WELL-POSEDNESS OF KDV TYPE EQUATIONSS

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Abstract. In this work, we study the initial value problems associated to some linear perturbations of the KdV equations. Our focus is in the well-posedness issues for the initial data given in the L^2-based Sobolev spaces. We develop a method that allows us to treat the problem in the Bourgain's space associated to the KdV equation. With this method, we can use the multilinear estimates developed in the KdV context thereby getting analogous well-posedness results for the linearly perturbed equations.

1. Introduction

In this paper we consider the initial value problems (IVPs)

$$\begin{cases} v_t + v_{xxx} + \eta L v + (v^{k+1})_x = 0, \quad x \in \mathbb{R},\, t \geq 0, k \in \mathbb{Z}^+, \\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t + u_{xxx} + \eta L u + (u_x)^{k+1} = 0, \quad x \in \mathbb{R},\, t \geq 0, k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where $\eta > 0$ is a constant; $u = u(x,t)$, $v = v(x,t)$ are real valued functions and the linear operator L is defined via the Fourier transform by $\widehat{Lf}(\xi) = -\Phi(\xi)\hat{f}(\xi)$.

The Fourier symbol

$$\Phi(\xi) = \sum_{j=0}^{n}\sum_{i=0}^{2m} c_{i,j}\xi^i|\xi|^j, \quad c_{i,j} \in \mathbb{R},\ c_{2m,n} = -1, \tag{1.3}$$

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is a real valued function which is bounded above, i.e., there is a constant C such that $\Phi(\xi) < C$.

We observe that, if u is a solution to the IVP (1.2) then $v = u_x$ is a solution to the IVP (1.1) with initial data $v_0 = (u_0)_x$. That is why (1.1) is called as the derivative equation of (1.2).

In this work, we are interested in investigating the well-posedness results to the IVPs (1.2) and (1.1) for given data in the low regularity Sobolev spaces $H^s(\mathbb{R})$. Recall that, for $s \in \mathbb{R}$, the L^2-based Sobolev spaces $H^s(\mathbb{R})$ are defined by

$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s} := \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\Big)^{1/2},$$

and $\hat{f}(\xi)$ is the usual Fourier transform given by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

The factor $\frac{1}{\sqrt{2\pi}}$ in the definition of the Fourier transform does not alter our analysis, so we will omit it.

The notion of well-posedness we use is the standard one. We say that an IVP for given data in a Banach space X is locally well-posed, if there exists a certain time interval $[-T, T]$ and a unique solution depending continuously upon the initial data and the solution satisfies the persistence property, i.e., the solution describes a continuous curve in X in the time interval $[-T, T]$. If the above properties are true for any time interval, we say that the IVP is globally well-posed.

Before stating the main results of this work, we present some particular examples that belong to the class considered in (1.1) and (1.2) and discuss the known well-posedness results about them.

The first examples belonging to the classes (1.1) and (1.2) are

$$\begin{cases} v_t + v_{xxx} - \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) + (v^{k+1})_x = 0, \quad x \in \mathbb{R},\, t \geq 0,\, k \in \mathbb{Z}^+, \\ v(x,0) = v_0(x), \end{cases} \tag{1.4}$$

and

$$\begin{cases} u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + (u_x)^{k+1} = 0, & x \in \mathbb{R},\, t \geq 0,\, k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x), \end{cases} \tag{1.5}$$

respectively, where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}g(x) = \text{ P.V.}\frac{1}{\pi}\int \frac{g(x-\xi)}{\xi}d\xi;$$

$u = u(x,t)$, $v = v(x,t)$ are real-valued functions and $\eta > 0$ is a constant.

The equation in (1.4) with $k = 1$ was derived by Ostrovsky et al. [19] to describe the radiational instability of long waves in a stratified shear flow. Recently, Carvajal and Scialom [8] considered the IVP (1.4) and proved the local well-posedness results for given data in H^s, $s \geq 0$ when $k = 1, 2, 3$. They also obtained an *a priori* estimate for given data in $L^2(\mathbb{R})$ there by proving global well-posedness result. The earlier well-posedness results for the IVP (1.4) with $k = 1$ can be found in [1], where for given data in $H^s(\mathbb{R})$, local well-posedness when $s > 1/2$ and global well-posedness when $s \geq 1$ have been proved. In [1], the IVP (1.5), when $k = 1$, is also considered to prove global well-posedness for given data in $H^s(\mathbb{R})$, $s \geq 1$.

Another model that fits in the classes (1.2) and (1.1) respectively are the Korteweg-de Vries-Kuramoto Sivashinsky (KdV-KS) equation

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + (u_x)^2 = 0, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.6}$$

and its derivative equation

$$\begin{cases} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + vv_x = 0, & x \in \mathbb{R},\, t \geq 0, \\ v(x,0) = v_0(x), \end{cases} \tag{1.7}$$

where $u = u(x,t)$, $v = v(x,t)$ are real-valued functions and $\eta > 0$ is a constant.

The KdV-KS equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (see [11, 21]). The KdV-KS equation is very interesting in the sense that it combines the dispersive characteristics of the Korteweg-de Vries equation and dissipative characteristics of the Kuramoto-Sivashinsky equation. Also, it is worth noticing that the equation (1.7) is a particular case of the Benney-Lin equation [2, 21], i.e.

$$\begin{cases} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + \beta v_{xxxxx} + vv_x = 0, \quad x \in \mathbb{R}, \, t \geq 0, \\ v(x,0) = v_0(x), \end{cases} \tag{1.8}$$

when $\beta = 0$.

The IVPs (1.6) and (1.7) were studied by Biagioni, Bona, Iorio and Scialom [3]. The authors in [3] proved that the IVPs (1.6) and (1.7) are locally well-posed for given data in H^s, $s \geq 1$ with $\eta > 0$. They also constructed appropriate *a priori* estimates and used them to prove global well-posedness too. The limiting behavior of solutions as the dissipation tends to zero (i.e., $\eta \to 0$) has also been studied in [3]. The IVP (1.8) associated to the Benney-Lin equation is also widely studied in the literature [2, 4, 21]. Regarding well-posedness issues for the IVP (1.8) the work of Biagioni and Linares [4] is worth mentioning, where they proved global well-posedness for given data in $L^2(\mathbb{R})$.

