UNIVERSALITY OF KPZ EQUATION AND RENORMALIZATION TECHNIQUES IN INTERACTING PARTICLE SYSTEMS

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1. Introduction

In the middle eighties, Kardar, Parisi and Zhang in [7] proposed a phenomenological model for the stochastic evolution of the profile of a growing interface $h_t(x)$. The Kardar, Parisi and Zhang (KPZ) equation has the following form in one dimension: $\partial_t h = D\Delta h + a\big(\nabla h\big)^2 + \sigma \mathcal{W}_t$, where $\mathcal{W}_t$ is a space-time white noise and the constants D, a, σ are related to some thermodynamic properties of the interface. The quantity $h_t(x)$ represents the *height* of the interface at the point $x \in \mathbb{R}$. From a mathematical point of view, this equation is ill-posed, since the solutions are expected to look locally like a Brownian motion, and in this case the nonlinear term does not make sense, at least not in a classical sense.

In dimension $d = 1$, a conservative version of the KPZ equation can be obtained by defining $\mathcal{Y}_t = \nabla h_t$: $\partial_t \mathcal{Y}_t = D\Delta \mathcal{Y}_t + a\nabla \mathcal{Y}_t^2 + \sigma \nabla \mathcal{W}_t$. This equation has spatial white noise as an invariant solution. In this case is even clearer that some procedure is needed in order to define $\mathcal{Y}_t^2$ in a proper way. It is widely believed in the physics community that the KPZ equation governs the large-scale properties of one-dimensional, weakly asymmetric, conservative systems in great generality. The microscopic details of each model should only appear through the values of the constants D, a and σ. In this work we provide a new approach which is robust enough to apply for a wide family of one-dimensional weakly asymmetric systems. As a stochastic partial differential equation, the main problem with the KPZ equation is the definition of the square $\mathcal{Y}_t^2$.

Our first contribution is the notion of *energy solutions* of the KPZ equation, which we introduce in order to state in a rigorous way our second contribution. Take a one-dimensional, weakly asymmetric conservative particle system and consider the rescaled space-time fluctuations of the density field $\mathcal{Y}_t^n$. When the strength of the asymmetry is of order $1/\sqrt{n}$, we prove that any limit point of $\mathcal{Y}_t^n$ is an energy solution of the KPZ equation. The only ingredients needed in order to prove this result are a sharp estimate on the spectral gap of the dynamics of the particle system restricted to finite boxes and a strong form of the equivalence of ensembles for the stationary distribution. Therefore, our approach works, modulo technical modifications, for any one-dimensional, weakly asymmetric conservative particle system satisfying these two properties. As a consequence, we say that energy solutions of the KPZ equation are *universal*, in the sense that they arise as the scaling limit of the density in one-dimensional,

Date: November 2010.

weakly asymmetric conservative systems satisfying fairly general, minimal assumptions.

In order to prove this result, we introduce a new mathematical tool, which we call *second-order Boltzmann-Gibbs principle*. The usual Boltzmann-Gibbs principle, introduced in [1] and proved in [3] in our context, basically states that the space-time fluctuations of any field associated to a conservative model can be written as a linear functional of the density field $\mathcal{Y}_t^n$. A stronger Boltzmann-Gibbs Principle was derived in [4], which implies that for strength asymmetry less than $1/\sqrt{n}$, the limit field falls into the Edwars-Wilkinson universality class [5]. Our second-order Boltzmann-Gibbs principle states that the first-order correction of this limit is given by a singular, quadratic functional of the density field. It has been proved that in dimension $d \geq 3$, this first order correction is given by a white noise [2]. As a consequence for strength asymmetry $1/\sqrt{n}$ the system fall into the KPZ universality class [6].

