposition 5.1 and (5.4).

As N is a non-singular M-matrix, then (see [7]) there is $d = (d_1, \ldots, d_n) > 0$ such that $Nd > 0$, i.e.,

$$\beta_i d_i > \sum_{j=1}^{n} l_{ij} d_j, \quad i = 1, \ldots, n,$$

hence there is $\delta > 0$ such that

$$\beta_i d_i > \sum_{j=1}^{n} d_j \sum_{p=1}^{P} \zeta_{ij}^{(p)} + \mu_{ij}^{(p)} \sigma_{ij}^{(p)} (1+\delta), \quad i = 1, \ldots, n. \tag{5.5}$$

Since $\int_{-\infty}^{0} d\eta_{ij}^{(p)}(s) < 1 + \delta$ for $i, j = 1, \ldots, n$, $p = 1, \ldots, P$, from Lemma 4.1 in [6] we conclude that there is $g : (-\infty, 0] \to [1, \infty)$ satisfying (g1)-(g3) such that

$$\int_{-\infty}^{0} g(s)\, d\eta_{ij}^{(p)}(s) < 1 + \delta \quad \text{and} \quad g(-\tau) = 1.$$

The change $y_i(t) = d_i^{-1} x_i(t)$ transforms (5.1)-(5.2) into the system

$$\dot{y}_i(t) = -a_i(d_i y_i(t)) d_i^{-1} \Bigg[b_i(d_i y_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} \Bigg(h_{ij}^{(p)}(d_j y_j(t - \tau_{ij}^{(p)}(t))) + \\ + f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(d_j y_j(t+s))\, d\eta_{ij}^{(p)}(s) \right) \Bigg) \Bigg], \; 0 \le t \ne t_k, \tag{5.6}$$

$$\Delta(y_i(t_k)) = d_i^{-1} I_{ik}(d_i y_i(t_k^-)), \quad i = 1, \ldots n,\, k \in \mathbb{N}, \tag{5.7}$$

for which we consider PC_g as the phase space. For each $i \in \{1, \ldots, n\}$, define

$$\bar{f}_i(t, \phi) = d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \left(h_{ij}^{(p)} \left(d_j \phi_j(-\tau_{ij}^{(p)}(t)) \right) + f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(d_j \phi_j(s))\, d\eta_{ij}^{(p)}(s) \right) \right),$$

$$\bar{a}_i(u) = a_i(d_i u), \quad \bar{b}_i(u) = d_i^{-1} b_i(d_i u), \quad \bar{I}_{ik}(u) = d_i^{-1} I_{ik}(d_i u),$$

for $\phi \in PC_g$, $t \ge 0$, and $u \in \mathbb{R}$. System (5.6)-(5.7) is written as

$$\begin{aligned} &\dot{y}_i(t) = -\bar{a}_i(y_i(t))[\bar{b}_i(y_i(t)) + \bar{f}_i(t, y_t)], \quad 0 \le t \ne t_k, \\ &\Delta(y_i(t_k)) = \bar{I}_{ik}(y_i(t_k^-)), \quad i = 1, \ldots n,\, k \in \mathbb{N}, \end{aligned} \tag{5.8}$$

which has the form (4.1). Clearly b_i and I_{ik} satisfy (A1), (A4), and (A5) if and only if $\bar{b}_i$ and $\bar{I}_{ik}$ satisfy (A1), (A4), and (A5) with the same constants β_i and $\widehat{\gamma}_{ik}$.