Now, we state the main results of this work. The first result deals with the local well-posedness results for the IVP (1.1), while the second result deals the same for the IVP (1.2), with low regularity data.

Theorem 1.1. *Let $\eta > 0$ be fixed and $\Phi(\xi)$ be as given by* (1.3)*, then the IVP* (1.1) *is locally well-posed for any data $v_0 \in H^s(\mathbb{R})$, in the following cases:*

$$\begin{cases} k = 1, & s > -3/4, \\ k = 2, & s > 1/4, \\ k = 3, & s > -1/6, \\ k = 4, & s > 0. \end{cases}$$

Theorem 1.2. *Let $\eta > 0$ be fixed and $\Phi(\xi)$ be as given by* (1.3)*, then the IVP* (1.2) *is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, in the following cases:*

$$\begin{cases} k = 1, & s > 1/4, \\ k = 2, & s > 5/4, \\ k = 3, & s > 5/6, \\ k = 4, & s > 1. \end{cases}$$

The first main result, Theorem 1.1, deals with the quite general Fourier symbol and generalized nonlinearity. As discussed above, some particular cases are studied in the recent literature. In particular, the result of Theorem 1.1 improves the local well-posedness result for (1.4) with $k = 3$ obtained in [8]. It is worth noticing that, when $\eta = 0$ and $k = 2$, the IVP (1.1) turns out the modified KdV equation. We know that for the modified KdV equation local well-posedness holds for data in H^s, $s \geq 1/4$ and we have ill-posedness for $s < 1/4$. However, for $k = 2$, $\Phi(\xi) = |\xi| - |\xi|^3$ and $\eta > 0$ it has been proved in [8] that the local well-posedness holds for $s \geq 0$. Therefore, it would really be interesting to study the limiting behavior when $\eta \to 0$. As noted in [8], it is still an open problem.

At this point, we would like to note that the first main result for $k = 1$ is just the reproduction of our earlier result in [7]. Although the result presented in Theorem 1.1 in [7] is correct, in the due course of time, we found a misleading argument employed in the proof. More precisely, the estimate (2.5) in [7] was not as it should have been. In this work, this flaw has been corrected (see Lemma 2.3, below). This correction leads us to develop the contraction mapping scheme in the space $X_{s-p(b-\frac{1}{2}),b}$.

The second main result, Theorem 1.2, in particular, improves the local well-posedness results for (1.5) with $k = 1$ obtained in [1] and for (1.6) obtained in [3].

To prove the main results we follow the techniques used in [7]. The main idea is to use the theory developed by Bourgain [5] and Kenig, Ponce and Vega [17]. The main ingredients in the proof are estimates in the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$ (see IVPs (1.14) and (1.13) below). The main idea is to use the usual Bourgain space associated to the KdV equation instead of that associated to the linear part of the IVPs (1.1) and (1.2). To carry out this scheme, the Proposition 2.3 plays a fundamental role which permits us to use a multilinear estimates for $\partial_x(u^2)$, $\partial_x(u^3)$ $\partial_x(u^4)$ and $\partial_x(u^5)$ proved respectively in [17]), [20], [15] and [18].

As noted earlier, the IVPs (1.2) and (1.1) are globally well-posed for given data in $H^s(\mathbb{R})$, $s \geq 1$. As the models under consideration do not have conserved quantities, the global well-posedness have been proved by constructing appropriate *a priori* estimates. However, for given data in $H^s(\mathbb{R})$, $s < 1$ no *a priori* estimates are available. Also, the lack of conserved quantities prevent us to use the recently introduced $I - method$ [12, 13], to obtain global solution for the low regularity data.

Now we introduce function spaces that will be used to prove the main results. We consider the following IVP associated to the linear KdV equation

$$\begin{cases} w_t + w_{xxx} = 0, \quad x,\, t \in \mathbb{R}, \\ w(0) = w_0. \end{cases} \tag{1.9}$$

The solution to the IVP (1.9) is given by $w(x,t) = [U(t)w_0](x)$, where the unitary group $U(t)$ is defined as

$$\widehat{U(t)w_0}(\xi) = e^{it\xi^3}\widehat{w_0}(\xi). \tag{1.10}$$

For $s, b \in \mathbb{R}$, we define the space $X_{s,b}$ as the completion of the Schwartz space $S(\mathbb{R}^2)$ with respect to the norm

$$\begin{split} \|w\|_{X_{s,b}} \equiv \|U(-t)w\|_{H_{s,b}} :=& \|\langle\tau\rangle^b\langle\xi\rangle^s\widehat{U(-t)w}(\xi,\tau)\|_{L^2_\tau L^2_\xi} \\ =& \|\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\widehat{w}(\xi,\tau)\|_{L^2_\tau L^2_\xi}, \end{split} \tag{1.11}$$

where $\widehat{w}(\xi,\tau)$ is the Fourier transform of w in both space and time variables, and $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. The space $X_{s,b}$ is the usual Bourgain space for the KdV equation (see [5]) and using the Sobolev embedding theorem one has that $X_{s,b} \subset C(\mathbb{R}; H^s(\mathbb{R}))$, whenever $b > 1/2$.

Note that, the IVPs (1.2) and (1.1) are defined only for $t \geq 0$. To use Bourgain's type space, we should be able to write these IVPs for all $t \in \mathbb{R}$. For this, we define

$$\eta(t) \equiv \eta\,\mathrm{sgn}(t) = \begin{cases} \eta & \text{if } t \geq 0, \\ -\eta & \text{if } t < 0 \end{cases} \tag{1.12}$$

and write the IVPs (1.1) and (1.2) in the following forms

$$\begin{cases} v_t + v_{xxx} + \eta(t)Lv + (v^{k+1})_x = 0, \quad x,\, t \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ v(0) = v_0, \end{cases} \tag{1.13}$$

and

$$\begin{cases} u_t + u_{xxx} + \eta(t)Lu + (u_x)^{k+1} = 0, \quad x,\, t \in \mathbb{R},\ k \in \mathbb{Z}^+, \\ u(0) = u_0, \end{cases} \tag{1.14}$$

respectively. From here onwards we consider the IVPs (1.13) and (1.14) instead of (1.1) and (1.2) respectively.

