Order and chaos: interactive computational activities for the classroom

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Abstract: It has long been believed that typical students learn better through contemporary approaches to questions originated by physics problems that allow experiments. This belief motivated us to develop interactive computational didactic materials about contemporaneous mathematics that can be used both in the classroom and in mathematics clubs in school. Dynamical Systems, the study of how physical systems evolve with time, inspired the activities developed. They share a key goal of understanding the order/chaos relationship in natural phenomena, human behaviour and social systems. Another goal to achieve is to give mathematics an experimental/laboratorial component, which rarely is present. In fact, all the interactive computational didactic materials developed include simulations and the capability to generate wonderful pictures, from which students can enjoy the beauty of mathematics.

Introduction

With today's available web page technology, it is no difficult to recognize the limitations of traditional text-based instructional techniques: not only it is now possible to complement a lecture session with some multimedia support, so often better in communicating a concept, but also, with its asynchronous and distance character, is changing forever the school's traditional role. Moreover, the use of Java applets is giving us, educators, the possibility to introduce the student to an experimental or laboratorial vision of mathematics, that, it is our belief, will complement its traditional learning process. With this work, we present a web page constructed around three contemporaneous ideas of science, whose development were strongly supported by mathematical or computational models. Since all of them use topics of secondary school mathematics curriculum, we think it will motivate secondary school students to use and even explore some of its concepts.

Naturally Complex

The times they are a changing

Bob Dylan

In a report, published in 1989, by the Mathematical Sciences Education Board, [1], one can read that "at the end of the nineteenth century, the axiomatization of mathematics on a foundation of logic and sets made possible grand theories of algebra, analysis, and topology whose synthesis dominated mathematics research and teaching for the first two thirds of the twentieth century. These traditional areas have now been supplemented by major developments in other mathematical sciences - in number theory, logic, statistics, operations research, probability, computation, geometry, and combinatorics". Furthermore, "much more than arithmetic and geometry, mathematics today is a diverse discipline that deals with data, measurements, and observations from science; with inference, deduction, and proof; and with mathematical models of natural phenomena, of human behaviour, and of social systems", i.e., "the process of doing mathematics is far more than just calculation and deduction; it involves observation of patterns, testing of conjectures, and estimation of results". Being unquestionable that, since its beginning, mathematics has been a science of patterns and order, one can say that, in the last decades, these two words are used with a totally new meaning: complex fractal patterns and chaos, now found copiously in nature, become exciting subjects, studied by mathematicians, scientists, and philosophers alike.

Shapes

Explanation: Looking for fossils can be a quite unique and exciting outdoor experience, most of all due to the expectation that you can always run into a record left behind by an ancient living creature. However, there are many features in rocks that, for centuries, were misidentified as fossils. Dendrites, usually more complex and less regular in shape and without the vein structures found in leaves, are purely mineral growths whose branching pattern gives them an organic appearance; they are pseudo fossils.

Figure 1. A dendrite with its typical branching tree-like form.

Built on a model introduced twenty years ago, today we have computer models that mimic the complex and disorderly pattern formation due to the process of self-assembly of small particles to form larger structures via Brownian motion. Yet, the original diffusion-limited aggregation (DLA) computational model of T.A. Witten and L.M. Sander is considered one of the most striking examples of complex pattern generation by a simple algorithm. Being developed at a time when interest in fractals was growing rapidly, the DLA model turned out to be the paradigm for pattern generation far-from-equilibrium. Under this theme, the students will use regression analysis and the logarithm function from their mathematics curriculum.

Mathematical/Computational modelling: In the simple model proposed by Witten and Sander the growing structures are represented by clusters of filled sites on a lattice: starting from a large empty lattice, a simulation begins simply by filling a site, usually chosen at the lattice's origin; then, the growth process takes place with the launching of a random walker far from the occupied site and allowing it to follow its path until it reaches a site adjacent to the occupied site, which is then filled and becomes part of the growing cluster. The process is repeated with a new random walker and its path until a perimeter site of the new two-particle aggregate is selected and filled. Since a random walker can always find its way to the aggregate, the growth process will never stop (although is highly recommended to stop a random walker whenever its path is gone too far from the region occupied by the cluster, and start a new one).

Experimental mathematics: The students will use an applet to simulate the growth process over and over again and watch the complexity of the obtained clusters. Even though, due to the obvious random character of its growth, every single cluster is different from any other, it is fundamental to emphasize its similarities.

