THE CROSSOVER TO THE KPZ EQUATION

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ABSTRACT. In this paper we consider the one-dimensional weakly asymmetric simple exclusion process under the invariant state ν_ρ: the Bernoulli product measure of parameter $\rho \in (0,1)$. We show that the limit density field is governed by an Ornstein-Uhlenbeck process for strength asymmetry $n^{2-\gamma}$ if $\gamma \in (1/2,1)$, while for $\gamma = 1/2$ it is an energy solution of the KPZ equation. From this result we obtain that the fluctuations of the current of particles are Gaussian distributed for $\gamma \in (1/2,1)$, while for $\gamma = 1/2$ the limit distribution is written in terms of the KPZ equation.

1. INTRODUCTION

We consider the one-dimensional weakly asymmetric simple exclusion process (wasep), i.e. our microscopic dynamics is given by a stochastic lattice gas with hard core exclusion. This process arises as a simple model for the growing of random interfaces. The presence of weak asymmetry in the microscopic dynamics, breaks down the detailed balance condition, which implies the system to exhibit a non trivial behavior even in the stationary situation. The dynamical scaling exponent has been established by the physicists as being $z = 3/2$ and one of the challenging problems is to establish the limit distribution for the density and the current of particles, see Spohn (1991). We take the process with asymmetry given by $an^{2-\gamma}$ and we want to analyze the effect of strengthening the asymmetry in the limit distribution of the density field.

The wasep was studied in Masi et al (1986) and in Dittrich and Gartner (1991), for $\gamma = 1$; and in Bertini and Giacomin (1997) for $\gamma = 1/2$. The equilibrium density fluctuations (for $\gamma = 1$) are given by an Ornstein-Uhlenbeck process. For $\gamma = 1/2$ (which corresponds to strength asymmetry n^z), Bertini and Giacomin (1997) used the Cole-Hopf transformation to derive the non-equilibrium fluctuations of the current of particles. By removing the drift to the system, there is no effect of the strength of the asymmetry on the limit distribution of the density field. By strengthening the asymmetry the limit distribution "feels" the effect of this strengthening, by developing a non linear term in the limit distribution. In this case the limit density field is a solution of the Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation was proposed in [8] to model the growth of random interfaces. Denoting by h_t the height of the interface, this equation reads as

$$\partial_t h = D\Delta h + a(\nabla h)^2 + \sigma \mathcal{W}_t,$$

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where D, a, σ are constants related to the thermodynamical properties of the interface and $\mathcal{W}_t$ is a space-time white noise with covariance $E[\mathcal{W}_t(u)\mathcal{W}_s(v)] = \delta(t-s)\delta(u-v)$. According to z, a non-trivial behavior occurs under re-scaling $h_n(t,x) = n^{-1/2}h(tn^{3/2}, x/n)$. This means, roughly speaking, that in our case, for $\gamma = 1/2$ a non trivial behavior is expected even in the stationary situation and in this case, the model belongs to the universality class of the KPZ equation. Here we provide the characterization of the transition from the Edwards-Wilkinson class to the KPZ class, for the wasep. We prove that the transition depends on the strength of the asymmetry without having any other intermediate state and by establishing precisely the strength in order to have the crossover. From this result we obtain the crossover regime for the current of particles across a characteristic. Our method relies on a stronger Boltzmann-Gibbs Principle introduced in [4] and is robust enough in the sense that it can be applied for general interacting particle systems.

2. Equilibrium fluctuations

2.1. **The process.** Let η_t be the wasep evolving on $\mathbb{Z}$ and with space state $\Omega = \{0,1\}^{\mathbb{Z}}$. In this process, each particle waits a mean one exponential time after which jumps to an empty neighboring site according to a transition rate that has a weak asymmetry to the right. The process is taken on the diffusive time scale n^2 so that $\eta_t^n = \eta_{tn^2}$ and the transition rate to the right is $1/2 + 1/n^\gamma$ and to the left is $1/2 - 1/n^\gamma$. We notice that if we decrease the value of γ, this corresponds to speeding up the asymmetric part of the dynamics on longer time scales as $n^{2-\gamma}$. A stationary measure for this process is the Bernoulli product measure on Ω of parameter ρ, that we denote by $\{\nu_\rho : \rho \in [0,1]\}$.

2.2. **The density field.** We denote by π_t^n the empirical measure as the positive measure in $\mathbb{R}$ defined by

$$\pi_t^n(dx) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \eta_t^n(x)\delta_{x/n}(dx),$$

where for $u \in \mathbb{R}$, δ_u is the Dirac measure at u. We are interested in establishing the fluctuations of the empirical measure from the stationary state ν_ρ. From now on, fix a density ρ and take η_t^n moving in a reference frame with constant velocity given by $(1-2\rho)n^{2-\gamma}$. Let $\mathcal{Y}_t^n$ be the *density fluctuation field* on $H \in \mathcal{S}(\mathbb{R})$ as:

$$\mathcal{Y}_t^{n,\gamma}(H) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} T_t^\gamma H_x\big(\eta_t^n(x) - \rho\big), \tag{2.1}$$

where $T_t^\gamma H(\cdot) = H(\cdot - (1-2\rho)tn^{1-\gamma})$. From now on we use the denotation $H_x = H(x/n)$.

2.3. **On the hydrodynamic scale parameter.** For $\gamma = 1$, it is not hard to show that $\{\mathcal{Y}_t^{n,\gamma}; n \in \mathbb{N}\}$ converges to $\mathcal{Y}_t^\gamma$ solution of the Ornstein-Uhlenbeck equation:

$$d\mathcal{Y}_t^\gamma = \frac{1}{2}\Delta\mathcal{Y}_t^\gamma dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t, \tag{2.2}$$

where $\mathcal{W}_t$ is a space-time white noise. So, for $\gamma = 1$ the system belongs to the Edwards-Wilkinson universality class.