Now we consider the IVP associated to the linear parts of (1.14) and (1.13)

$$\begin{cases} w_t + w_{xxx} + \eta(t)Lw = 0, \quad x,\, t \in \mathbb{R}, \\ w(0) = w_0. \end{cases} \tag{1.15}$$

The solution to (1.15) is given by $w(x,t) = V(t)w_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)w_0}(\xi) = e^{it\xi^3 + \eta|t|\Phi(\xi)}\widehat{w_0}(\xi). \tag{1.16}$$

Observe that, defining $\widetilde{U}(t)$ by $\widehat{\widetilde{U}(t)u_0}(\xi) = e^{\eta|t|\Phi(\xi)}\widehat{u_0}(\xi)$, the semigroup $V(t)$ can be written as $V(t) = U(t)\widetilde{U}(t)$ where $U(t)$ is the unitary group associated to the KdV equation (see (1.10)).

This paper is organized as follows: In Section 2, we prove some preliminary estimates and in Section 3 we prove the main results.

2. Preliminary estimates

This section is devoted to obtain some preliminary estimates that are essential in the proof of the main results.

Before going to details, we consider a cut-off function $\psi \in C^\infty(\mathbb{R})$, such that $0 \le \psi(t) \le 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \le 1, \\ 0 & \text{if } |t| \ge 2. \end{cases} \tag{2.1}$$

Also, we define $\psi_T(t) \equiv \psi(\frac{t}{T})$.

Let $p = 2m + n$, observe that the Fourier symbol given in (1.3) can be written as

$$\begin{aligned} \Phi(\xi) &= -\,|\xi|^p + \sum_{\substack{0 \le i \le 2m,\ 0 \le j \le n, \\ (i,j) \ne (2m,n)}} c_{i,j}\xi^i|\xi|^j, \quad c_{i,j} \in \mathbb{R}, \\ &= -\,|\xi|^p + \Phi_1(\xi), \end{aligned} \tag{2.2}$$

where the degree of Φ_1 is less than p. In what follows, we present some elementary lemmas.

Lemma 2.1. *There exists $M > 0$ such that for all $|\xi| \ge M$, one has that*

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi) < -1. \tag{2.3}$$

Proof. The inequality (2.3) is a direct consequence of

$$\lim_{|\xi|\to\infty} \frac{\Phi_1(\xi)+1}{|\xi|^p} = 0.$$

□

Lemma 2.2. *The Fourier symbol $\Phi(\xi)$ satisfies the following estimate*

$$\langle \Phi(\xi) \rangle \leq c\langle \xi \rangle^p. \tag{2.4}$$

Proof. It's not difficult see that

$$\begin{aligned} \langle \Phi(\xi)\rangle \leq & \langle |\xi|^p\rangle + \langle \Phi_1(\xi)\rangle \\ \leq & \langle \xi\rangle^p + \sum_{\substack{0\leq i\leq 2m,\ 0\leq j\leq n,\\ (i,j)\neq(2m,n)}} |c_{i,j}|\langle \xi^i|\xi|^j\rangle \\ \leq & \langle \xi\rangle^p + \sum_{\substack{0\leq i\leq 2m,\ 0\leq j\leq n,\\ (i,j)\neq(2m,n)}} |c_{i,j}|\langle \xi\rangle^{i+j} \\ \leq & \langle \xi\rangle^p \left(1 + \sum_{\substack{0\leq i\leq 2m,\ 0\leq j\leq n,\\ (i,j)\neq(2m,n)}} |c_{i,j}|\right). \end{aligned}$$

□

Lemma 2.3. *Let $0 < T \leq 1$, $1/2 \leq b \leq 1$ and $a \leq B$. Then we have*

$$\|\Psi_T(\cdot)e^{a|\cdot|}\|_{H_t^b} \leq ce^{2B}\big(T^{\frac{1}{2}-b} + |a|^{b-1/2}\big). \tag{2.5}$$

Proof. Let $h(t) = \Psi(t)\, e^{a|t|T}$, so that $h_T(t) = \Psi_T(t)\, e^{a|t|}$. A straight forward calculation yields

$$\big\|\Psi_T(\cdot)e^{a|\cdot|}\big\|_{H_t^b} = \|h_T\|_{H_t^b} \leq c\,T^{1/2}\|h\|_{L^2} + c\,T^{1/2-b}\big\|D_t^b h\big\|_{L^2}. \tag{2.6}$$

We know that

$$\|h\|_{L^2}^2 = \int_{-2}^{2} |\Psi(t)|^2\, e^{2a|t|T}\, dt \leq 4\, e^{4BT}\|\Psi\|_{L^\infty}^2. \tag{2.7}$$

To bound the term $\left\|D_t^b h\right\|_{L^2}$, we explore $\widehat{h}(\tau)$ by integrating by parts two times, and get

$$\begin{aligned}
\widehat{h}(\tau) &= \int_0^{+\infty} \Psi(t) e^{aTt} e^{-it\tau}\, dt + \int_{-\infty}^{0} \Psi(t) e^{-aTt} e^{-it\tau}\, dt \\
&= \frac{-1}{aT - i\tau}\left(1 + \int_0^{+\infty} \frac{d\Psi}{dt}(t)\, e^{t(aT-i\tau)}\, dt\right) - \frac{1}{aT + i\tau}\left(1 - \int_{-\infty}^{0} \frac{d\Psi}{dt}(t)\, e^{-t(aT+i\tau)}\, dt\right) \\
&= \frac{-2aT}{(aT)^2 + \tau^2} + \frac{1}{(aT - i\tau)^2}\int_0^{+\infty} \frac{d^2\Psi}{dt^2}(t)\, e^{t(aT-i\tau)}\, dt + \frac{1}{(aT + i\tau)^2}\int_{-\infty}^{0} \frac{d^2\Psi}{dt^2}(t)\, e^{-t(aT+i\tau)}\, dt.
\end{aligned}$$