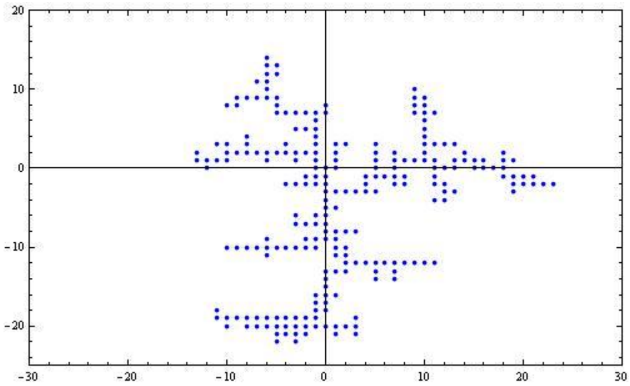

Figure 2. A DLA growth model computational simulation.

At this point, it is important to invite the students to choose different values for the cluster size, in order to appreciate the scaling irrelevance of this DLA growth process.

Measuring complexity: Since it is fundamental to give the students an idea of how intricate these organic-like structures are, it will be indispensable to introduce them to the concept of box dimension of a set. After an initial brief explanation, the students will be invited to use an applet to get a few points on the plane "length of the box" versus "number of boxes necessary to cover the cluster" from which a regression line will be drawn, and the box dimension of the cluster computed. Again, it should be noted how close these complexity measures are for different simulations, being therefore a characteristic of this kind of pattern formation.

Social Network

Explanation: The complexity observed in the inanimate world, it is also evident in living organisms. In this second theme, we suggest the students to study the time evolution of a very simple social network, from a mathematical model analogous to the previous DLA model. The mathematics curriculum topics that it aims to introduce, develop and explore are graph theory, probability and combinatorics.

Mathematical/Computational modelling: The Passion Madness Network: consider a group of people. For the sake of simplicity, we will say that they are seated around a table and that someone is either not in love, or madly in love. Furthermore, we will say that everyone follows the same time evolution rule, depending solely on the love state of both his/her left and right neighbours:

- someone not in love between two people also not in love, then the passion remains switched off and he/she will be not in love the following moment;
- someone not in love with one neighbour madly in love, then the passion inflames and he/she will be madly in love the following moment;
- someone not in love between two people madly in love, then it is too confusing, the passion remains switched off and he/she will be not in love the following moment;
- someone madly in love between two people not in love, then the passion turns off and he/she will be not in love the following moment;
- someone madly in love with one neighbour also madly in love, then the passion remains light and he/she will be madly in love the following moment;
- someone madly in love between two people madly in love, then it is too confusing, the passion burns out and ... extinguishes and he/she will be not in love the following moment.

The question we ask is about the collective behaviour of such network of people, that is, given a group of people, what is the future of each one's love state? To answer this question, we invite the students to consider the following mathematical/computational model: consider a set of cells positioned in a circle. Since each cell can only assume one of two states, we choose two colours to represent each of the states: light grey for the not in love state, and orange for the madly in love one. In Figure 1 (left), we represent a group of 40 people with everyone's love state clearly identified by its colour. Then, all we have to do is to follow the rules. In order to see the changes of the love state of the cells, it is convenient to represent both configurations of the system: it is usual to represent the next moment system configuration under the initial one, as shown in Figure 1 (center). If we continue to apply the rules, we get the time evolution of each cell love state. In Figure 1 (right) we show the time evolution of the system for 6 moments.

Figure 3. Everyone's love state time evolution, from a chosen initial configuration.

Since this 3-dimensional graphs is time and memory demanding, we invite the students to consider its 2-dimensional version: do the maths with the cells positioned in a circle, but choose a position, cut the circle, and show the set of cells as positioned in a line segment. Since we represent the next moment system configuration under the initial one, they should be careful and do the cut at the same position as before.

<u>Experimental mathematics</u>: Given the model and its graphical representation, next we allow the students to use an applet to explore it: for systems with small number of elements, can we identify its most probable future? And what happens if we choose a not so small number of elements? How can we describe its collective behaviour?

<u>Creative disorder</u>: By slightly modifying the rules describing the interaction between different elements, but still depending only on its own and its left and right neighbours previous state, the students will be able to build systems whose most probable collective behaviour is very different from the love madness, i.e., systems which collective behaviour is clearly characterized by its total organization, Figure 4. (left). Yet, for a few choices of the rules, the system will have a collective behaviour completely different from those two, highly organized but with some components of disorder, Figure 4. (right). Is this the kind of behaviour one can observe on an ant colony? And can we say the same about the activity of our brain, so much complicated, and yet, highly organized?

