On finitely based pseudovarieties of the forms $V \ast D$ and $V \ast D_n$ \(^1\)

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Abstract

Let $D_n$ be the pseudovariety of all finite semigroups in which products of length $n$ are right zeros and let $D = \bigcup_{n \geq 1} D_n$. It is shown in this paper that, if $V$ is a pseudovariety of semigroups whose global $g$ is finitely based, then $V \ast D_n$ $(n \geq 1)$ and $V \ast D$ are also finitely based. Moreover, if $V$ is itself finitely based and contains the aperiodic five-element Brandt semigroup, then $gV$ is also finitely based. As a further application, it is proved that the finite basis properties for $gV$, $V \ast D$ and $V \ast D_n$ $(n \geq 1)$ are all equivalent for an arbitrary non-group monoidal pseudovariety $V$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recall that a pseudovariety of semigroups is a class of finite semigroups closed under taking divisors and finite products. The semidirect product $V \ast W$ of two pseudovarieties of semigroups $V$ and $W$ is the pseudovariety consisting of all divisors of semidirect products $S \ast T$ with $S \in V$ and $T \in W$. The pseudovariety of all finite semigroups whose idempotents are right zeros is denoted by $D$. In turn, the pseudovariety defined by the pseudoidentity $yx_1 \cdots x_n = x_1 \cdots x_n$ is denoted by $D_n$.

The semidirect products of the form $V \ast D_{(p)}$ have deserved much attention [14, 8, 15, 17, 1, 4, 2]. In [1] a representation of the relatively free semigroups over semi-
direct products $V \ast W$ when $W$ is \textit{locally finite} (in the sense that its relatively free finitely generated semigroups are finite) is introduced and this is applied to the study of $V \ast \mathcal{D}_n$. Such a representation has been extended to arbitrary $W$ [5].

At the S. Petersburg Conference on “Semigroups with Applications, including Semigroup Rings”, in June 1995, P. Trotter presented some recent results on semidirect products in which the second factor is the $(\varepsilon)$-variety of right zero semigroups ($\mathcal{D}_1$ in the notation of this paper). He showed that, if an $\varepsilon$-variety (or pseudovariety) $V$ is “local” and finitely based, then $V \ast \mathcal{D}_1$ is again finitely based. He also gave an example of a non-local, finitely based $V$, namely $V = \langle xyz = yzx \rangle$, such that $V \ast \mathcal{D}_1$ is not finitely based [21].

The purpose of this paper is to reconsider these results in the light of techniques introduced recently by Almeida and Weil [6] and to determine to what extent they can be put in a more general framework.

The results of [6] involve Tilson’s theory of pseudovarieties of categories and semigroupoids. For a pseudovariety $V$ of semigroups, denote by $gV$ the pseudovarieties of semigroupoids generated by the elements of $V$ viewed as one-vertex semigroupoids, and by $lV$ the pseudovariety of all finite semigroupoids whose finite one-vertex subsemigroupoids with nonempty edge set lie in $V$. It follows from the definitions that a basis of pseudoidentities for $lV$ can be easily obtained from a given basis for $V$. Now, Almeida and Weil [6] have shown how a basis of pseudoidentities for $V \ast W$ can be obtained from a basis of pseudoidentities for $gV$ and the knowledge of when a pseudoidentity is valid in $W$. Hence, it would be useful to have also an easy way to compute a basis of pseudoidentities for $gV$ given such a basis for $V$. In particular the problem is easy if $gV = lV$, that is if the pseudovariety $V$ is local.

In this paper we obtain a finite basis for $V \ast \mathcal{D}$ and for $V \ast \mathcal{D}_n$ ($n \geq 1$) if a finite basis for $gV$ is given. Conversely, for a pseudovariety of monoids containing nontrivial semilattices, given a finite basis of either $V \ast \mathcal{D}$ or $V \ast \mathcal{D}_n$, a finite basis for $gV$ is found.

In case $V = [\Sigma]$, where $\Sigma$ is a set of pseudoidentities, is a pseudovariety of semigroups containing the semigroup $B_2 = \langle a, b; a^2 = b^2 = 0, aba = a, bab = b \rangle$, we obtain a basis of pseudoidentities for $gV$ with the same cardinal as $\Sigma$. In particular, if $V$ is finitely based and contains $B_2$, then $gV, V \ast \mathcal{D}$ and $V \ast \mathcal{D}_n$ ($n \geq 1$) are all finitely based. Furthermore, this leads to the equivalence of these three finite basis properties for an arbitrary non-group monoidal pseudovariety $V$.

2. Preliminaries

For general background and undefined terms, the reader is referred to [2]. In particular, recall that, for a pseudovariety $V$ of semigroups, $LV$ denotes the pseudovariety consisting of all finite semigroups $S$ such that $eSe \in V$ for every idempotent $e$ of $S$. Denote by $I$ the pseudovariety consisting only of singleton semigroups.
By a graph we mean a partial algebra \( G \) with a partition \( G = V(G) \cup E(G) \) into two sorts of elements, respectively vertices and edges (or arrows), and two unary operations \( \alpha, \omega: E(G) \to V(G) \). A graph morphism is a function \( \phi: G \to H \) between two graphs respecting sorts and operations. A subgraph \( H \) of a graph \( G \) is a graph contained in \( G \) such that the inclusion \( H \to G \) is a graph morphism. For a graph \( G \) and two vertices \( c, d \) we denote by \( G(c,d) \) the set of edges \( s \) such that \( \alpha(s) = c \) and \( \omega(s) = d \).

In a natural way one can define the product and the coproduct (disjoint union) of a family of graphs.

**Definition 2.1.** A semigroupoid is a graph \((S, \alpha, \omega)\) endowed with a binary partial operation on \( E(S) \) such that: for \( s, t \in E(S) \), \( st \) is defined if and only if \( \omega(s) = \alpha(t) \), and in this case \( \alpha(st) = \alpha(s) \) and \( \omega(st) = \omega(t) \).