¿From this we have that

$$|\widehat{h}(\tau)| \leq \frac{2|a|T}{(aT)^2 + \tau^2} + \frac{4\, e^{2BT}\left\|\frac{d^2\Psi}{dt^2}\right\|_{L^\infty}}{(aT)^2 + \tau^2}, \tag{2.8}$$

and

$$|\widehat{h}(\tau)| \leq 4\, e^{2BT}\|\Psi\|_{L^\infty} \leq ce^{2B}. \tag{2.9}$$

From (2.8) and (2.9), we obtain that

$$|\widehat{h}(\tau)| \leq \frac{2|a|T + ce^{2B}}{1 + (aT)^2 + \tau^2}. \tag{2.10}$$

Multiplying (2.10) by $|\tau|^b$, taking square and integrating on $\mathbb{R}$, we get

$$\begin{aligned}
\left\|D_t^b h\right\|_{L^2}^2 = \left\||\tau|^b \widehat{h}(\tau)\right\|_{L^2}^2 &\leq ca^2T^2 \int_{\mathbb{R}} \frac{|\tau|^{2b}}{(1 + a^2T^2 + \tau^2)^2}\, d\tau + ce^{4B}\int_{\mathbb{R}} \frac{|\tau|^{2b}}{(1 + a^2T^2 + \tau^2)^2}\, d\tau \\
&\leq ca^2T^2 \int_{\mathbb{R}} \frac{|\tau|^{2b}}{(a^2T^2 + \tau^2)^2}\, d\tau + ce^{4B}\int_{\mathbb{R}} \frac{|\tau|^{2b}}{(1 + \tau^2)^2}\, d\tau \\
&\leq c|aT|^{2b-1} + ce^{4B} \\
&\leq c\, e^{4B}\langle aT\rangle^{2b-1},
\end{aligned} \tag{2.11}$$

where in the second inequality we used $\tau = |a|Tx$. Thus

$$\|\Psi_T(\cdot)e^{a|\cdot|}\|_{H_t^b} \leq ce^{2B}\left(T^{1/2} + T^{1/2-b} + |a|^{b-1/2}\right). \tag{2.12}$$

Since $T \leq 1$, we conclude (2.5) from (2.6), (2.7), (2.11) and (2.12). □

Remark. *Considering* $T = 1$, *the estimate* (2.5) *yields*

$$\|\Psi_T(\cdot)e^{a|\cdot|}\|_{H_t^b} \leq ce^{2B}\langle a\rangle^{b-1/2}. \tag{2.13}$$

I what follows we present some results from the earlier works [9] and [7]. Before providing the exact announcement we gather some elementary estimates.

Proposition 2.1. *For any functions φ, g such that $\varphi g \in H^1$ and $supp\, \varphi \subset [-L, L]$ we have*

$$\|\varphi g\|_{L^2} \leq CL\|\frac{d}{dt}(\varphi g)|\|_{L^2}, \tag{2.14}$$

where C is independent of g, φ, L.

Proof. We have

$$\|\varphi g\|_{L^2}^2 = \int_{-L}^{L} |g(x)\,\varphi(x)|^2 dx \leq 2\,L\|\varphi\, g\|_{L^\infty}^2.$$

Now, using the known inequality $\|u\|_{L^\infty}^2 \leq c\|u\|_{L^2}\|u'\|_{L^2}$, we get

$$\|g\,\varphi\|_{L^2}^2 \leq C\,L\,\|g\,\varphi\|_{L^2}\|\frac{d}{dt}(g\,\varphi)\|_{L^2},$$

thereby getting the required estimate.

□

Lemma 2.4. *The following estimate holds true*

$$\|\Psi_T g\|_{H^1} \leq C\|\Psi_{2T} g\|_{H^1}. \tag{2.15}$$

Proof. We have

$$\|\Psi_T g\|_{H^1} \sim \|\Psi_T g\|_{L^2} + \|\frac{d}{dt}(\Psi_T g)\|_{L^2}.$$

It is obvious that $\|\Psi_T g\|_{L^2} \leq \|\Psi_{2T} g\|_{L^2}$. Thus to get the desired estimate (2.15) it is enough to prove that

$$\|\frac{d}{dt}(\Psi_T g)\|_{L^2} \leq C\|\frac{d}{dt}(\Psi_{2T} g)\|_{L^2}. \tag{2.16}$$

In order to prove (2.16), observe that in the support of Ψ_T one has $g = g\Psi_{2T}$. On the other hand

$$\begin{aligned} \|\frac{d}{dt}(\Psi_T g)\|_{L^2} &= \|\frac{d}{dt}(\Psi_T) g + \Psi_T \frac{d}{dt}(g)\|_{L^2} \\ &\leq \|\frac{d}{dt}(\Psi_T) g\|_{L^2} + \|\Psi_T \frac{d}{dt}(g)\|_{L^2}. \end{aligned} \tag{2.17}$$

¿From the observation above ($g = g\Psi_{2T}$ in the support of Ψ_T) we get

$$\|\Psi_T \frac{d}{dt}(g)\|_{L^2} = \|\Psi_T \frac{d}{dt}(g\Psi_{2T})\|_{L^2} \leq \|\frac{d}{dt}(g\Psi_{2T})\|_{L^2}. \tag{2.18}$$

We have that

$$\begin{aligned}\|\Psi'(\frac{t}{T})g\|_{L^2}^2 &= \int_{\mathbb{R}} |\Psi'(\frac{t}{T})|^2 |g(t)\Psi_{2T}|^2 dt\\ &\leq \|g\Psi_{2T}\|_{L^\infty}^2 \int_{\mathbb{R}} |\Psi'(\frac{t}{T})|^2 dt\\ &= T\|g\Psi_{2T}\|_{L^\infty}^2 \int_{\mathbb{R}} |\Psi'(\tau)|^2 d\tau\\ &\leq C_{\Psi'} T\|g\Psi_{2T}\|_{L^\infty}^2. \end{aligned}\tag{2.19}$$

