The Moore-Penrose inverse of a companion matrix*  

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Abstract

Necessary and sufficient conditions are given for the Moore-Penrose inverse of a companion matrix over an arbitrary ring to exist.

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1 Introduction

Let $R$ be a ring with 1 and involution $(\bar{\cdot})$. That is, for all $a, b \in R$, the equalities $1 = 1$, $(a + b) = \bar{a} + \bar{b}$ and $(ab) = \bar{b}\bar{a}$ hold. The involution $(\bar{\cdot})$ in $R$ endows an involution $\bar{\cdot}$ in the set $\mathcal{M}(R)$ of (finite) matrices over $R$, defined as $[a_{ij}]^\ast = [\bar{a}_{ji}]$.

A matrix $A$ is said to be Moore-Penrose invertible with respect to $\ast$ provided there is $A^\dagger$ such that

$AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^\ast = AA^\dagger, (A^\dagger A)^\ast = A^\dagger A.$

If such a matrix $A^\dagger$ exists, then it is well known it is unique (see [1]).

We say $a \in R$ is regular if $a \in aRa$, or equivalently $axa = a$ is a ring consistent equation. A particular solution is denoted by $a^-$ and called a von Neumann inverse of $a$. A regular ring is a ring whose elements are regular. It is a standard fact that if $R$ is a regular ring then the ring of $m \times m$ matrices over $R$ is again regular (see, for instance, [2]).

We will use the following known fact:

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**Lemma 1.1.** Given $x, y \in R$, then $1 + xy$ is a unit if and only if $1 + yx$ is a unit, and in this case

$$(1 + xy)^{-1} = 1 - x(1 + yx)^{-1}y.$$ 

Lemma 1.1 has a useful extension for rectangular matrices which we will need later on. Given $n \times k$ matrices $B$ and $C$, then

$$I_n + BC^T$$

is invertible if and only if $I_k + C^TB$ is invertible. (1)

Versions of this relation for generalized inverses can be found in [3] and [4].

Using von Neumann inverses, it was shown in [6], [8], [9] how to characterize the existence of a Moore-Penrose inverse by means of units. The equivalence between the existence of $M^\dagger$, the invertibility of $U = MM^* + I - MM^*$, and the invertibility of $V = M^*M + I - M^*M$ will play an important role throughout this paper.

**Theorem 1.1.** Let $a \in R$ be a regular element, and $a^-$ a von Neumann inverse of $a$. The following conditions are equivalent:

(a) $a^\dagger$ exists;

(b) $s = a^-a^- + 1 - a^-a$ is a unit;

(c) $h = a^-a^-a + 1 - a^-a$ is a unit;

(d) $v = a^- + 1 - a^-a$ is a unit;

(e) $u = a^- + 1 - a^-a$ is a unit.

In this case,

$$a^\dagger = (s^{-1}a) = (ah^{-1}) = (u^{-1}a) = (av^{-1}).$$

**Proof.** The equivalences (a) ⇔ (b) ⇔ (c) follow from [8, Theorem 2], as well as the first two expressions for $a^\dagger$.

(b) ⇔ (d). Write $s = a^-a^- + 1 - a^- = 1 - a(-a^-a^- + a^-) = 1 - xy$ with $y = a$ and $x = -a^-a^- + a^-$. Then $v = 1 - xy = a^- + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $v^{-1} = 1 + xy^{-1} = 1 + (-a^-a^- + a^-)s^{-1}a = a^-s^{-1}a + 1 - a^-a^{-1}a$ and $s^{-1} = (1 - xy)^{-1} = 1 + y(1 - xy)^{-1}x = 1 + av^{-1}(-a^-a^- + a^-) = av^{-1}a^- + 1 - av^{-1}a^-a^-$. (c) ⇔ (e). Now, write $u = a^- + 1 - a^-a = 1 - a(-a^-a^- + a^-) = 1 - xy$ with $x = a$ and $y = -a^-a^- + a^-$. Then $h = 1 - xy = 1 - (-a^-a^- + a^-)a = a^-a^-a + 1 - a^-a$ and the equivalence follows using Lemma 1.1, with $u^{-1} = 1 + xh^{-1}y = 1 + ah^{-1}(-a^-a^- + a^-) = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - ah^{-1}a^-a^- = ah^{-1}a^- + 1 - a^-a^-a^-a^-.
We now derive the expressions for the $a^\dagger$. From [8, Theorem 2], $a^\dagger = (ah^{-1})$. Since
\[
ah^{-1} = a a^{-1} - a - a a^{-1} u
= a a^{-1} u - uu^{-1} a - a a^{-1} u
= (a a^{-1} - a a + u) u^{-1} a
= u^{-1} a
\]
then $a^\dagger = (u^{-1} a)$.