For $\varphi, \phi \in PC_g$ and $t \ge 0$, since $h_{ij}^{(p)}$, $f_{ij}^{(p)}$, $g_{ij}^{(p)}$ are Lipschitz functions and $\eta_{ij}^{(p)}$ are non-decreasing, we have

$$|\bar{f}_i(t,\phi)-\bar{f}_i(t,\varphi)| \le d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(\zeta_{ij}^{(p)}d_j\left|\phi_j(-\tau_{ij}^{(p)}(t))-\varphi_j(-\tau_{ij}^{(p)}(t))\right|$$

$$+\mu_{ij}^{(p)}\int_{-\infty}^{0}\left|g_{ij}^{(p)}(d_j\phi_j(s))-g_{ij}^{(p)}(d_j\varphi_j(s))\right|d\eta_{ij}^{(p)}(s)\Bigg)$$

$$\le d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(\zeta_{ij}^{(p)}d_j\frac{|(\phi_j-\varphi_j)(-\tau_{ij}^{(p)}(t))|}{g\left(-\tau_{ij}^{(p)}(t)\right)}g(-\tau)$$

$$+d_j\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^{0}g(s)\frac{|(\phi_j-\varphi_j)(s)|}{g(s)}\,d\eta_{ij}^{(p)}(s)\Bigg)$$

$$\le\left[d_i^{-1}\sum_{j=1}^{n}d_j\sum_{p=1}^{P}\left(\zeta_{ij}^{(p)}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^{0}g(s)\,d\eta_{ij}^{(p)}(s)\right)\right]\|\phi-\varphi\|_g.$$

This means that

$$|\bar{f}_i(t,\phi)-\bar{f}_i(t,\varphi)|\le l_i\|\phi-\varphi\|_g,\quad i=1,\dots,n,$$

with $l_i := d_i^{-1}\sum_{j=1}^{n}d_j\sum_{p=1}^{P}\left(\zeta_{ij}^{(p)}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}(1+\delta)\right)$, then each $\bar{f}_i$ satisfies (A2). On the other hand, (5.5) implies that (A3) holds. The conclusion follows now from Theorem 4.1. □

Theorem 5.2. *Consider (5.1)-(5.2) under the hypotheses above. In addition, suppose that:*
(i) the functions a_i satisfy (A7);
(ii) there is $k^*\in\mathbb{N}$ such that the conditions

$$\int_{-\infty}^{0}e^{-\gamma s}\,d\eta_{ij}^{(p)}(s)<\infty,\quad i,j=1,\dots,n,\ p=1,\dots,P,$$

hold for some $\gamma>\eta:=\sup_{k\ge k^*}\left(\frac{\log\left(\max\{1,\widehat{\gamma}_k\}\right)}{t_k-t_{k-1}}\right)$, where $\widehat{\gamma}_k:=\max_{1\le i\le n}\widehat{\gamma}_{ik}$;
(iii) the matrix

$$M=diag\left(\beta_1-\frac{\eta}{\underline{a_1}},\dots,\beta_n-\frac{\eta}{\underline{a_n}}\right)-[n_{ij}],$$

where $n_{ij}=\sum_{p=1}^{P}\left(\zeta_{ij}^{(p)}e^{\eta\tau_{ij}^{(p)}}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^{0}e^{-\eta s}\,d\eta_{ij}^{(p)}(s)\right)$, is a non-singular M-matrix.

Then, there is a unique equilibrium point x^* of (5.1)-(5.2) which is globally exponentially stable.

Proof. Let N be defined as in (5.3). As $N\ge M$ and M is a non-singular M-matrix, then N is also a non-singular M-matrix. From Proposition 5.1 and assuming (5.4), we conclude that the impulsive system (5.1)-(5.2) has a unique equilibrium point $x^*\in\mathbb{R}^n$.

Since M is a non-singular M-matrix, there is $d=(d_1,\dots,d_n)>0$ such that (see [7])

$$\left(\beta_i-\frac{\eta}{\underline{a_i}}\right)d_i-\sum_{j=1}^n d_j\sum_{p=1}^P\left(\zeta_{ij}^{(p)}e^{\eta\tau_{ij}^{(p)}}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^0 e^{-\eta s}\,d\eta_{ij}^{(p)}(s)\right)>0,$$

for all $i=1,\dots,n$, which is equivalent to

$$C_i(\eta):=\eta-\underline{a_i}\left(\beta_i-d_i^{-1}\sum_{j=1}^n d_j\sum_{p=1}^P\left(\zeta_{ij}^{(p)}e^{\eta\tau_{ij}^{(p)}}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^0 e^{-\eta s}\,d\eta_{ij}^{(p)}(s)\right)\right)<0.$$