Now, using the known inequality $\|u\|_{L^\infty}^2 \leq c\|u\|_{L^2}\|u'\|_{L^2}$; from (2.19) and (2.14) it follows that

$$\begin{aligned}\|\Psi'(\frac{t}{T})g\|_{L^2}^2 &\leq C_{\Psi'} T\|g\Psi_{2T}\|_{L^2}\|\frac{d}{dt}(g\Psi_{2T})\|_{L^2}\\ &\leq C_{\Psi'} T^2\|\frac{d}{dt}(g\Psi_{2T})\|_{L^2}^2,\end{aligned}$$

and this completes the proof of the lemma. □

Proposition 2.2. *Let $0 \leq b \leq 1$, $B_1 \leq B_2 \leq 0$. Then,*

$$\left\|\Psi_T(t)\int_0^t e^{B_1|t-x|} f(x)\,dx\right\|_{H^b} \leq C\,(1+T)\left\|\Psi_{2T}(t)\int_0^t e^{B_2\,|t-x|} f(x)\,dx\right\|_{H^b}, \tag{2.20}$$

where $C = C_\Psi = C\max\Big\{\|\Psi\|_{L^\infty}, \|\frac{d\Psi}{dt}\|_{L^\infty}\Big\}$ is a constant independent of B_1, B_2 and f.

Proof. For the proof of this result follows by using estimate (2.15) from Lemma 2.4. For details we refer to [9]. □

Lemma 2.5. *Let $-1/2 < b' \leq 0$, $1/2 < b \leq b'/3 + 2/3$, $T \in (0,1]$, $|a| < B$. Then*

$$\|\psi_T(t)\int_0^t e^{a|t-t'|} f(t')dt'\|_{H_t^b} \leq c_{B,\psi} T^{1+b'/2-3b/2}\|f\|_{H^{b'}}, \tag{2.21}$$

where $c_{B,\psi}$ is a constant independent of a, f and T.

Proof. A detailed proof of this lemma has been presented in [7], so omit it. □

We start with following Proposition that plays a central role in the proof of the main results of this work. The result of this Proposition allows us to work in the usual $X_{s,b}$ space associated to the KdV group $U(t)$ defined by (1.10) instead of the Bourgain space associated to the group $V(t)$ defined by (1.16).

Proposition 2.3. *Let $b > 1/2$ and $-1/2 < b' \le 0$, $T \in (0,1]$. Then we have*

$$\|\psi(t)V(t)u_0\|_{X_{s,b}} \le c\|u_0\|_{s+p(b-1/2)}. \tag{2.22}$$

If $1/2 < b \le b'/3 + 2/3$, $s \in \mathbb{R}$ then

$$\|\psi_T(t)\int_0^t V(t-t')F(t')dt'\|_{X_{s,b}} \le c\,T^{1+b'/2-3b/2}\|F\|_{X_{s,b'}}, \tag{2.23}$$

where c is a constant.

Proof. In order to prove (2.22), we have

$$\|\psi(t)V(t)u_0\|_{X_{s,b}} = \|\langle\xi\rangle^s\widehat{u_0}(\xi)\|\psi(t)e^{\Phi(\xi)\eta|t|}\|_{H_t^b}\|_{L_\xi^2}.$$

Using Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}\|\psi(t)V(t)u_0\|_{X_{s,b}} &\le \|\langle\xi\rangle^s\widehat{u_0}(\xi)\,\big\|\psi(t)e^{\Phi(\xi)\eta|t|}\big\|_{H_t^b}\|_{L_\xi^2}\\ &\le c\|\langle\xi\rangle^s\widehat{u_0}(\xi)\langle\Phi(\xi)\rangle^{b-\frac{1}{2}}\|_{L_\xi^2}\\ &\le c\|\langle\xi\rangle^s\widehat{u_0}(\xi)\langle\xi\rangle^{p(b-\frac{1}{2})}\|_{L_\xi^2},\end{aligned}$$

and this proves (2.22).

Now, in order to prove (2.23), let M be as in Lemma 2.1. From definition of Bourgain's space, we have that:

$$\begin{aligned}\|\psi_T(t)\int_0^t V(t-t')F(t')dt'\|_{X_{s,b}} &= \|\langle\xi\rangle^s\|\psi_T(t)\int_0^t e^{-it'\xi^3}e^{\Phi(\xi)\eta|t-t'|}\widehat{F(t')}(\xi)dt'\|_{H_t^b}\|_{L_\xi^2}\\ \le&\|\langle\xi\rangle^s\|\psi_T(t)\int_0^t e^{-it'\xi^3}e^{\Phi(\xi)\eta|t-t'|}\widehat{F(t')}(\xi)dt'\|_{H_t^b}\|_{L_\xi^2(|\xi|<M)}\\ &+\|\langle\xi\rangle^s\|\psi_T(t)\int_0^t e^{-it'\xi^3}e^{\Phi(\xi)\eta|t-t'|}\widehat{F(t')}(\xi)dt'\|_{H_t^b}\|_{L_\xi^2(|\xi|\ge M)}\\ =:&I_1+I_2.\end{aligned} \tag{2.24}$$

To estimate I_1, note that for $|\xi| < M$, one has

$$|\Phi(\xi)| \le \sum_{j=0}^{n}\sum_{i=0}^{2m}|c_{i,j}|\,|\xi|^i|\xi|^j \le \sum_{j=0}^{n}\sum_{i=0}^{2m}|c_{i,j}|\,M^{i+j} =: c_M.$$

Therefore, using Lemma 2.5, we obtain

$$\begin{aligned}I_1 \le& c_{M,\psi}T^{1+b'/2-3b/2}\|\langle\xi\rangle^s\|e^{-it\xi^3}\widehat{F(t)}\|_{H_t^{b'}}\|_{L_\xi^2(|\xi|\le M)}\\ \le& c_{M,\psi}T^{1+b'/2-3b/2}\|F\|_{X_{s,b'}}.\end{aligned}$$

To estimate I_2, we observe that for $|\xi| \geq M$, one can write the Fourier symbol as $\Phi(\xi) = (\Phi(\xi)+1) - 1$, where from Lemma 2.1 $\Phi(\xi) + 1 < 0$. Now, using Proposition 2.2 and Lemma 2.5, we get

$$\begin{aligned} I_2 \leq & a(1+T)\|\langle\xi\rangle^s\|\psi_{2T}(t)\int_0^t e^{-it'\xi^3}e^{-\eta|t-t'|}\widehat{F(t')}(\xi)dt'\|_{H_t^b}\|_{L^2_\xi(|\xi|\geq M)} \\ \leq & c_\psi T^{1+b'/2-3b/2}\|F\|_{X_{s,b'}}. \end{aligned} \tag{2.25}$$

□

In what follows we record the familiar multilinear estimate in the Bourgain's space associated to the KdV group.