2. THE RESULTS

2.1. The process. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the state space of a continuous-time Markov chain η_t which we will define as follows. We say that a function $f : \Omega \to \mathbb{R}$ is *local* if there exists $R = R(f) > 0$ such that $f(\eta) = f(\xi)$ for any $\eta, \xi \in \Omega$ such that $\eta(x) = \xi(x)$ whenever $|x| \geq R$. Let $c : \Omega \to \mathbb{R}$ be a non-negative function. We assume the following conditions on c:

i) *Ellipticity*: There exists $\epsilon_0 > 0$ such that $\epsilon_0 \leq c(\eta) \leq \epsilon_0^{-1}$ for any $\eta \in \Omega$.
ii) *Finite range*: The function $c(\cdot)$ is local.
iii) *Reversibility*: For any $\eta, \xi \in \Omega$ such that $\eta(x) = \xi(x)$ whenever $x \neq 0, 1$, $c(\eta) = c(\xi)$.

For any $x \in \mathbb{Z}$ let $\tau_x f(\eta) = f(\tau_x \eta)$ for any $\eta \in \Omega$, where $\tau_x \eta$ denotes the space translation by x. We will also assume a fourth condition, which is the most restrictive one:

iv) *Gradient condition*: There exists a local function $h : \Omega \to \Omega$ such that $c(\eta)(\eta(1) - \eta(0)) = \tau_1 h(\eta) - h(\eta)$ for any $\eta \in \Omega$.

In this work, we consider the Markov process $\{\eta_t^n; t \geq 0\}$ generated by the operator L_n acting over local functions $f : \Omega \to \mathbb{R}$ as

$$L_n f(\eta) = n^2 \sum_{x \in \mathbb{Z}} \tau_x c(\eta) \big\{ p_n \eta(x)(1 - \eta(x+1)) + q_n \eta(x+1)(1 - \eta(x)) \big\} \nabla_{x,x+1} f(\eta),$$

where $n \in \mathbb{N}$, $\nabla_{x,x+1} f(\eta) = f(\eta^{x,x+1}) - f(\eta)$, p_n and q_n are non-negative constants such that $p_n + q_n = 1$ (and $p_n - q_n = a/\sqrt{n}$ with $a \neq 0$) and $\eta^{x,x+1}$ is given by $\eta^{x,x+1}(x) = \eta(x+1)$, $\eta^{x,x+1}(x+1) = \eta(x)$, otherwise $\eta^{x,x+1}(z) = \eta(z)$.

For $\rho \in [0,1]$ let ν_ρ be the Bernoulli product measure in Ω of parameter ρ. Under condition iii), the measures $\{\nu_\rho; \rho \in [0,1]\}$ are invariant and reversible with respect to the evolution of η_t^n. Under condition i), these measures are also ergodic with respect to the evolution of η_t^n.

2.2. Equilibrium Fluctuations. In this work we are interested in a central limit theorem for the density of particles starting from the equilibrium state ν_ρ. Let us fix a density $\rho \in (0,1)$ and let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of test functions and let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions in $\mathbb{R}$, which corresponds to the topological dual of $\mathcal{S}(\mathbb{R})$. The fluctuation field $\{\mathcal{Y}_t^n; t \geq 0\}$ is defined as the $\mathcal{S}'(\mathbb{R})$-valued process given by

$$\mathcal{Y}_t^n(G) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \big(\eta_t^n(x) - \rho\big) G(x/n - v(\rho) t n^{1/2})$$

where $v(\rho) = a\beta'(\rho)$. Our main result in this work says that the sequence of processes $\{\{\mathcal{Y}_t^n; t \in [0,T]\}; n \in \mathbb{N}\}$ is tight in $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ and any limit point is a stationary energy solution of the KPZ equation:

$$d_t \mathcal{Y}_t = \frac{\varphi'(\rho)}{2} \Delta \mathcal{Y}_t dt - \frac{a\beta''(\rho)}{2} \nabla \mathcal{Y}_t^2 dt + \sqrt{\chi(\rho)\varphi'(\rho)} \nabla d\mathcal{W}_t. \tag{2.1}$$