As the function $F_{ij}^{(p)}(t):=\int_{-\infty}^0 e^{-ts}\,d\eta_{ij}^{(p)}(s)$ is continuous (see [6], proof of Theorem 4.3) and non-decreasing on $[\eta,\gamma]$, we conclude that each C_i is also continuous and non-decreasing on $[\eta,\gamma]$ and therefore there is $\varepsilon>\eta$ such that

$$C_i(\varepsilon)<0,\quad i=1,\dots,n. \tag{5.9}$$

As in the proof above, the change $y_i(t)=d_i^{-1}x_i(t)$ transforms (5.1)-(5.2) into the system (5.6)-(5.7), now considering PC_ε as the phase space, which has the form of (5.8) with the same functions $\bar{a}_i$, $\bar{\beta}_i$, $\bar{f}_i$, and $\bar{I}_{ik}$. Similar computations lead to

$$|\bar{f}_i(t,\phi)-\bar{f}_i(t,\varphi)|\le l_i\|\phi-\varphi\|_\varepsilon,\quad i=1,\dots,n,$$

for all $\phi,\varphi\in PC_\varepsilon$ and $t\ge 0$, with $l_i:=d_i^{-1}\sum_{j=1}^n d_j\sum_{p=1}^P\left(\zeta_{ij}^{(p)}e^{\varepsilon\tau_{ij}^{(p)}}+\mu_{ij}^{(p)}\sigma_{ij}^{(p)}\int_{-\infty}^0 e^{-\varepsilon s}d\eta_{ij}^{(p)}(s)\right)$. This means that (A2) holds and from (5.9) the hypothesis (A8) also holds. Now the conclusion follows from Theorem 4.2. □

Remark 5.1. In the previous two theorems, it is specially relevant the situation where $\widehat{\gamma}_k>1$ for large k (that is, in Theorem 5.2 the constant η is strictly positive), since this allows the solution to be further away from the equilibrium point after the impulsive moments, than before their occurence. Although this situation happens quite often in the real world, it does not fit into the setting of most of the literature. In fact, in many papers [2] ,[19], [24], [25], [31], [33], the impulsive operators I_{ik} are defined as

$$I_{ik}(u)=-\alpha_{ik}(u-x_i^*),\quad u\in\mathbb{R},\ \text{with } 0<\alpha_{ik}<2, \tag{5.10}$$

where $x^*=(x_1^*,\dots,x_n^*)\in\mathbb{R}^n$ is the equilibrium point of the neural network model. From (5.10), we conclude that

$$|x_i(t_k^+)-x_i^*|=|1-\alpha_{ik}|\,|x_i(t_k)-x_i^*|<|x_i(t_k)-x_i^*|,$$

for each $k\in\mathbb{N}$ and $i=1,\dots,n$, which imposes what seems to be (in spite of its applications to some control problems) a very severe and unrealistic constraint: the solutions of the impulsive model must be nearer of the equilibrium point than the solutions of the model without impulses.

On the other hand, this gives rise to an interesting question. First, we note that, in the absence of impulses, the criteria for stability provided by Theorems 4.1 and 4.2 coincide with the ones established in [5], [6], [28]. In many real world situation, one however expects that impulses satisfying (5.10) are introduced to control the behaviour of solutions and force them to converge to the equilibrium. This suggests that, if the conditions (5.10) are satisfied, maybe a criterion for the attractivity of the equilibrium point will be achieved under weaker conditions on f than the ones above.