Proposition 2.4. *Let $k = 1, 2, 3, 4$ and $s > a_k$. There exist $\gamma \in (\frac{1}{2}, 1)$ and $r(s) > 0$ such that if b, b' are two numbers satisfying $1/2 < b \leq b' + 1 < \gamma$ and $b' + 1/2 \leq r(s)$, then for $u \in X_{s,b}$, the following estimate holds*

$$\|(u^{k+1})_x\|_{X_{s,b'}} \leq c\,\|u\|^{k+1}_{X_{s,b}}, \tag{2.26}$$

where,

$$a_1 = -\frac{3}{4}, \quad a_2 = \frac{1}{4}, \quad a_3 = -\frac{1}{6}, \quad a_4 = 0. \tag{2.27}$$

Proof. For the proof we refer to [17, 14], [20], [15] and [18] respectively for $k = 1$, $k = 2$, $k = 3$ and $k = 4$. □

Before providing another multilinear estimate to prove Theorem 1.2, we introduce some new notations from [20] and auxiliary results.

For any abelian additive group Z with an invariant measure $d\xi$, we use $\Gamma_k(Z)$ to denote the hyperplane

$$\Gamma_k(Z) := \{(\xi_1, \cdots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\}, \quad k \geq 2,$$

endowed with the measure

$$\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) d\xi_1 \cdots d\xi_{k-1}.$$

We define a $[k; Z]$ – *multiplier* to be any function $m : \Gamma_k(Z) \to \mathbb{C}$ and also define $\|m\|_{[k;Z]}$ to be the best constant such that the inequality

$$|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^k f_j(\xi_j)| \leq \|m\|_{[k;Z]}\prod_{j=1}^k \|f_j\|_{L^2(Z)},$$

holds for all test functions f_j on Z. Note that, in our case the abelian group Z will be Euclidean space $\mathbb{R}^{n+1}$ with Lebesgue measure.

In what follows, we state in the form of Lemmas, some properties satisfied by the $[k;Z]-$ *multiplier*, whose proof can be found in [20].

Lemma 2.6 (Comparison principle). *If m and M are $[k;Z]$ multipliers such that $|m(\xi)| \leq M(\xi)$ for all $\xi \in \Gamma_k(Z)$, then $\|m\|_{[k;Z]} \leq \|M\|_{[k;Z]}$ and*

$$\|m(\xi)\prod_{j=1}^{k} a_j(\xi_j)\|_{[k;Z]} \leq \|m\|_{[k;Z]}\prod_{j=1}^{k}\|a_j\|_{\infty},$$

where $a_1, \cdots, a_k$ are functions from Z to $\mathbb{R}$.

Lemma 2.7. *For any $[k;Z]-$ multiplier $m : Z^k \to \mathbb{R}$, the following properties hold true.*

(1) *TT^* identity:*

$$\|m(\xi_1,\dots,\xi_k)\overline{m(-\xi_{k+1},\dots,-\xi_{2k})}\|_{[2k;Z]} = \|m(\xi_1,\dots,\xi_k)\|^2_{[k+1;Z]}.$$

(2) *Translation invariance:*

$$\|m(\xi)\|_{[k;Z]} = \|m(\xi+\xi_0)\|_{[k;Z]},$$

for any $\xi_0 \in \Gamma_k(Z)$.

(3) *Averaging:*

$$\|m*\mu\|_{[k;Z]} \leq \|m\|_{[k;Z]}\|\mu\|_{L^1(\Gamma_k(Z))},$$

for any finite measure μ on $\Gamma_k(Z)$.

The following proposition is crucial in proving multilinear estimates that are essential in the proof of the second main result of this work.

Proposition 2.5. *Let $k = 2,3,4,5$. Under the hypothesis of the Proposition 2.4, we have*

$$\|\prod_{j=1}^{k} u_j\|_{X_{s,b'}} \leq c\prod_{j=1}^{k}\|u_j\|_{X_{s,b}}, \quad s > s_k, \tag{2.28}$$

where $s_2 = -3/4$, $s_3 = 1/4$, $s_4 = -1/6$, $s_5 = 0$.

Proof. To prove the estimate (2.28), we will use the techniques developed by Tao in [20] on $[k, Z]$ multipliers.

Consider $u_j \in X_{s,b}$ for $j = 1, \dots k$, $u_{k+1} \in X_{-s,-b'}$ and use properties of the Fourier transform, to obtain

$$\int_{\mathbb{R}^2} (\prod_{j=1}^{k} u_j)(\xi,\tau)\overline{u_{k+1}}(\xi,\tau)d\xi d\tau = \int_{\mathbb{R}^2}\int_{\mathbb{R}^{(k-1)\times(k-1)}} \widehat{u_1}(\xi_1,\tau_1)\widehat{u_2}(\xi_2,\tau_2)\dots$$

$$\widehat{u_k}(\xi - \sum_{j=1}^{k-1}\xi_j, \tau - \sum_{j=1}^{k-1}\tau_j)\widehat{u_{k+1}}(-\xi,-\tau)d\xi_1 d\tau_1 \dots d\xi_{k-1}d\tau_{k-1}d\xi d\tau$$

$$=: \int_{\substack{\xi_1+\xi_2+\dots+\xi_{k+1}=0\\ \tau_1+\tau_2+\dots+\tau_{k+1}=0}} \prod_{j=1}^{k+1} \widehat{u_j}(\xi_j,\tau_j)d\xi_1 d\tau_1\dots d\xi_4 d\tau_4,$$