A category is a semigroupoid \( S \) admitting, for all \( c \in V(S) \), an edge \( 1_c \) such that for all \( s \in E(S) \), if \( \omega(s) = c \) then \( s1_c = s \), and if \( \alpha(s) = c \) then \( 1_cs = s \).

Note that a semigroup can be seen as a semigroupoid with only one vertex.

**Definition 2.2.** For a semigroup \( S \), let \( S_E \) be the semigroupoid such that:

- \( V(S_E) = \{ e \in S: e^2 = e \} \);
- \( E(S_E) = \{ (e, esf, f) \in S^3: e^2 = e, f^2 = f \} \);
- \( \alpha((e, esf, f)) = e, \) \( \omega((e, esf, f)) = f \);
- \( \alpha((e, esf, f))(f, fg, g) = (e, esftg, g) \).

A morphism of semigroupoids is a graph morphism respecting the partial binary operation.

A morphism of semigroupoids \( \varphi: C \to D \) is a quotient if it is surjective and it is injective when restricted to \( V(C) \); \( \varphi \) is faithful if, for all \( c, d \in V(C) \), \( \varphi \) restricted to \( C(c, d) \) is injective. We say that a semigroupoid \( C \) divides a semigroupoid \( D \) and we write \( C \rhd D \) if there exists a semigroupoid \( E \), a quotient morphism \( E \to C \) and a faithful morphism \( E \to D \).

**Definition 2.3.** A class of finite semigroupoids is a pseudovariety if it is closed under taking divisors, finite products and coproducts, and contains the semigroupoid with only one vertex and one edge.

We denote by \( Sd \) the pseudovariety of all semigroupoids. For a pseudovariety \( V \) of semigroups, denote by \( gV \) the pseudovariety of semigroupoids generated by the elements of \( V \) viewed as one-vertex semigroupoids (or equivalently, the class of semigroupoids dividing some semigroup of \( V \)), and by \( lV \) the pseudovariety of all finite semigroupoids whose one-vertex subsemigroupoids with nonempty edge set lie in \( V \). In particular, \( gV \) is always contained in \( lV \). When \( gV \) is equal to \( lV \) we say that \( V \) is local.
A pseudovariety of semigroups is said to be \textit{monoidal} if it is generated by its monoids. For a semigroupoid \( S \), denote by \( S^c \) the least category containing \( S \), namely the category obtained by adjoining local identities at the vertices where they are missing. It is easy to show that, if \( S \) and \( T \) are semigroupoids and \( S \approx T \), then \( S^c \approx T^c \). So, in particular, if \( S \) is a semigroupoid and \( V \) is a monoidal pseudovariety, then \( S \in gV \) if and only if \( S^c \in gV \) and so, in particular, \( gV \) is generated by its categories.

For a semigroupoid \( S \) and a finite graph \( A \), denote by \( G(A;S) \), the set of all graph morphisms \( A \to S \).

\textbf{Definition 2.4.} Let \( A \) be a finite connected graph. An \( A \)-ary implicit operation over a pseudovariety of semigroupoids \( V \) is defined as a family \((\pi;A) = (\pi_S)_{S \in V}\) where for each \( S \in V \), \( \pi_S : G(A;S) \to S \) is a function, and for each homomorphism \( \phi : S \to T \) with \( S, T \in V \), \( \phi(\pi_S(f)) = \pi_T(\phi(f)) \), for all \( f \in G(A;S) \).

It can be easily proved that every implicit operation assumes only edge values or only vertex values. We therefore call them, respectively, \textit{edge} and \textit{vertex} implicit operations.

For a finite connected graph \( A \), the set of all \( A \)-ary implicit operations on \( V \) is denoted by \( \Omega_A V \) and has a natural structure of semigroupoid with the above choice of edges and vertices.

A formal identity between two cterminal edge implicit operations on the same finite connected graph is called a \textit{pseudoidentity}. We say that a semigroupoid \( S \) satisfies a pseudoidentity \((\pi;A) = (\rho;A)\) (or simply, \((\pi = \rho;A)\)), on the graph \( A \), and write \( S \models \pi = \rho \) if \( \pi_S = \rho_S \). Given a set \( \Sigma \) of pseudoidentities we denote by \( [\Sigma] \) the class of all finite semigroupoids satisfying all the pseudoidentities of \( \Sigma \).

Now we are ready to state the analog, for finite semigroupoids, of Reiterman’s Theorem.

\textbf{Theorem 2.5} (Jones [9]; Almeida and Weil [6]). A class \( V \) of semigroupoids is a pseudovariety if and only if there exists a set \( \Sigma \) of pseudoidentities over \( Sd \) on finite connected graphs such that \( V = [\Sigma] \).

Given a set of pseudoidentities defining a pseudovariety of semigroups, it is easy to obtain a basis of pseudoidentities for \( IV \), but in general it is not easy for \( gV \). So, finding a basis for \( gV \) is easy for local \( V \), although it is in general difficult to prove locality of \( V \). For example the pseudovarieties of semigroups \( Sl = [x^2 = x, \ xy = yx] \) [7], \( CR = [x^{n+1} = x] \) [10], and \( DA = [(xy)^n(xy)^n(xy)^n = (xy)^n, x^{n+1} = x^n] \) [3] are all local. The following is a list of pseudovarieties of the form \( gV \) (with \( V \) not local) for which bases of pseudoidentities are known.
Example 2.6.

- \( g[x = y] = [x = y; \quad \bullet y \quad \bullet] \).

- \( g[xy = yx] = [xyz = zyx; \quad \bullet x \quad \bullet \quad \bullet y \quad \bullet] \) \quad [16].

- \( g[(xy)^{\alpha}, x^{\alpha+1} = x^{\alpha}] = [(xy)^{\alpha}x\alpha(ux)^{\alpha} = (xy)^{\alpha}(ux)^{\alpha}; \quad \bullet x \quad \bullet \quad \bullet y \quad \bullet] \) \quad [11].