Finally, since $ua = a a = av$ then $u^{-1} a = av^{-1}$ and $a^\dagger = (av^{-1})$. 

Consider the $(n+1) \times (n+1)$ companion matrix
\[
M = \begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix},
\]
with $a \in R$ and $b \in R^n$. In this paper, we are interested on characterizing the existence of $M^\dagger$ by means of units in $R$. For the group inverse of $M$ the reader is referred to [5] and [10].

We will reduce the Moore-Penrose inverse of the companion matrix $M$ to the lower triangular case, by using the factorization $M = AP$ where
\[
A = \begin{bmatrix}
a & 0 \\
b & I_n
\end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix}
0 & 1 \\
I_n & 0
\end{bmatrix}.
\]
Since $M$ is unitarily equivalent to $A$, then $M$ has a Moore-Penrose inverse exactly when $A$ is Moore-Penrose invertible. Futhermore,
\[
M^\dagger = P^* A^\dagger.
\]

In this paper, we will assume $a$ to be regular in $R$, that is, there exists $a^- \in R$ for which $aa^-a = a$. Given solutions (possibly distinct) $a^-, a^=$ to $axa = a$ in $R$, then one can construct a reflexive inverse of $a$, that is, a common solution to $axa = a$ and $xax = x$, by taking $a^+ = a^- a a^-$. Note that the Moore-Penrose invertibility of $A$ does not imply $a$ is Moore-Penrose invertible. Indeed, consider $R$ the ring of $2 \times 2$ complex matrices with transposition as the involution, and set
\[
a = \begin{bmatrix}
1 & i \\
0 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
a & 0 \\
b & I_2
\end{bmatrix} = \begin{bmatrix}
1 & i & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}.
\]

Using [11], $A^\dagger$ exists since $rk(A) = rk(AA^T) = rk(A^T A) = 3$, but $a^\dagger$ does not since $aa^T = 0$. Consequently, the Moore-Penrose invertibility of the companion matrix $M$ does not imply the Moore-Penrose invertibility of $a$. 

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On the other hand, $a$ may be Moore-Penrose invertible and $M^\dagger$ may not exist. As an example, consider $R$ the ring of $2 \times 2$ complex matrices with transposition as the involution,

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a^\dagger, \quad b = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & a \\ I_2 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & i \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$ 

Then $rk(M) = 3 \neq 2 = rk(M^T M)$ for $M$ as a $4 \times 4$ complex matrix, and $M^\dagger$ does not exist.

We will divide this paper in two parts. In the first, we will assume $a^\dagger$ exists, and in the second we just assume regularity of $a$.

2 The case $a^\dagger$ exists

Suppose $a^\dagger$ exists and consider the unit

$$u = a\bar{a} + 1 - aa^\dagger, \quad \text{with } u^{-1} = \bar{a}a^\dagger + 1 - aa^\dagger.$$ 

Note that $u^{-1}a = \bar{a}$ and $\bar{a}u^{-1} = a^\dagger$.

The matrix

$$A = \begin{bmatrix} a & 0 \\ b & I_n \end{bmatrix}$$

is Moore-Penrose invertible if and only if $U = AA^* + I_{n+1} - AA^-$ is invertible for one, and hence, all choices of von Neumann inverses $A^-$ of $A$, by Theorem 1.1. Applying [7, Theorem 1], we may take

$$A^- = \begin{bmatrix} a^\dagger & 0 \\ -ba^\dagger & I_n \end{bmatrix},$$

for which choice we obtain

$$AA^- = \begin{bmatrix} aa^\dagger & 0 \\ 0 & I_n \end{bmatrix}$$

and

$$U = \begin{bmatrix} a\bar{a} + 1 - aa^\dagger & ab^* \\ b\bar{a} & bb^* + I_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b\bar{a}u^{-1} & I_n \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} 1 & u^{-1}ab^* \\ 0 & I_n \end{bmatrix},$$

where

$$Z = \begin{bmatrix} bb^* + I_n - b\bar{a}u^{-1}ab^* \\ b\bar{a} \end{bmatrix} = \begin{bmatrix} I_n + b(1 - a\bar{a}^{-1}a)b^* \\ I_n + b(1 - a^\dagger a)b^* \end{bmatrix}.$$
Now, the invertibility of $Z$ is equivalent to $z = 1 + b^*b(1 - a^\dagger a)$ being a unit of $R$, by the equivalence (1). Writing $b = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, this is the same as

$$z = 1 + \sum_{i=1}^n b_i(1 - a^\dagger a) \tag{2}$$

being a unit of $R$.