Therefore, using duality proving (2.28) is equivalent to proving

$$\int_{\substack{\xi_1+\xi_2+\dots+\xi_{k+1}=0\\ \tau_1+\tau_2+\dots+\tau_{k+1}=0}} \prod_{j=1}^{k+1} \widehat{u_j}(\xi_j,\tau_j)d\xi_1 d\tau_1\dots d\xi_{k+1} d\tau_{k+1} \lesssim \prod_{j=1}^{k} \|u_j\|_{X_{s,b}}\|u_{k+1}\|_{X_{-s,-b'}}.$$

Let

$$\langle\xi_j\rangle^s\langle\tau_j-\xi_j^3\rangle^b\widehat{u_j}(\xi,\tau) = \widehat{f_j}(\xi,\tau), \quad j=1,\dots,k,$$
$$\langle\xi_{k+1}\rangle^{-s}\langle\tau_{k+1}-\xi_{k+1}^3\rangle^{-b'}\widehat{u_{k+1}}(\xi,\tau) = \widehat{f_{k+1}}(\xi,\tau).$$

Now with these considerations, proving (2.28) is equivalent to proving

$$\int_{\substack{\xi_1+\xi_2+\dots+\xi_{k+1}=0\\ \tau_1+\tau_2+\dots+\tau_{k+1}=0}} m((\xi_1,\tau_1),\dots(\xi_{k+1},\tau_{k+1}))\prod_{j=1}^{k+1} \widehat{f_j}(\xi_j,\tau_j)d\xi_1 d\tau_1\dots d\xi_{k+1} d\tau_{k+1} \lesssim \prod_{j=1}^{k+1}\|f_j\|_{L^2_{x,t}}, \tag{2.29}$$

where

$$m((\xi_1,\tau_1),\dots(\xi_{k+1},\tau_{k+1})) = \frac{\langle\xi_{k+1}\rangle^s}{\prod_{j=1}^{k}\langle\xi_j\rangle^s\prod_{j=1}^{k+1}\langle\tau_j-\xi_j^3\rangle^{b_j}}, \tag{2.30}$$

and $b_1 = \dots = b_k = b$, $b_{k+1} = -b'$.

So, we need to prove that the $[k+1,\mathbb{R}^2] - \textit{multiplier}$ estimate is finite, i.e.,

$$\|m\|_{[k+1;\mathbb{R}^2]} < \infty.$$

We know from Proposition 2.4 that the following $[k+1,\mathbb{R}^2] - \textit{multiplier}$ estimate

$$\tilde{m}((\xi_1,\tau_1),\dots(\xi_{k+1},\tau_{k+1})) = \frac{|\xi_{k+1}|\langle\xi_{k+1}\rangle^s}{\prod_{j=1}^{k}\langle\xi_j\rangle^s\prod_{j=1}^{k+1}\langle\tau_j-\xi_j^3\rangle^{b_j}}, \tag{2.31}$$

where $b_1 = \dots = b_k = b$, $b_{k+1} = -b'$; is finite.

Observe that we may restrict the multiplier (2.30) to the region $|\xi_{k+1}| \geq 1$, (since the general case then follows by an averaging over unit time scales). The $|\xi_j| \leq 1$ behavior of m is usually identical to its $|\xi_j| \sim 1$ behavior, see Section 4 on $X_{s,b}$ spaces in [20] pag 17).

In the high frequencies, we have

$$m \leq \tilde{m},$$

and the Comparison principle implies that $\|m\|_{[k+1;\mathbb{R}^2]} < \infty$ as required. □

Remark. *We note that the multilinear estimates without derivative hold in the $X_{s,b}$ spaces with low regularity than that with derivative. For example, in the case $k = 3$ the inequality* (2.28) *holds true for $s > -1/4$, see* [6], *and with derivative holds for $s \geq 1/4$, see* (2.26) *in Proposition 2.4 above.*

The following Lemma is an immediate consequence of Propositions 2.4 and 2.5 and will be used in the proof of Theorem 1.2.

Lemma 2.8. *Let $k = 1, 2, 3, 4$. Under the hypothesis of the Proposition 2.4, we have*

$$\|(u_x)^{k+1}\|_{X_{s,b'}} \leq c\|u\|_{X_{s,b}}^{k+1}, \tag{2.32}$$

whenever,

$$\begin{cases} k = 1, & s > 1/4, \\ k = 2, & s > 5/4, \\ k = 3, & s > 5/6, \\ k = 4, & s > 1. \end{cases} \tag{2.33}$$

Proof. Let $k = 1, 2, 3, 4$, and consider s satisfying (2.33). As $\langle\xi\rangle^s = \langle\xi\rangle^{s-1}\langle\xi\rangle$, we have

$$\|(u_x)^{k+1}\|_{X_{s,b'}} \leq \|D_x(u_x)^{k+1}\|_{X_{s-1,b'}} + \|(u_x)^{k+1}\|_{X_{s-1,b'}}. \tag{2.34}$$

For the first term we have

$$\|D_x(u_x)^{k+1}\|_{X_{s-1,b'}} \leq c\,\|u_x\|_{X_{s-1,b}}^{k+1} \leq c\|u\|_{X_{s,b}}^{k+1}, \tag{2.35}$$

where in the first inequality the bilinear estimate (2.26) has been used.

In order to estimate the second term in (2.34), we use (2.28) to obtain

$$\|(u_x)^{k+1}\|_{X_{s-1,b'}} \leq c\|u_x\|_{X_{s-1,b}}^{k+1} \leq c\|u\|_{X_{s,b}}^{k+1}, \tag{2.36}$$

which completes the proof of (2.32). □

3. Proof of the main results.

Proof of Theorem 1.1. As discussed in the introduction, we will use Bourgain's space associated to the KdV group to prove well-posedness for the IVP (1.1), therefore we need to consider the IVP (1.13) that is defined for all t. Now consider the IVP (1.13) in its equivalent integral form

$$v(t) = V(t)v_0 - \int_0^t V(t-t')(v^{k+1})_x(t')dt', \tag{3.1}$$

where $V(t)$ is the semigroup associated with the linear part given by (1.16).