Definition 2.7. For pseudovarieties of semigroups \( V \) and \( W \), let \( V \ast W \) be the pseudovariety generated by the semidirect products of the form \( S \ast T \), with \( S \in V \) and \( T \in W \). As in [2], we only consider semidirect products \( S \ast T \) determined by monoid homomorphisms \( T^1 \rightarrow \text{End } S \).

It is well known that the semidirect product operation \( \ast \) on pseudovarieties is associative (cf. [2, Proposition 10.1.4]).

The following theorem will be used to compute a basis of pseudoidentities for a semidirect product \( V \ast W \) of pseudovarieties of semigroups.

Theorem 2.8 (Almeida and Weil [6]). Let \( V \) and \( W \) be pseudovarieties of semigroups and \( \Sigma \) a basis of pseudoidentities for \( gV \). Then \( V \ast W \) is defined by the pseudoidentities of the form \( \pi_{uv}(u) = \pi_{uv}(v) \), where \( (u = v; A) \in \Sigma \), with say \( u, v \in E(\overline{\Omega}_A Sd) \), \( \pi_q \in (\overline{\Omega}_A Sd)^1 \) \( (q \in V(A)) \) and \( \rho_s \in \overline{\Omega}_B S \) \( (s \in E(A)) \), \( W \models \pi_{uv}(u) = \pi_{uv}(v) \) for all \( s \in E(A) \) and \( \varepsilon : \overline{\Omega}_A Sd \rightarrow \overline{\Omega}_B S \) is the continuous morphism of semigroupoids such that \( \varepsilon(s) = \rho_s \) for \( s \in \overline{E}(A) \).

As a corollary we have the following.

Corollary 2.9 (Almeida and Weil [6]). If \( V \) and \( W \) are pseudovarieties of semigroups, with \( V \) local, and \( \Sigma \) is a basis of pseudoidentities for \( V \), then \( V \ast W \) is defined by the pseudoidentities of the form \( \pi_{uv}(\rho_1, \ldots, \rho_r) = \pi_{uv}(\rho_1, \ldots, \rho_r) \), where \( (u = v; A) \in \Sigma \), \( u, v \in \overline{\Omega}_A S \) \( (r \geq 1) \), \( \rho_1, \ldots, \rho_r \in \overline{\Omega}_B S \), \( \pi \in (\overline{\Omega}_A S)^1 \) and \( W \models \pi_{uv}(\rho_1, \ldots, \rho_r) = \pi_{uv}(\rho_1, \ldots, \rho_r) \).

As a further consequence we obtain the following.

\(^2\)Actually, this pseudoidentity defines the pseudovariety of categories obtained by intersecting with the class of all finite categories. However, the original pseudovariety of semigroups is monoidal so that the global is generated by its categories. Hence, for the setting of semigroupoids we are considering in this paper, the pseudoidentity should be taken as representing all pseudoidentities which can be obtained from it by removing edges (and identifying the end vertices of such edges).
Corollary 2.10. If \((U_i)_{i \in I}\) is a family of semigroup pseudovarieties such that \(g(\bigcap_i U_i) = \bigcap_i gU_i\) then, for every pseudovariety \(W\), \((\bigcap_i U_i) * W = \bigcap_i (U_i * W)\).

Note that in general we do not have the equality \(g(U \cap V) = gU \cap gV\). For instance, if \(G\) and \(A\) are, respectively, the pseudovarieties of finite groups and finite aperiodic semigroups, we have

\[
g(A \cap G) = g1 \neq gA \cap gG
\]

since say the semigroupoid \(\begin{array}{c} s \\ \downarrow \alpha \end{array} \begin{array}{c} t \\ \downarrow \beta \end{array}\) lies in both \(gA = lA\) and \(gG\). We have also that \((A \cap G) * D ( = D)\) is different from \((A * D) \cap (G * D) ( = L1)\).

3. The case of \(V * D\)

It is known that the decidability of \(V * D\), for a pseudovariety of semigroups \(V\) containing a nontrivial monoid, is equivalent to the decidability of \(gV\). More precisely, we have the following result, due to Tilson [17] (see [6] for a syntactical proof).

Theorem 3.1 (Delay Theorem). A finite semigroup \(S\) lies in \(V * D\) if and only if \(SE\) belongs to \(gV\).

In this section we will describe, given a (finite) basis of \(gV\), a (finite) basis of \(V * D\). This result can be regarded as a straightforward consequence of [6].

Theorem 3.2. Let \(V\) be a pseudovariety not contained in \(L1\) (i.e., \(V\) contains a nontrivial monoid).

If \(gV = \langle \Sigma \rangle\) (where \(\Sigma\) is a set of pseudoidentities) then \(V * D\) is defined by the pseudoidentities of the form

\[
\sigma(u) = \varepsilon(v),
\]

where \((u = v; A) \in \Sigma\), \(\varepsilon : \overline{\Omega}_B SD \to \overline{\Omega}_B S\) is the continuous morphism defined by \(\varepsilon(s) = x_{\sigma s}^\sigma y_s x_{o s}^\sigma\), where \(B\) is such that \(|B| = |V(A)| + |E(A)|\).

In particular, if \(gV\) is finitely based, then the same is true for \(V * D\).