**Theorem 2.1.** Given $a \in R$ such that $a^\dagger$ exists and $b = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$, then the following are equivalent:

(a) The companion matrix $M = \begin{bmatrix} 0 & a \\ I_n & b \end{bmatrix}$ is Moore-Penrose invertible.

(b) $1 + (1 - a^\dagger a)b^*b(1 - a^\dagger a)$ is a unit of $R$.

(c) $1 + b^*b(1 - a^\dagger a)$ is a unit of $R$.

(d) $1 + (1 - a^\dagger a)b^*b$ is a unit of $R$.

We now carry out the construction of the Moore-Penrose inverse of the companion matrix, in the case $a^\dagger$ exists.

Using Theorem 1.1,

$$A^\dagger = (U^{-1}A)^*$$

which leads to

$$\begin{bmatrix} 0 & a \\ I_n & b \end{bmatrix}^\dagger = \begin{bmatrix} 0 & I_n \\ I_n & 1 \end{bmatrix} A^\dagger = \begin{bmatrix} 0 & I_n \\ 1 & 0 \end{bmatrix} (U^{-1}A)^*.$$

Note that $u$, $U$ and $Z$ are symmetric, and hence also are their inverses. Therefore,

$$U^{-1} = \begin{bmatrix} 1 & -\bar{a}^\dagger b^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & Z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b^\dagger & I_n \end{bmatrix} = \begin{bmatrix} u^{-1} + \bar{a}^\dagger b^*Z^{-1}b^\dagger & -\bar{a}^\dagger b^*Z^{-1} \\ -Z^{-1}b^\dagger & Z^{-1} \end{bmatrix},$$

with $Z^{-1} = (I_n + b(1 - a^\dagger a)b^*)^{-1} = I_n - b(1 - a^\dagger a)z^{-1}(1 - a^\dagger a)b^*$ and $z = 1 + (1 - a^\dagger a)b^*b(1 - a^\dagger a)$. Then

$$A^\dagger = A^* (U^*)^{-1}$$

$$= \begin{bmatrix} \bar{a}u^{-1} + a^\dagger ab^*Z^{-1}b^\dagger & -a^\dagger ab^*Z^{-1} + b^*Z^{-1} \\ -Z^{-1}b^\dagger & Z^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} a^\dagger - (1 - a^\dagger a)b^*Z^{-1}b^\dagger & (1 - a^\dagger a)b^*Z^{-1} \\ -Z^{-1}b^\dagger & Z^{-1} \end{bmatrix}.$$
Finally,
\[
\begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix}^\dagger = \begin{bmatrix}
0 & I_n \\
1 & 0
\end{bmatrix} A^\dagger
\]
\[= \begin{bmatrix}
-Z^{-1}b a^\dagger & Z^{-1} \\
a^\dagger - (1 - a^\dagger a) b^* Z^{-1} b a^\dagger & (1 - a^\dagger a) b^* Z^{-1}
\end{bmatrix}.
\]

3 The case $a$ is regular

We note that the companion matrix
\[
\begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix}
\]
is regular if and only if $a$ is regular. This follows from the factorization
\[
\begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
I_n & 0
\end{bmatrix} \begin{bmatrix}
I_n & 0 \\
0 & a
\end{bmatrix} \begin{bmatrix}
I_n & b \\
0 & 1
\end{bmatrix}.
\]