Example 5.1. Consider the following Cohen-Grossberg neural networks model with impulses:

$$\dot{x}_i(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t)))$$
$$-\sum_{j=1}^{n} c_{ij} \quad h_j\left(\int_{-\infty}^{0} K_{ij}(-s)x_j(t+s)\,ds\right) + J_i\Bigg],\ 0 \le t \ne t_k, \qquad (5.11)$$

$$\Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)),\ i = 1, \dots n,\ k \in \mathbb{N}, \qquad (5.12)$$

where $a_{ij}, b_{ij}, c_{ij} \in \mathbb{R}$, $a_i : \mathbb{R} \to (0,\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$, and $\tau_{ij} : [0,\infty) \to [0,\infty)$ are continuous functions, with $\tau_{ij}(t) \le \tau_{ij} \le \tau$, f_j, g_j, and h_j are Lipschitz functions with Lipschiz constants F_j, G_j, and H_j respectively, and K_{ij} are nonnegative continuous functions such that

$$\int_0^\infty K_{ij}(t)\,dt = 1, \quad i,j = 1, \dots, n.$$

Since the system (5.11)-(5.12) is a particular case of (5.1)-(5.2), Theorem 5.1 applied to (5.11)-(5.12) gives the following result:

Corollary 5.1. *Consider (5.11)-(5.12) under the hypotheses considered above, and, for $i,j = 1, \dots, n$, assume that:*
(i) b_i satisfy (A1);
(ii) I_{ik} satisfy (A4) and (A5);
(iii) (A6) holds;
(iv) the matrix $N = B - [l_{ij}]$, where $B = diag(\beta_1, \dots, \beta_n)$ for β_i as in (A1) and $l_{ij} = |a_{ij}|F_j + |b_{ij}|G_j + |c_{ij}|H_j$, is a non-singular M-matrix.
Then there is a unique equilibrium point x^* of (5.11)-(5.12), which is globally asymptotically stable.

As a consequence of Theorem 5.2, we have the following result:

Corollary 5.2. *Consider (5.11)-(5.12) under the hypotheses considered above, and, for $i,j = 1, \dots, n$, assume that:*
(i) b_i satisfy (A1);
(ii) I_{ik} satisfy (A4);
(iii) a_i satisfy (A7);
(iv) there is $k^* \in \mathbb{N}$ such that the conditions

$$\int_0^\infty K_{ij}(t)e^{\gamma t}\,dt < \infty, \quad i,j = 1, \dots, n,$$

hold for some $\gamma > \eta := \sup_{k \ge k^*} \left(\frac{\log(\max\{1, \widehat{\gamma}_k\})}{t_k - t_{k-1}}\right)$, *where* $\widehat{\gamma}_k := \max_{1 \le i \le n} \widehat{\gamma}_{ik}$.
If the matrix

$$M = D - [n_{ij}],$$

where $D = diag\left(\beta_1 - \frac{\eta}{\underline{a_1}}, \dots, \beta_n - \frac{\eta}{\underline{a_n}}\right)$ *and* $n_{ij} = |a_{ij}|F_j + |b_{ij}|e^{\eta\tau_{ij}}G_j + |c_{ij}|H_j \int_{-\infty}^{0} K_{ij}(-s)e^{-\eta s}ds$, *is a non-singular M-matrix, then there is a unique equilibrium point* x^* *of (5.11)-(5.12), which is globally exponentially stable.*

Remark 5.2. System (5.11)-(5.12) was studied in [21]. We note that this system is only a particular case of (5.1)-(5.2), hence our Theorem 5.2 is more general than the main result in [21].

Example 5.2. The following impulsive Cohen-Grossberg-type BAM neural network model is also a particular case of the impulsive model (5.1)-(5.2):

$$\begin{cases} \dot{x}_i(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) + \sum_{j=1}^{m}\Big(c_{ij}g_j(y_j(t))) + \\ \qquad + d_{ij}\int_{-\infty}^{0} K_{ij}(-s)g_j(y_j(t+s))\,ds\Big) + J_i\Big],\ 0 \le t \ne t_k, \\ \Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)),\ i = 1, \dots n,\ k \in \mathbb{N}; \\ \dot{y}_j(t) = -\bar{a}_j(y_j(t))\Big[\bar{b}_j(y_j(t)) + \sum_{i=1}^{n}\Big(\bar{c}_{ji}f_i(x_i(t))) + \\ \qquad + \bar{d}_{ji}\int_{-\infty}^{0} \bar{K}_{ji}(-s)f_i(x_i(t+s))\,ds\Big) + \bar{J}_j\Big],\ 0 \le t \ne t_k, \\ \Delta(y_j(t_k)) = \bar{I}_{jk}(y_j(t_k^-)),\ j = 1, \dots m,\ k \in \mathbb{N} \end{cases} \tag{5.13}$$