Proof. Let \(X\) be the pseudovariety we want to prove to be equal to \(V * D\). Considering, for \(s \in E(A)\), \(\pi_{gs} = x_{gs}^s\) and \(\pi_{os} = x_{os}^s\), and noting that \(\varepsilon(u) = x_{\pi gu}^{\sigma u} \varepsilon(u)\), \(\varepsilon(v) = x_{\pi gv}^{\sigma v} \varepsilon(v)\), we see that the pseudoidentities defining \(X\) are some of the pseudoidentities used in Theorem 2.8 to calculate \(V * D\). In particular, \(V * D\) is contained in \(X\).
Then, using Theorem 2.8, we have, for a semigroup $S$:

$$ S \in X \Rightarrow S_E \in gV \quad \text{by definition of } S_E $$

$$ \Leftrightarrow S \in V \ast D \quad \text{by the Delay Theorem} $$

$$ \Rightarrow S \in X \quad \text{as } V \ast D \subseteq X. $$

So, we conclude that $V \ast D = X$. \hfill \Box

Applying this result to the pseudovariety $\text{Com}$ and using the already quoted result that $g\text{Com} = \langle xyz = zyx; x = y \rangle$, we obtain

$$ \text{Com} \ast D = [a^\omega xb^\omega yd^\omega zb^\omega = a^\omega zb^\omega ya^\omega xb^\omega] \quad [16, 1]. $$

Other examples will be given in Section 5.

4. The case of $V \ast D_n$

In this section we construct, given a basis $\Sigma$ of pseudoidentities of $gV$, a basis of pseudoidentities for $V \ast D_n$, which is finite if $\Sigma$ is finite.

For $\pi \in \Omega_bS$ and a positive integer $n$, denote by $t_n(\pi)$ the longest word of length at most $n$ which is a suffix of $\pi$. Dually, $i_n(\pi)$ is defined as the longest word of length at most $n$ which is a prefix of $\pi$. It is well known that $D_n$ satisfies a pseudoidentity $\pi = \rho$ if and only if $t_n(\pi) = t_n(\rho)$.

**Theorem 4.1.** If $V$ is a pseudovariety of semigroups such that $gV$ is finitely based, then, for all $n \geq 1$, $V \ast D_n$ is also finitely based.

**Proof.** Let $gV = [\Sigma]$. Using the notation of Theorem 2.8 define, for each $(u = v; A) \in \Sigma$, the set $\Sigma_{(u = v; A) \in \Sigma}$ of all pseudoidentities of the form

$$ \pi_{xu} = \pi_{xv}, $$

where $\pi_q \in (\Omega_bS)^I\ (q \in V(A))$ and $D_n \models \pi_{xu} = \pi_{xs} \ (s \in E(A))$.

By Theorem 2.8, we know that

$$ V \ast D_n = \bigcap_{(u = v; A) \in \Sigma} \Sigma_{(u = v; A)}. $$

Fix $(u = v; A) \in \Sigma$ and, for each continuous morphism $\varepsilon : \Omega_bSd \to \Omega_bS$ and each family $(\pi_q)_{q \in V(A)}$ as in Theorem 2.8, define $\mu^\varepsilon(q) = t_n(\pi_q)$ and, for $s \in E(A)$,

$$ \delta^\varepsilon(s) = \begin{cases} y_s t_n(\varepsilon(s)) & \text{if } t_n(\varepsilon(s)) \neq \varepsilon(s) \text{ (where } y_s \text{ is a new variable)}, \\ t_n(\varepsilon(s)) & \text{otherwise.} \end{cases} $$
Extend $\delta_c$ to a continuous morphism of semigroupoids on $\Omega_sSd$ and let

$$\Sigma_{(u=v,A)} = \{ \mu'(zu)\delta_c(u) = \mu'(zu)\delta_c(v) : e \text{ and } (\pi_q)_q \text{ as above} \}.$$ 

Note that $\Sigma_{(u=v,A)}$ is a finite subset of $\Sigma_{(u=v,A)}$ and, if $t_a(\rho_s) \neq \rho_s$, then there exists an implicit operation $\sigma_s$ such that $\rho_s = \sigma_st_a(\rho_s)$.

On the other hand, if we substitute, for each $s \in E(A)$ such that $t_a(\delta(s)) \neq \rho_s$, $y_s$ by $\sigma_s$, we obtain the pseudoidentity $\pi_{zu}\delta(u) = \pi_{zu}\delta(v)$ and therefore $\Sigma_{(u=v,A)}$ and $\Sigma_{(u=v,A)}$ define the same pseudovariety.

As $V \ast D_n = \bigcap_{(u,v,A) \in \Sigma} [\Sigma_{(u=v,A)}]$, we conclude that, if $\Sigma$ is finite, then $V \ast D_n$ is also finitely based. \qed

It should be stressed that the proof of Theorem 4.1 is constructive. The following are some examples of application of the construction described in the proof. It must also be stressed that, for the construction to work, a finite basis of pseudoidentities for $gV$ must be given, a task which is in general quite hard. Moreover, it is important to note that, in some of the examples below, the calculation of $gV$. Only cases in which $V$ is local are presented because otherwise too many pseudoidentities would have to be written to justify including it here. (In fact, the pseudovariety $G$ is not local as a pseudovariety of semigroups but only as a pseudovariety of monoids. However, as has been observed in [6, Section 5], for the calculation of semidirect products, with proper care, $G$ behaves as though it were local.)

**Example 4.2.** (1) Since $gSL$ is finitely based, so is $SL \ast D_n$. A finite basis for a pseudovariety closely related with $SL \ast D_n$ was given in [20] (see [2, Ch. 10] for extensions). Following the procedure in the proof of Theorem 4.1, we consider the two identities $xy = yx$ and $x^2 = x$, both on one-vertex graphs, and we must find, up to equivalence, all identities which can be obtained from these of the form $w\delta_e(\pi) = w\delta_e(\rho)$ where $\pi = \rho$ is the given identity, $e$ chooses an implicit operation for each of the edges $x$ and $y$, $\delta_e$ is defined as in the proof of Theorem 4.1, $w$ is a word of length $n$, and $t_a(w\delta_e(z)) = w$ ($z \in \{x, y\}$). Note that $\delta_e$ of an edge is a word of length at most $n + 1$.