Note that, if for all $t \in \mathbb{R}$, $v(t)$ satisfies

$$v(t) = \psi(t)V(t)v_0 - \psi_T(t)\int_0^t V(t-t')(v^{k+1})_x(t')dt',$$

with $T \in (0,1]$, then $v(t)$ satisfies (3.1) in $[-T,T]$. We define an application

$$\Psi(v)(t) = \psi(t)\, V(t)v_0 - \psi_T(t)\, \int_0^t V(t-t')(v^{k+1})_x(t')dt'.$$

Let $k \in \{1,2,3,4\}$, and $s > a_k$, where a_k is given by (2.27). Let $v_0 \in H^s$ and let us define $b := 1/2+\epsilon$, $b' := -1/2+4\epsilon$, with $0<\epsilon \ll 1$ satisfying

$$0 < \epsilon < \min\Big\{\frac{s-a_k}{p}, \frac{1}{4}\Big(\frac{3}{4} - \frac{s_1}{3}\Big)\Big\}, \tag{3.2}$$

where γ and $r(s)$ are as in Proposition 2.4. With this choose of b and b' it is easy to verify that all the conditions of Propositions 2.3 and 2.4, and Lemma 2.5 are satisfied. For $M>0$, let us define a ball

$$X^M_{s-p(b-\frac12),b} = \{f \in X_{s-p(b-\frac12),b};\ \|f\|_{X_{s-p(b-\frac12),b}} \le M\}.$$

We will prove that there exists M such that the application Ψ maps $X^M_{s-p(b-\frac12),b}$ into $X^M_{s-p(b-\frac12),b}$ and is a contraction. Let $v \in X^M_{s-p(b-\frac12),b}$. By using Proposition 2.3, we get

$$\|\Psi(v)\|_{X_{s-p(b-\frac12),b}} \le c\|v_0\|_{H^s} + c\,T^\alpha \|(v^{k+1})_x\|_{X_{s-p(b-\frac12),b'}}, \tag{3.3}$$

where $\alpha := 1 + \frac{b'}{2} - \frac{3b}{2} = \frac{\epsilon}{2} > 0$. The use of Proposition 2.4 in (3.3) yields

$$\|\Psi(v)\|_{X_{s-p(b-\frac12),b}} \le c\|v_0\|_{H^s} + c\,T^\alpha \|v\|^{k+1}_{X_{s-p(b-\frac12),b}}, \tag{3.4}$$

whenever

$$\begin{cases} s-p(b-\frac{1}{2}) > -3/4, & \text{for, } k=1,\\ s-p(b-\frac{1}{2}) > 1/4, & \text{for, } k=2,\\ s-p(b-\frac{1}{2}) > -1/6, & \text{for, } k=3,\\ s-p(b-\frac{1}{2}) > 0, & \text{for, } k=4, \end{cases} \tag{3.5}$$

holds, which is true because of the choice of b and arbitrarily small ϵ satisfying (3.2). Now, using the definition of $X^M_{s-p(b-\frac{1}{2}),b}$, one obtains

$$\|\Psi(v)\|_{X_{s-p(b-\frac{1}{2}),b}} \leq \frac{M}{4} + cT^\alpha M^{k+1} \leq \frac{M}{2}, \tag{3.6}$$

where we have chosen $M = 4c\|v_0\|_{H^s}$ and $cT^\alpha M^k = 1/4$. Therefore, from (3.6) we see that the application Ψ maps $X_{s-p(b-\frac{1}{2}),b}$ into itself. A similar argument proves that Ψ is a contraction. Hence Ψ has a fixed point v which is a solution of the IVP (1.1) such that $u \in C([-T,T], H^{s-p(b-\frac{1}{2})})$.

Since $\epsilon > 0$ is arbitrarily small satisfying (3.2) and $b = \frac{1}{2} + \epsilon$, this concludes the proof of the theorem. □

Proof of Theorem 1.2. This proof is analogous to that of Theorem 1.1. The only difference is that, in this case, we use Lemma 2.8 instead of Proposition 2.4. □

4. A priori estimate: global solutions

In this section we find an *a priori* estimate that leads to conclude global well-posedness of the IVPs (1.1) and (1.2).

Lemma 4.1. *Let $v_0 \in H^3(\mathbb{R})$ and $v \in C([0,T], H^3(\mathbb{R}))$ be the solution of (1.1) with initial data $v(x,0) = v_0$. Then the following a priori estimate*

$$\|v(t)\|_{L^2} \leq C\|v_0\|_{L^2} e^{C\eta T}, \tag{4.1}$$

holds true.

Proof. We multiply the equation (1.1) by v and integrate by parts to get

$$\frac{1}{2}\frac{d}{dt}\int v^2(x)dx + \eta\int v(x)Lv(x)dx = 0. \tag{4.2}$$

Now using our assumption on the Fourier symbol Φ of L from (1.3), Plancherel's identity we obtain from (4.2) that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = \eta\int \widehat{v}(\xi)\Phi(\xi)\bar{\widehat{v}}(\xi)d\xi \leq C\eta\int\int \widehat{v}(\xi)\bar{\widehat{v}}(\xi)d\xi = C\eta\|v(t)\|_{L^2}^2. \tag{4.3}$$

Now, integrating (4.3) in $[0,t]$ for $t\in[0,T]$, and applying Gronwall's inequality, we obtain the required an *a priori* estimate (4.1). □

Remark. *As in Lemma 4.1, differentiating equation* (1.2) *with respect to* x*, multiplying the resulting equation by* u_x *and the integrating by parts and using Plancherel's identity and Gronwall's inequality, we obtain the following an* a priori *estimate*

$$\|\partial_x u(t)\|_{L^2} \leq C\|\partial_x u_0\|_{L^2} e^{C\eta T}. \tag{4.4}$$

Now, with the *a priori* estimates (4.1) and (4.4) at hand, one can prove the following global results for the IVPs (1.1) and (1.2) for some particular values of k.

Theorem 4.1. *Let* $k = 1, 3$*, and* $v_0 \in H^s(\mathbb{R})$*,* $s \geq 0$*, then the local solution to the IVP* (1.1) *obtained in Theorem 1.1 can be extended globally in time.*

Theorem 4.2. *Let* $k = 1, 3$*, and* $u_0 \in H^s(\mathbb{R})$*,* $s \geq 1$*, then the local solution to the IVP* (1.2) *obtained in Theorem 1.2 can be extended globally in time.*

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