where $t_k \nearrow \infty$ as $k \to \infty$, $c_{ij}, \bar{c}_{ji}, d_{ij}, \bar{d}_{ji} \in \mathbb{R}$, $a_i, \bar{a}_j : \mathbb{R} \to (0,\infty)$, $b_i, \bar{b}_j, I_{ik}, \bar{I}_{jk} : \mathbb{R} \to \mathbb{R}$, are continuous functions, $g_j, f_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschiz constants G_j, F_i, respectively, and $K_{ij}, \bar{K}_{ji} : [0,\infty) \to [0,\infty)$ are nonnegative continuous functions such that

$$\int_0^\infty K_{ij}(t)dt = \int_0^\infty \bar{K}_{ji}(t)dt = 1.$$

for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.

Corollary 5.3. *For the above system (5.13), for* $i = 1, \dots, n$, $j = 1, \dots, m$ *assume that:*
(i) $b_i, \bar{b}_j$ *satisfy (A1) with constants* $\beta_i, \bar{\beta}_j$, *respectively;*
(ii) $I_{ik}, \bar{I}_{jk}$ *satisfy (A4) with constants* $\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}$, *respectively;*
(iii) for $\widehat{\gamma}_k := \max_{1\le i\le n, 1\le j\le m}\{\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}\}$, $\prod_{k=1}^{\infty}\max\{1, \widehat{\gamma}_k\} < \infty$;
(iv) (A6) holds;
(v) the matrix

$$N = \begin{pmatrix} B & 0 \\ 0 & \bar{B} \end{pmatrix}_{(n+m)\times(n+m)} - \begin{pmatrix} 0 & (|C| + |D|)G \\ (|\bar{C}| + |\bar{D}|)F & 0 \end{pmatrix}_{(n+m)\times(n+m)}$$

where $B = diag(\beta_1, \dots, \beta_n)$, $\bar{B} = diag(\bar{\beta}_1, \dots, \bar{\beta}_m)$ *for* $\beta_i, \bar{\beta}_j$ *as in (A1) and* $|C| = [|c_{ij}|]_{n\times m}$, $|\bar{C}| = [|\bar{c}_{ji}|]_{m\times n}$, $|D| = [|d_{ij}|]_{n\times m}$, $|\bar{D}| = \left[|\bar{d}_{ji}|\right]_{m\times n}$, $F = diag(F_1, \dots, F_n)$, *and* $G = diag(G_1, \dots, G_m)$, *is a non-singular M-matrix.*

Then there is a unique equilibrium point $(x_1^*, \dots, x_n^*, y_1^*, \dots, y_m^*) \in \mathbb{R}^{n+m}$ *of (5.13), which is globally asymptotically stable.*

The exponential stability of the equilibrium point of (5.13) was recently studied in [22], with the p-norm in $\mathbb{R}^n$, $p \geq 1$. (The case $p = \infty$ was not treated in [22]). As a consequence of Theorem 5.2, we have the following stability criterion with the ∞-norm, which complements the result in [22].