Let $u = \delta_e(x)$ and $v = \delta_e(y)$. Then $w = t_a(wu) = t_a(wv)$. Say, in the case of $u$, (a) either $|u| > n$, and $u = yw$, or (b) $u$ is a suffix of $w$, in which case $w = uu'w^m$ for some $m \geq 1$ and some suffix $u'$ of $u$. If only the case (a) occurs, we may take $w = x_1 \cdots x_n$ where the $x_i$ are distinct variables. If the case (b) occurs, take the shortest of $u$ and $v$ for which it holds to obtain $w = (x_1 \cdots x_r)^m x_1 \cdots x_p$ where $r = \min\{|u|, |v|\}$ and $m$ and $p$ are respectively the quotient and the remainder of the integer division of $n$ by $r$. This gives rise to the following identities:

$$z : x_1 \cdots x_n y x_1 \cdots x_n x_1 x_1 \cdots x_n = x_1 \cdots x_n x_1 x_1 \cdots x_n y x_1 \cdots x_n$$

$$z_r : (x_1 \cdots x_r)^m x_1 \cdots x_p y (x_1 \cdots x_r)^{m+1} x_1 \cdots x_p = (x_1 \cdots x_r)^{m+1} x_1 \cdots x_p y (x_1 \cdots x_r)^m x_1 \cdots x_p$$

$$(1 \leq r \leq n)$$
\[ \begin{align*}
\beta_r : & \quad (x_1 \ldots x_r)^{m+2} x_1 \ldots x_p = (x_1 \ldots x_r)^{m+1} x_1 \ldots x_p \quad (1 \leq r \leq n) \\
\beta_{r+1} : & \quad x_1 \ldots x_n y x_1 \ldots x_n y x_1 \ldots x_n = x_1 \ldots x_n y x_1 \ldots x_n
\end{align*} \]

Now, using \( \beta_r \) and \( \alpha \), it is easy to deduce \( \alpha \), so that the identities \( \alpha \) may be dropped. We thus obtain precisely the basis of identities for \( \text{SI} \ast \text{D}_n \), described in [2, Section 10.8] consisting of \( \alpha \) and \( \beta_r \) \((1 \leq r \leq n + 1)\).

(2) Similarly, a basis for \( \text{G} \ast \text{D}_n \) is given by the pseudoidentities

\[
(x_1 \ldots x_r)^{m+1+\alpha} x_1 \ldots x_p = (x_1 \ldots x_r)^{m+1} x_1 \ldots x_p \quad (1 \leq r \leq n, n = mr + p, 0 \leq p < r).
\]

5. Calculation of \( gV \), if \( B_2 \in V \)

In this section we are going to establish a process to compute a basis of semigroupoid pseudoidentities for \( gV \), from a basis of semigroup pseudoidentities for a semigroup pseudovariety \( V \) such that \( B_2 \in V \). Recall that \( B_2 \) is the five-element Brandt semigroup given by the presentation \( \langle a, b : aba = a, bab = b, a^2 = b^2 = 0 \rangle \) and that \( V(B_2) = \{ x^2 = x^2, x^2 y^2 = y^2 x^2, x(yx)^2 = x(yx) \} \) [18, 19].

We begin by establishing a simple connection between \( V \) and \( gV \), using the consolidation operation. To each semigroupoid \( S \) we associate a semigroup \( S_{cd} \) which consists of the set \( \{ (i, s, t) : s \in E(S), \alpha(s) = i, \omega(s) = t \} \) with a zero 0 adjoined if \( S \) has at least two arrows which are not composable. The operation in \( S_{cd} \) is defined by taking, for \((i, s, t), (i', s', t') \in S_{cd},\)

\[
(i,s,t)(i',s',t') = \begin{cases} 
(i,ss',t') & \text{if } t = t', \\
0 & \text{otherwise}
\end{cases}
\]

and letting 0 multiply as a zero in case such an element is needed. Obviously \( S \) divides \( S_{cd} \) and the natural division \( d : S \rightarrow S_{cd} \) that associates to the arrow \( s \in E(S) \) the element \( d(s) = (\alpha(s), s, \omega(s)) \) is injective on \( E(S) \).

**Proposition 5.1** (Tilson [17]). If \( V \) is a pseudovariety of semigroupoids such that \( B_2 \in V \), then a semigroupoid \( S \) is in \( gV \) if and only if \( S_{cd} \) belongs to \( V \).

Using this proposition, Theorem 2.8 and Corollary 2.10 we obtain the following result.

**Corollary 5.2.** If \((V_i)_{i \in I}\) is a family of semigroup pseudovarieties containing \( B_2 \) then

(i) \( g(\bigcap_{i \in I} V_i) = \bigcap_{i \in I} gV_i \);
(ii) \( (\bigcap_{i \in I} V_i) \ast W = \bigcap_{i \in I} V_i \ast W \).

For a pseudovariety \( \text{H} \) of groups, denote by \( \overline{\text{H}} \) the pseudovariety consisting of all finite semigroups whose subgroups (subsemigroups which are groups) belong to \( \overline{\text{H}} \).
Denote by $\mathbf{ER}$ the pseudovariety of all finite semigroups whose subsemigroups generated by the idempotents are, $\mathcal{R}$-trivial.

The following is a special case of Corollary 5.2 which solves, in particular, Problem 33 in [2].

**Corollary 5.3.** Let $\mathbf{H}$ be a pseudovariety of groups. Then $(\mathbf{ER} \cap \mathbf{H}) * \mathbf{D} = \mathbf{L} \mathbf{ER} \cap \mathbf{H} = \mathbf{L}(\mathbf{ER} \cap \mathbf{H})$.

**Proof.** Since the semigroup $B_2$ belongs to both $\mathbf{ER}$ and $\mathbf{H}$, by Corollary 5.2 we obtain the equality $(\mathbf{ER} \cap \mathbf{H}) * \mathbf{D} = (\mathbf{ER} * \mathbf{D}) \cap (\mathbf{H} * \mathbf{D})$. On the other hand, $\mathbf{ER} * \mathbf{D} = \mathbf{LER}$ (cf. [8, Section V.12]) while $\mathbf{H} * \mathbf{D} = \mathbf{H}$ since $\mathbf{LH} = \mathbf{H}$ (cf. [2, Proposition 10.6.13]). This proves the first equality in the statement of the corollary. The second equality follows immediately from the definitions. $\square$

The word problem in the variety of inverse semigroups generated by $B_2$ is decidable [13, 12], and the corresponding proofs associate to each word a special graph (or automaton). Extending the arguments of Reilly to pseudoidentities of semigroups, we associate a graph $A_\pi$ to each implicit operation $\pi \in \mathcal{O}_X\mathbf{S}$.