Suppose $a$ is regular and let $a^+$ be any reflexive inverse of $a$.
The matrix
\[
A = \begin{bmatrix}
a & 0 \\
b & I_n
\end{bmatrix}
\]
is Moore-Penrose invertible if and only if $V = A^* A + I_{n+1} - A^{-} A$ is invertible for one, and hence, all choices of von Neumann inverses $A^{-}$ of $A$, by Theorem 1.1. Applying [7, Theorem 1], we may take
\[
A^{-} = \begin{bmatrix}
a^+ & 0 \\
-b a^+ & I_n
\end{bmatrix},
\]
for which choice we obtain
\[
A^{-} A = \begin{bmatrix}
a^+ a & 0 \\
-b a^+ a + b & I_n
\end{bmatrix}
\]
and
\[
V = \begin{bmatrix}
\bar{a} a + 1 - a^+ a + b^* a & b^* \\
b a^+ a & I_n
\end{bmatrix} = \begin{bmatrix}
1 & b^*  \\
0 & I_n
\end{bmatrix} \begin{bmatrix}
\zeta & 0 \\
0 & I_n
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
b a^+ a & I_n
\end{bmatrix},
\]
where
\[
\zeta = \bar{a} a + 1 - a^+ a + b^* b (1 - a^+ a)
\]
\[= \bar{a} a + 1 - a^+ a + \sum_{i=1}^{n} b_i b_i (1 - a^+ a),
\]
with $b = \begin{bmatrix}
b_1 & b_2 & \cdots & b_n
\end{bmatrix}^T$. 

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Theorem 3.1. Given \( a \in R \) and \( b = [b_1 \ b_2 \ \cdots \ b_n]^T \), then the companion matrix
\[
M = \begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix}
\] is Moore-Penrose invertible if and only if \( a \) is regular and, for some reflexive inverse \( a^+ \) of \( a \), the element
\[
\zeta = \bar{a}a + 1 - a^+a + \sum_{i=1}^{n} b_ib_i(1 - a^+a)
\] (3)
is a unit of \( R \).

We now construct the Moore-Penrose inverse of the companion matrix, in the case \( a \) is regular.

Using Theorem 1.1, the Moore-Penrose inverse of \( A \) is given by
\[
A^\dagger = (AV^{-1})^\ast
\]
where
\[
V^{-1} = (A^*A + I_{n+1} - A^-)^{-1}
\]
\[
= \begin{bmatrix}
1 & 0 \\
-ba^+a & I_n
\end{bmatrix}^{-1}
\]
and \( \zeta = \bar{a}a + 1 - a^+a + b^*b(1 - a^+a) \). Then
\[
(V^{-1})^\ast = \begin{bmatrix}
\bar{\zeta}^{-1} & -\zeta^{-1}(a^+a)b^* \\
-b\bar{\zeta}^{-1} & I_n + b\bar{\zeta}^{-1}(a^+a)b^*
\end{bmatrix}.
\]
Substituting in the expression of \( A^\dagger \),
\[
A^\dagger = (V^{-1})^\ast A^*
\]
\[
= \begin{bmatrix}
\bar{\zeta}^{-1} & -\zeta^{-1}(a^+a)b^* \\
-b\bar{\zeta}^{-1} & I_n + b\bar{\zeta}^{-1}(a^+a)b^*
\end{bmatrix} \begin{bmatrix}
\bar{a} & b^* \\
0 & I_n
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\zeta^{-1}a & \zeta^{-1}(1 - (a^+a))b^* \\
-b\bar{\zeta}^{-1}a & I_n - b\bar{\zeta}^{-1}(1 - (a^+a))b^*
\end{bmatrix}
\]
from which we deduce
\[
\begin{bmatrix}
0 & a \\
I_n & b
\end{bmatrix}^\dagger = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} A^\dagger
= \begin{bmatrix}
-b\bar{\zeta}^{-1}a & I_n - b\bar{\zeta}^{-1}(1 - (a^+a))b^* \\
\bar{\zeta}^{-1}a & \zeta^{-1}(1 - (a^+a))b^*
\end{bmatrix}
\]
4 Questions and remarks

1. If \( a^\dagger \) exists and \( b^*b \in a^\dagger b^*bR \) then \( \begin{bmatrix} 0 & a \\ I_n & b \end{bmatrix}^\dagger \) exists. Indeed, if \( b^*b = a^\dagger b^*bx \) for some \( x \) in \( R \) then \( a^\dagger b^*b = a^\dagger b^*bx = b^*b \).

2. If \( a \) is regular and \( b_i \in Rb \) then \( \begin{bmatrix} 0 & a \\ I_n & b_i \end{bmatrix}^\dagger \) exists, with \( b = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]^T \), if and only if \( a^\dagger \) exists. Indeed, if \( b_i = xb_i a \) then \( b_i a^\dagger a = xb_i a = b_i \), from which the element \( \zeta \) in equation (3) collapses to \( \zeta = \bar{a}a + 1 - a^\dagger a \), which is a unit exactly when \( a^\dagger \) exists.

3. How can the invertible elements defined in equations (2) and (3) be directly related, in the case \( a^\dagger \) exists?

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References