Corollary 5.4. *For the above system (5.13), for $i = 1, \ldots, n$, $j = 1, \ldots, m$ assume that:*
(i) $b_i, \bar{b}_j$ satisfy (A1) with constants $\beta_i, \bar{\beta}_j$, respectively;
(ii) $I_{ik}, \bar{I}_{jk}$ satisfy (A4) with constants $\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}$, respectively;
(iii) $a_i, \bar{a}_j$ satisfy (A7) with constants $\underline{a_i}, \underline{\bar{a}_j}$, respectively;
(iv) there is $k^* \in \mathbb{N}$ such that

$$\int_0^\infty K_{ij}(t) e^{\gamma t}\, dt < \infty, \quad \int_0^\infty \bar{K}_{ji}(t) e^{\gamma t}\, dt < \infty$$

hold for some $\gamma > \eta := \sup_{k \geq k^*} \left(\dfrac{\log \left(\max\limits_{i,j} \{1, \widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}\} \right)}{t_k - t_{k-1}} \right).$

If the matrix

$$M = \begin{pmatrix} B - E & 0 \\ 0 & \bar{B} - \bar{E} \end{pmatrix}_{(n+m)\times(n+m)} - \begin{pmatrix} 0 & (|C| + |D_K|)G \\ (|\bar{C}| + |\bar{D}_K|)F & 0 \end{pmatrix}_{(n+m)\times(n+m)}$$

with B, $\bar{B}$, $|C|$, F, G defined as in the above corollary, $E = diag\left(\frac{\eta}{\underline{a_1}}, \ldots, \frac{\eta}{\underline{a_n}}\right)$, $\bar{E} = diag\left(\frac{\eta}{\underline{\bar{a}_1}}, \ldots, \frac{\eta}{\underline{\bar{a}_m}}\right)$, $|D_K| = \left[|d_{ij}| \int_{-\infty}^0 K_{ij}(-s) e^{-\eta s}\, ds\right]_{n\times m}$, $|\bar{D}_K| = \left[|\bar{d}_{ji}| \int_{-\infty}^0 \bar{K}_{ji}(-s) e^{-\eta s}\, ds\right]_{m\times n}$, is a non-singular M-matrix, then there is a unique equilibrium point $(x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*) \in \mathbb{R}^{n+m}$ of (5.13), which is globally exponentially stable.

Example 5.3. In [33], the following BAM neural network model with impulses was considered:

$$\begin{cases} \dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^m \Big(c_{ij} g_j(y_j(t)) + d_{ij} g_j(y_j(t - \tau_{ij})) \Big) + J_i, \ 0 \leq t \neq t_k, \\ \Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)), \ i = 1, \ldots n, \ k \in \mathbb{N}; \\ \dot{y}_j(t) = -\bar{b}_j y_j(t) + \sum_{i=1}^n \Big(\bar{c}_{ji} f_i(x_i(t)) + \bar{d}_{ji} f_i(x_i(t - \sigma_{ji})) \Big) + \bar{J}_j, \ 0 \leq t \neq t_k, \\ \Delta(y_j(t_k)) = \bar{I}_{jk}(y_j(t_k^-)), \ j = 1, \ldots m, \ k \in \mathbb{N} \end{cases}, \qquad (5.14)$$

where $b_i, \bar{b}_j \in (0, \infty)$, $c_{ij}, \bar{c}_{ji}, d_{ij}, \bar{d}_{ji}, J_i, \bar{J}_j \in \mathbb{R}$, $\tau_{ij}, \sigma_{ji} \in [0, \infty)$, $I_{ik}, \bar{I}_{jk} : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f_i, g_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_i, G_j, respectively, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Clearly, system (5.14) is still a particular case of (5.1)-(5.2), and from Theorems 5.1 and 5.2 we obtain the next two results.

Corollary 5.5. *Consider (5.14) under the hypotheses above, and, for $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $k \in \mathbb{N}$ assume that:*
(i) $I_{ik}, \bar{I}_{jk}$ satisfy (A4) with constants $\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}$, respectively;

(ii) for $\widehat{\gamma}_k := \max_{1\le i\le n, 1\le j\le m}\{\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}\}$*,* $\prod_{k=1}^{\infty}\max\{1,\widehat{\gamma}_k\} < \infty$*;*

(iii) (A6) holds;