**Definition 5.4.** Given an alphabet $X$, and $\pi \in \mathcal{O}_X\mathbf{S}$, we define the equivalence relation $\delta_\pi$ over the set $Y = X \cup X^{-1}$ (where $X^{-1} = \{x^{-1} : x \in X\}$ is a disjoint copy of $X$) generated by the pairs $(x^{-1}, y) \in Y^2$ such that $xy$ is a factor of $\pi$.

Since factors of length 2 can be recognized by finite semigroups, the correspondence $\pi \mapsto \delta_\pi$ defines a continuous function $\delta : \mathcal{O}_X\mathbf{S} \to \mathcal{P}(Y^2)$ into the power set of $Y^2$, viewed as a discrete space.

**Definition 5.5.** Given $\pi \in \mathcal{O}_X\mathbf{S}$, we define the graph $A_\pi$ by

$$V(A_\pi) = Y / \delta_\pi,$$

$$A_\pi([w], [y]) = \{ w \in X : (w, x) \in \delta_\pi \text{ and } (w^{-1}, y) \in \delta_\pi \},$$

where $[w]$ denotes the $\delta_\pi$-class of $w \in Y$.

The edges of $A_\pi$ are elements of $X$, so a path of $A_\pi$ is an element of $X^*$ and every word factor of $\pi$ is a path of $A_\pi$. As an example, consider the alphabet $X = \{x, y, z, t\}$ and the implicit operation $\pi = xty^uyxz$. Then $A_\pi$ is the following graph:

![Graph](image)

where $v_1 = \{x, y^{-1}\}$, $v_2 = \{x^{-1}, y, z, t, t^{-1}\}$ and $v_3 = \{z^{-1}\}$. 
The following is the announced extension to pseudoidentities of Reilly’s result.

**Proposition 5.6.** Let \( u = v \) be a pseudoidentity of semigroups on a finite alphabet \( X \) over a pseudovariety containing \( B_2 \). Then \( B_2 \models u = v \) if and only if:

(i) \( \delta_u = \delta_v \) (or \( A_u = A_v \));
(ii) \( i_1(u) \delta_u i_1(v) \);
(iii) \( (i_1(u))^{-1} \delta_u (i_1(v))^{-1} \).

**Proof.** Given two words \( u, v \in X^+ \), as \( u, v \in Y^+ \), we can decide whether \( B_2 \) satisfies the identity \( u = v \) using Theorem 3.3 in [13], where Reilly gives a solution for the word problem in the inverse semigroup variety generated by \( B_2 \). Note that given a word \( w = x_1 x_2 \ldots x_n \in X^+ \), its canonical form as an inverse semigroup word is

\[
\left[ (x_1 \ldots x_n)(x_1 \ldots x_n)^{-1} \right](x_1 x_1^{-1})(x_1 x_2)(x_2 x_2)^{-1} \ldots
\]

So the relation \( \delta_u \) is exactly the same relation defined by Reilly for the word \( w \), and the necessary and sufficient (three) conditions of Theorem 3.3 in [13] are exactly the conditions (i)–(iii). Hence \( B_2 \) satisfies the identity \( u = v \) if and only if (i)–(iii) are verified.

In general, \( u, v \in \Omega_\lambda \) and \( \pi \) is a path in \( A_\pi \) and the free profinite semigroupoid \( \Omega_\lambda J \) it generates. Since \( X^+ \) is dense in \( \Omega_\lambda \), there is a sequence \( (w_n)_n \) of words in \( X^+ \) converging to \( \pi \). For \( \pi \) to be a path, we assume, in particular, that \( B_2 \models w_n = \pi \) for all \( n \). By Proposition 5.6, it follows that \( A_{w_n} = A_\pi \) for all \( n \). Thus, all words \( w_n \) are paths in \( A_\pi \) and, again by Proposition 5.6, they are coterminal paths. We claim the sequence \( (w_n)_n \) of these paths converges in \( \Omega_\lambda J \). To prove the claim, given a morphism \( \varphi: \Omega_\lambda J \to S \) into a finite semigroupoid, we define an associated homomorphism \( \theta: \Omega_\lambda J \to S_{cd} \) by \( \theta(x) = d(\varphi(x)) \) for \( x \in X \). Since each \( w_n \) is a path in \( A_\pi \), \( \theta(w_n) \neq 0 \). Since \( (w_n)_n \) converges to \( \pi \) in \( \Omega_\lambda J \), \( \theta(\pi) = \lim_{n \to \infty} \theta(w_n) = \theta(\pi) \neq 0 \). Hence the sequence \( (\varphi(w_n))_n \) is eventually constant, which proves the claim and the following lemma.

**Lemma 5.7.** Given \( \pi \in \Omega_\lambda J \), there is a unique limit in \( \Omega_\lambda J \) of sequences \( (w_n)_n \) in \( X^+ \) converging to \( \pi \) in \( \Omega_\lambda J \), where, for large enough \( n \), each \( w_n \) is viewed as a path in \( A_\pi \).
We denote the limit in Lemma 5.7 again by \( \pi \). For instance, if \( \pi = x^{t_0}y^{x_0}z \) as in the above example, then \( x^{t_0}y^{x_0}z \) does indeed represent an element of \( \mathcal{O}_{A_n}Sd \) and precisely the element given by Lemma 5.7.

**Lemma 5.8.** Let \( u = v \) be a pseudoidentity of semigroups and suppose it is satisfied by \( B_3 \). Then a finite semigroupoid \( S \) satisfies \( (u = v; A_\nu) \) if and only if \( S_{cd} \) satisfies \( u = v \).