2.3. **Energy solutions of the KPZ equation.** The space $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ is the space on which the solutions of the KPZ equation (2.1) will live. For $\epsilon > 0$ we define $i_\epsilon(x) : \mathbb{R} \to \mathbb{R}$ by $i_\epsilon(x)(y) = \epsilon^{-1} \mathbf{1}(x < y \leq x + \epsilon)$. We say that a process $\{\mathcal{Y}_t; t \in [0,T]\}$ with trajectories in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ and adapted to some natural filtration $\{\mathcal{F}_t; t \in [0,T]\}$ is a *weak solution* of (2.1) if:

i) There exists a process $\{\mathcal{A}_t; t \in [0,T]\}$ with trajectories in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ and adapted to $\{\mathcal{F}_t; t \in [0,T]\}$ such that for any $G \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{R}} \mathcal{Y}_s(i_\epsilon(x))^2 \frac{G(x+\epsilon) - G(x)}{\epsilon} dx ds = \mathcal{A}_t(G). \tag{2.2}$$

ii) For any function $G \in \mathcal{S}(\mathbb{R})$ the process

$$M_t(G) = \mathcal{Y}_t(G) - \mathcal{Y}_0(G) - \frac{\varphi'(\rho)}{2} \int_0^t \mathcal{Y}_s(G'') ds - \frac{a\beta''(\rho)}{2} \mathcal{A}_t(G) \tag{2.3}$$

is a martingale of quadratic variation $\chi(\rho)\varphi'(\rho) t \int G'(x)^2 dx$.

Now we introduce a stronger notion of solution, which captures well some of the particularities of the solutions of (2.1). Let $\{\mathcal{Y}_t; t \in [0,T]\}$ be a weak solution of (2.1). For $0 \leq s < t \leq T$, let us define the fields

$$\mathcal{I}_{s,t}(G) = \int_s^t \mathcal{Y}_u(G'') du,$$

$$\mathcal{A}_{s,t}(G) = \mathcal{A}_t(G) - \mathcal{A}_s(G),$$

$$\mathcal{A}_{s,t}^\epsilon(G) = \int_s^t \int_{\mathbb{R}} \mathcal{Y}_u(i_\epsilon(x))^2 \frac{G(x+\epsilon) - G(x)}{\epsilon} dx du.$$

We say that $\{\mathcal{Y}_t; \in [0,T]\}$ is an *energy solution* of (2.1) if there exists a constant $\kappa > 0$ such that

$$E[\mathcal{I}_{s,t}(G)^2] \leq \kappa (t-s) \int G'(x)^2 dx$$

and

$$E[(\mathcal{A}_{s,t}(G) - \mathcal{A}_{s,t}^\epsilon(G))^2] \leq \kappa \epsilon (t-s) \int G'(x)^2 dx$$

for any $0 \leq s < t \leq T$, any $\epsilon \in (0,1)$ and any $G \in \mathcal{S}(\mathbb{R})$. We say that a weak solution $\{\mathcal{Y}_t; t \in [0,T]\}$ is a stationary solution if for any $t \in [0,T]$ the $\mathcal{S}'(\mathbb{R})$-valued random variable $\mathcal{Y}_t$ is a white noise of variance $\chi(\rho)$.

An immediate consequence of last result is the existence of weak solutions of the KPZ equation. Let $\mathcal{Y}_t$ be a limit point of $\mathcal{Y}_t^n$. Since the measure ν_ρ is invariant under the evolution of η_t^n, for any fixed time $t \in [0,T]$ the $\mathcal{S}'(\mathbb{R})$-valued random variable $\mathcal{Y}_t$ is a white noise of variance $\chi(\rho)$. As a consequence of the previous result we obtain that for any limit point $\{\mathcal{Y}_t; t \in [0,T]\}$ of

$\{\{\mathcal{Y}_t^n; t \in [0,T]\}; n \in \mathbb{N}\}$, there is a finite constant $c > 0$ such that the process $\{\mathcal{A}_t; t \in [0,T]\}$ defined as above satisfies the moment bound $E[\mathcal{A}_{s,t}(G)^2] \leq c|t-s|^{3/2} \int G'(x)^2 dx$. Moreover, for any $\gamma \in (0, 1/4)$ and any $G \in \mathcal{S}(\mathbb{R})$ the real-valued process $\{\mathcal{Y}_t(G); t \in [0,T]\}$ is Hölder-continuous of order γ.

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