(iv) the matrix

$$N = \begin{pmatrix} B & 0 \\ 0 & \bar{B} \end{pmatrix}_{(n+m)\times(n+m)} - \begin{pmatrix} 0 & (|C|+|D|)G \\ (|\bar{C}|+|\bar{D}|)F & 0 \end{pmatrix}_{(n+m)\times(n+m)}$$

where $B = diag(b_1,\dots,b_n)$*,* $\bar{B} = diag(\bar{b}_1,\dots,\bar{b}_m)$*,* $|C| = [|c_{ij}|]_{n\times m}$*,* $|\bar{C}| = [|\bar{c}_{ji}|]_{m\times n}$*,* $|D| = [|d_{ij}|]_{n\times m}$*,* $|\bar{D}| = [|\bar{d}_{ji}|]_{m\times n}$*,* $F = diag(F_1,\dots,F_n)$*, and* $G = diag(G_1,\dots,G_m)$*, is a non-singular M-matrix.*

Then there is a unique equilibrium point $(x_1^*,\dots,x_n^*,y_1^*,\dots,y_m^*) \in \mathbb{R}^{n+m}$ *of (5.14), which is globally asymptotically stable.*

Corollary 5.6. *Consider (5.14) under the hypotheses above, and, for* $i = 1,\dots,n$*,* $j = 1,\dots,m$*, and* $k \in \mathbb{N}$ *assume that:*

(i) $I_{ik}, \bar{I}_{jk}$ *satisfy (A4) with constants* $\widehat{\gamma}_{ik}, \widehat{\bar{\gamma}}_{jk}$*, respectively.*

(ii) For $\eta = \sup_{k\ge k^*}\left(\frac{\log\left(\max_{i,j}\{1,\gamma_{ik},\bar{\gamma}_{jk}\}\right)}{t_k - t_{k-1}}\right)$*, the matrix*

$$M = \begin{pmatrix} B & 0 \\ 0 & \bar{B} \end{pmatrix} - \begin{pmatrix} E & (|C|+|D_\eta|)G \\ (|\bar{C}|+|\bar{D}_\eta|)F & \bar{E} \end{pmatrix}$$

where B*,* $\bar{B}$*,* $|C|$*,* F*,* G *are as in the above result, and* $E = diag(\eta,\dots,\eta)_{n\times n}$*,* $\bar{E} = diag(\eta,\dots,\eta)_{m\times m}$*,* $|D_\eta| = [|d_{ij}|e^{\eta\tau_{ij}}]_{n\times m}$*,*$|\bar{D}_\eta| = [|\bar{d}_{ji}|e^{\eta\sigma_{ji}}]_{m\times n}$*, is a non-singular M-matrix.*

Then there is a unique equilibrium point $(x_1^*,\dots,x_n^*,y_1^*,\dots,y_m^*) \in \mathbb{R}^{n+m}$ *of (5.14), which is globally exponentially stable.*

Remark 5.3. In [33], the impulsive operators I_{ik}, $\bar{I}_{jk}$ are defined as

$$\begin{aligned} I_{ik}(u) &= -\alpha_{ik}(u - x_i^*), \quad \text{with } 0 < \alpha_{ik} < 2, \\ \bar{I}_{jk}(u) &= -\bar{\alpha}_{jk}(u - y_j^*), \quad \text{with } 0 < \bar{\alpha}_{jk} < 2, \end{aligned} \tag{5.15}$$

where $(x_1^*,\dots,x_n^*,y_1^*,\dots,y_m^*) \in \mathbb{R}^{n+m}$ is the equilibrium point of (5.14). Consequently, $\widehat{I}_{ik}(u) = (1-\alpha_{ik})u - \alpha_{ik}x_i^*$ and $\widehat{\bar{I}}_{jk}(u) = (1-\bar{\alpha}_{jk})u - \bar{\alpha}_{jk}y_j^*$ and therefore (A4) holds with $\gamma_{ik} = |1-\alpha_{ik}| < 1$ and $\bar{\gamma}_{jk} = |1-\bar{\alpha}_{jk}| < 1$ and $\eta = 0$. Then we conclude that Corollary 5.6 strongly improves the main result in [33].

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