**Proof.** By Proposition 5.6, \( A_\nu = A_\epsilon \). By Lemma 5.7, \( u \) and \( v \) denote well defined elements of \( \mathcal{O}_{A_n}Sd \). Moreover, by continuity of the functions \( t_1 \) and \( t_1 \) and Proposition 5.6, \( u \) and \( v \) are coterminal edges in \( \mathcal{O}_{A_n}Sd \). Hence, \( (u = v; A_\nu) \) is indeed a semigroupoid pseudoidentity.

Let \( X \) be a minimal finite alphabet such that \( u, v \in \mathcal{O}_{A_n}S \) (also known as the content of the words \( u \) and \( v \)). If \( (u_\nu)_n \) and \( (v_\nu)_n \) are sequences of words in \( X^+ \) converging respectively to \( u \) and \( v \) in \( \mathcal{O}_{X^+}S \), then we may assume that \( A_\nu = A_\nu = A_\nu = A_\nu \). \( S_{cd} \) satisfies \( u = u_\nu \), \( v = v_\nu \), and \( S \) satisfies \( u = u_\nu; A_\nu \), \( v = v_\nu; A_\nu \) for all \( n \). This reduces the proof of the lemma to the case where \( u, v \in X^+ \) which we consider for the remainder of the proof.

Suppose first that \( S_{cd} \models u = v \) and consider an arbitrary graph morphism \( \phi : A_\nu \to S \). Define a homomorphism \( \theta : X^+ \to S_{cd} \) by taking \( \theta(x) = d(\phi(x)) \) for each \( x \in X \). Since \( u \) and \( v \) are paths in \( A_\nu \), \( \theta(u) \) and \( \theta(v) \) are nonzero and they must be equal as \( S_{cd} \models u = v \). Hence, for the unique extension \( \tilde{\phi} \) of \( \phi \) to a semigroupoid morphism \( A_\nu^+ \to S \), as \( d \) is injective on \( E(S) \), \( \tilde{\phi}(u) = \tilde{\phi}(v) \) This shows that \( S \models (u = v; A_\nu) \).

Conversely, suppose that \( S \models (u = v; A_\nu) \) and let \( \theta : X \to S_{cd} \) be an arbitrary mapping. Let \( \tilde{\theta} \) be the unique extension of \( \theta \) to a semigroup homomorphism \( X^+ \to S_{cd} \). We distinguish two cases according to whether \( 0 \in \theta(X) \).

If \( 0 \notin \theta(X) \), say \( \theta(x) = 0 \), then \( \tilde{\theta}(u) = 0 = \tilde{\theta}(v) \) since the letter \( x \) occurs in both \( u \) and \( v \).

Suppose now that \( 0 \in \theta(X) \). Then we define a function \( \phi : E(A_\nu) \to S \) by taking \( \phi(x) = d^{-1}(\theta(x)) \) for each \( x \in X \) (\( = E(A_\nu) \)). If \( \phi \) is a graph morphism, then it extends uniquely to a semigroupoid morphism \( \tilde{\phi} : A_\nu^+ \to S \) and, since \( S \models (u = v; A_\nu) \), \( \tilde{\phi}(u) = \tilde{\phi}(v) \). Hence, since \( \tilde{\phi}, d \) and \( \tilde{\theta} \) all respect products, \( \tilde{\theta}(u) = d(\tilde{\phi}(u)) = d(\tilde{\phi}(v)) = \tilde{\theta}(v) \). We may therefore assume that \( \phi \) is not a graph morphism, i.e., that there are \( x, y \in E(A_\nu) \) such that \( \omega(x) = \omega(y) \) and \( \omega(\phi(x)) \neq \omega(\phi(y)) \). Then \( (x^{-1}, y) \in \delta_{\omega} = \delta_{\omega} \) and \( \theta(x) \theta(y) = 0 \). By definition of \( \delta_{\omega} \) (respectively \( \delta_{\omega} \)), it follows that there are \( n \geq 1 \) and \( z_1, \ldots, z_{2n} \in X \) such that

\[
xz_1, z_2z_1, z_2z_3, \ldots, z_2z_{2n}z_{2n-1}, z_{2n}y
\]

are factors of \( u \) (respectively \( v \)). We claim that the image under \( \tilde{\theta} \) of at least one of these factors is 0, thereby showing that \( \tilde{\theta}(u) = 0 = \tilde{\theta}(v) \). Indeed, if the claim fails, then \( \theta(x) \theta(z_1), \theta(z_2) \theta(z_1), \ldots, \theta(z_{2n}) \theta(y) \in S_{cd} \setminus \{0\} \) and this means that, in \( S \),

\[
\omega(\phi(x)) = \omega(\phi(z_{2k})) = \omega(\phi(z_{2k+1})) = \omega(\phi(y))
\]
for \( k \in \{1, \ldots, n\} \), in contradiction with the assumption. Hence \( \tilde{\theta}(u) = \tilde{\theta}(v) \) in all cases, showing that \( S_{cd} \models u = v \). □

In view of Proposition 5.1 and Corollary 5.2(i), we immediately obtain the following theorem from Lemma 5.8.

**Theorem 5.9.** Let \( V \) be a pseudovariety of semigroups containing \( B_2 \). If \( V = \langle (u_i = v_i)_{i \in I} \rangle \) then \( \bar{V} = \langle (u_i = v_i; A_{u_i})_{i \in I} \rangle \).

We may now establish the main theorem of this section which is a consequence of the previous one, Proposition 5.1, and of the fact that, for any pseudovariety \( V \) of semigroups, \( V = \bar{V} \cap S \).

**Theorem 5.10.** Let \( V \) be a semigroup pseudovariety containing \( B_2 \). Then:

(i) \( V \) is decidable if and only if \( \bar{V} \) is decidable;

(ii) \( V \) is finitely based if and only if \( \bar{V} \) is finitely based.