2.4. **Beyond the hydrodynamic scale parameter.** In order to see the effect of the asymmetry in the limit density field we increment the strength of the asymmetry by decreasing the value of γ. According to Bertini and Giacomin (1997), the effect of the asymmetry is presented in the limit field when $\gamma = 1/2$ and in that case $\mathcal{Y}_t^\gamma$ has a very different qualitatively behavior from the one obtained for $\gamma = 1$, namely the solution of (2.2).

In this work, we characterize the limit field $\mathcal{Y}_t^\gamma$ for the intermediate state, i.e. for $\gamma \in (1/2, 1)$, by showing that for this range of the parameter it also solves (2.2). As a consequence, for $\gamma \in (1/2, 1)$ the system still belongs to the Edwards-Wilkinson universality class. The idea of the proof of last result is to use Dynkin's formula so that

$$\mathcal{Y}_t^{n,\gamma}(H) = M_t^{n,\gamma}(H) + \mathcal{Y}_0^{n,\gamma}(H) + \mathcal{I}_t^{n,\gamma}(H) + \mathcal{A}_t^{n,\gamma}(H)$$

where $M_t^{n,\gamma}(H)$ is a martingale with respect to the natural filtration,

$$\mathcal{I}_t^{n,\gamma}(H) = \int_0^t \frac{1}{2\sqrt{n}} \sum_{x\in\mathbb{Z}} \Delta^n T_s^\gamma H_x(\eta_s^n(x) - \rho)ds,$$

$$\mathcal{A}_t^{n,\gamma}(H) = \int_0^t \frac{n^{1-\gamma}}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \nabla^n T_s^\gamma H_x \Big\{ \eta_s^n(x)(1-\eta_s^n(x+1)) - \chi(\rho) - (1-2\rho)(\eta_s^n(x)-\rho) \Big\} ds,$$

and Δ^n, ∇^n are the discrete laplacian and the discrete derivative, respectively.

Now we analyze the asymptotic behavior of the martingale and the integral terms above. The hard programme of this approach is to analyze the limit of the integral term $\mathcal{A}_t^{n,\gamma}(H)$. For that purpose, we derive a *stronger Boltzmann-Gibbs principle* as in Corollary 7.4 of Gonçalves (2008), which implies that $\mathcal{A}_t^{n,\gamma}(H)$ vanishes as $n \to \infty$. In [4] the result was obtained for the symmetric simple exclusion but is also true for the process we consider here. In fact that result can be stated as: if $\psi : \Omega \to \mathbb{R}$ is a local function, $\gamma \in (1/2, 1)$ and if $H \in \mathcal{S}(\mathbb{R})$ then

$$\lim_{n\to\infty} \mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^t \frac{n^{1-\gamma}}{\sqrt{n}} \sum_{x\in\mathbb{Z}} H_x \Big\{ \tau_x\psi(\eta_s^n) - E_{\nu_\rho}[\psi(\eta)] - \partial_\rho E_{\nu_\rho}[\psi(\eta)](\eta_s^n(x)-\rho) \Big\} ds \Big)^2\Big] = 0. \tag{2.3}$$

Last result together with some computations on the quadratic variation of the martingale, gives us that $\mathcal{Y}_t^\gamma$ is solution of (2.2).

2.5. **On the KPZ scale parameter.** From the previous arguments we have seen that if we want to see the effect of the asymmetry in the limit field, we go towards decreasing the value of γ, which, as mentioned above, corresponds to speeding up the asymmetric part of the dynamics. This is in agreement with the result of Bertini and Giacomin (1997) which says that for $\gamma = 1/2$ that is indeed the case.

Recently in Gonçalves and Jara (2010), it was shown that for $\gamma = 1/2$, $\{\mathcal{Y}_t^{n,\gamma}; n \in \mathbb{N}\}$ is tight and any limit point is an energy solution of the KPZ equation:

$$d\mathcal{Y}_t^\gamma = \frac{1}{2}\Delta\mathcal{Y}_t^\gamma dt + \nabla(\mathcal{Y}_t^\gamma)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t. \tag{2.4}$$

Since we are in the presence of a stronger asymmetry, the result in (2.3) is no longer true. In order to establish last result, a *second order Boltzmann-Gibbs principle* was derived in Gonçalves and Jara (2010). The ingredients invoked in order to derive this stronger replacement is a multi-scale argument introduced in Gonçalves (2008) combined with some fundamental features of the model: as a sharp spectral gap bound for the dynamics restricted to finite boxes, plus a second order expansion on the equivalence of ensembles.

The results beyond the hydrodynamic time scale, the crossover at $\gamma = 1/2$ and the KPZ class, are in fact true for a general class of weakly asymmetric exclusion processes see Gonçalves and Jara (2010).

As a consequence of last result and by relating the current of particles with the density field, we can obtain the crossover on the fluctuations of the current and we obtain that for $\gamma \in (1/2, 1)$ the limit is Gaussian, while for $\gamma = 1/2$ the limit is written in terms of the KPZ equation. For details we refer the interested reader to [6].

2.6. **Conclusion.** We point out that, our approach is robust enough in order to be applied to general one-dimensional interacting particle systems, as for example: the zero-range process and the Ginzburg-Landau model. Our approach also works for models with finite-range, non-nearest neighbor interactions with basically notational modifications. As mentioned above features of the model that is needed in order to obtain the results presented here are: a sharp spectral gap bound for the dynamics restricted to finite boxes, plus a second order expansion on the equivalence of ensembles, which are quite general.

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