Using Theorems 3.2 and 4.1 we obtain the following corollary.

**Corollary 5.11.** If \( V \) is a finitely based pseudovariety containing \( B_2 \) then, \( V \ast D \) and \( V \ast D_n \) are finitely based.

The calculation of \( V \ast D \) and \( V \ast D_n \) (if \( B_2 \in V \)) can be done without the effective calculation of the graphs associated with the pseudoidentities defining \( \bar{V} \). For instance, suppose \( V \) is defined by a set of identities \( \Sigma \). For each identity \( (u = v) \in \Sigma \) substitute each variable \( x \) in the content of \( u \) by \( a_x \) in such a way that if \( xy \) is a factor of \( u \), then \( b_x = a_y \). Then we obtain a basis of pseudoidentities for \( V \ast D \).

As a consequence we obtain that the pseudovarieties of the forms \([x^n = x^m]\) and \([x^n y^m = y^k x^s]\), with \( n, m, k, s \geq 2 \) are local.

**Example 5.12.**

- \( V(B_2) \ast D = \langle (exe)^3 = (exe)^2, (exe)^2(eye)^2 = (eye)^2(exe)^2, (exf y e)^2exf = exf \ ye \rangle \) where \( e = z^0 \) and \( f = t^0 \);
- \( V(B_2) \ast D_1 = \langle [a(xa)^2 = a(axa)^2, a^4 = a^2, a(xa)^2(ya)^2 = a(ya)^2(xa)^2, a(xa)^2a^2 = a^2(xa)^2, (axby)^2axb = (axby)axb, (aby)^2ab = (aby)ab, (axb)^3 = (axb)^2] \rangle \);
- \( [x^m = x^n] \ast D_2 = \langle [xy(xy)^m = x(yx)y^n, xy(xy)^m = xy(xy)^m] \rangle \), for \( n, m \geq 2 \).

In the case of monoidal pseudovarieties containing \( Sl \), we find the following tighter connection between various finite basis problems associated with it.

**Theorem 5.13.** Let \( V \) be a monoidal pseudovariety of semigroups containing \( Sl \). Then the following conditions are equivalent:

(i) for all \( n \geq 1 \), \( V \ast D_n \) is finitely based;

(ii) there exists \( n \geq 1 \) such that \( V \ast D_n \) is finitely based;
(iii) $V \ast D$ is finitely based;
(iv) $gV$ is finitely based.

**Proof.** (i) $\Rightarrow$ (ii) is obvious.

(iv) $\Rightarrow$ (i) is a particular case of Theorem 4.1.

(ii) $\Rightarrow$ (iii) First note that $B_2 \in \mathbf{Sl} \ast D_1 \subseteq V \ast D_n$. Then, by Theorem 3.2 we conclude that $(V \ast D_n) \ast D$, which by associativity of $\ast$ is equal to $V \ast D$, is finitely based.

(iii) $\Rightarrow$ (iv) Since $B_2 \in \mathbf{Sl} \ast D_1 \subseteq V \ast D$ and $V \ast D$ is assumed to be finitely based, by Theorem 5.10(b) $g(V \ast D)$ is defined by some finite set $\Sigma$ of semigroupoid pseudoidentities.

On the other hand, by Proposition 5.1, for a finite semigroupoid $S$, $S \in g(V \ast D)$ if and only if $S_{cd} \in V \ast D$, which, by the Delay Theorem, is in turn equivalent to $(S_{cd})_E \in gV$. Now, it is easy to show that, if $C$ is a finite category, the categories $C$ and $(C_{cd})_E$ divide each other (cf. [17, Proposition 16.1]). Hence, for a finite category $C$, $C \in g(V \ast D)$ if and only if $C \in gV$.

For each pseudoidentity in the set $\Sigma$, consider the pseudoidentities which can be obtained from it by collapsing edges to vertices, that is by removing edges of the underlying graph and identifying their end vertices. Let $\Sigma'$ be the set of all pseudoidentities obtained in this way from the pseudoidentities from $\Sigma$. Note that $\Sigma'$ is again a finite set. Then, as in [2, Proposition 7.1.5], one may show that, for a finite semigroupoid $S$, $S \models \Sigma'$ if and only if $S \models \Sigma'$.

Let now $S$ be a finite semigroupoid. Then

$$S \in gV \iff S^c \in gV \quad \text{since } V \text{ is monoidal}$$
$$\iff S^c \in g(V \ast D) \quad \text{as argued above}$$
$$\iff S^c \models \Sigma \quad \text{since } \Sigma \text{ is a basis for } g(V \ast D)$$
$$\iff S \models \Sigma' \quad \text{as argued above.}$$

Hence $gV = [\Sigma']$ and so $gV$ is finitely based. \[\square\]

### 6. Final remarks

While we are mostly concerned in this paper with the finite basis problem, it is worth observing, as suggested by the anonymous referee, that all our constructions of bases of pseudoidentities may be applied without a finite basis assumption so as to preserve computability. More precisely, if $gV$ is given by a basis of pseudoidentities which is a recursively enumerable set (within a suitable universe) and which consists of formal equalities between *computable* operations (i.e., which can be effectively computed on finite semigroupoids), then the bases for $V \ast D$ and $V \ast D_n$ constructed, respectively, in Sections 3 and 4, are also recursively enumerable sets and also consist of formal equalities between computable operations. Moreover, in each case, the complexity of the algorithms does not increase more than polynomially.
A similar remark applies to the construction of Section 5. Specifically, given a basis $\Sigma$ of pseudoidentities for a pseudovariety $V$ containing $B_2$ such that $\Sigma$ is a recursively enumerable set and each member of $\Sigma$ is a formal equality of computable operations, then the associated basis for $gV$ given by Theorem 5.10 is also a recursively enumerable set and consists of formal equalities between computable semigroupoid operations over (finite) computable graphs. Again, the construction does not increase complexity more than polynomially.

References