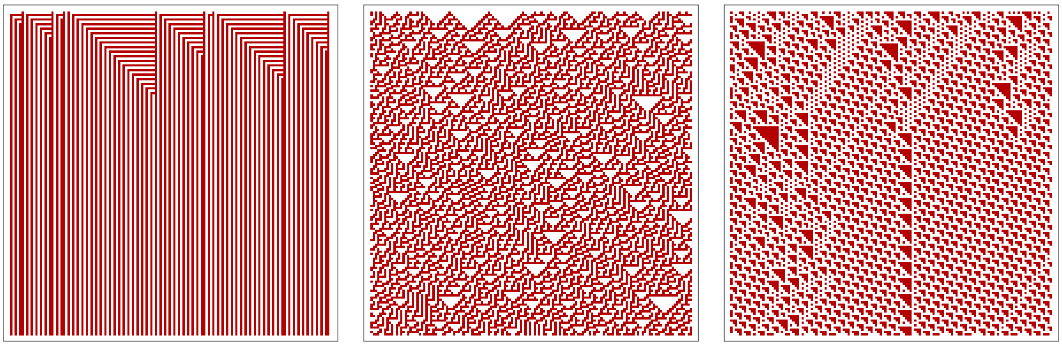

Figure 4. The different collective behaviour showed by this computer model.

This third type of collective behaviour, point out by computer simulation, has sparked an enormous interest at least because of its implications on business organization. Until now, was more or less common sense that a large company could only survive based on a rigid hierarchy, a solution to prevent that, to an increase in the number of employees and activities, inevitably follows a lack of coordination and the company's collapse as a supplier of products. Today, we can see social networks, as Facebook, and companies, like Google, adopting different strategies, by putting their efforts on an organization without the previous stiffness, attempting to keep a kind of ordered collective behaviour, but where minor changes are allowed and even stimulated. Google's success as a creative organization reveals that it is possible to take lessons from a computer simulation to manage a company. Except that, unlike our simple computer models, it is extraordinarily difficult to recognize the organization characteristics that are crucial to reach that desirable creative disorder behaviour.

Dynamics

Explanation: In his book, "An Essay on the Principle of Population; or, a View of its Past and Present Effects on Human Happiness; with an enquiry into our Prospects respecting the Future Removal or Mitigation of the Evils which it occasions", the revised second edition of a previous work, published in 1802, the Reverend Thomas Robert Malthus alerted the reader to the unequal nature of food supply to population growth: while food supply, he argued, is subject to an arithmetic/linear growth, population growth had a geometric/exponential nature. This divergence, that played a key role in the development of the theory of natural selection, by both Charles Darwin and Alfred Russell Wallace, was solved with a nonlinear model that takes into account the consequences of food shortage when the population grows over a certain number. But, what seemed to be a slight modification of Malthus law, brought to population dynamics the new and exciting mathematics of chaos theory. With this third theme, the students will be able to work extensively with the parabola, to draw its graph and to interpret some of its features from it.

Mathematical/Computational modeling: A perfectly reasonable model for the time evolution of the number of individuals of a population comes as P(t+1)=f(P(t)), where P(t) stands for the number of individuals at time t, P(t+1) for the number of individuals at the next moment, t+1, and f some parabola. However, f cannot be any parabola, since it is obvious that our model must reflect the fact that from a null population, at time t, we can never have something other than zero, at time t+1: therefore, our parabola is required to satisfy f(0)=0. A second condition for the parabola f is obtained when we demand that, if we have a critical overcrowded situation at time t, then the population faces extinction at time t+1. Since this critical value depends on the population and its habitat, it is preferable to work with the percentage of the population relative to the critical extinction value, instead of the number of individuals. Thus, our nonlinear model is given by X(t+1)=f(X(t)), with X(t) and X(t+1) the percentage of individuals of a population relative to its critical extinction value, at time t and at time t+1, respectively, and f a parabola satisfying both f(0)=0 and f(1)=0, which leads us to X(t+1)=A*X(t)(1-X(t+1)), with A a parameter to be chosen between 0 and 4.

Experimental mathematics: The students will be invited to use a first applet that will give them the graphical representation of X(t), for t=0 to a certain tmax, once chosen a value for the parameter A and the initial value X(0). After being familiarized with the concept of time series, a second applet will be presented, in which the students will have both the time series and the corresponding graphical analysis associated with the iteration of the parabola. In this case, we think it will be better to allow the students to choose only among a few values for the parameter A, being X(0) arbitrary, corresponding to asymptotic situations for the time evolution of X(t) easy to identify either as constant or as repeating a set of values.

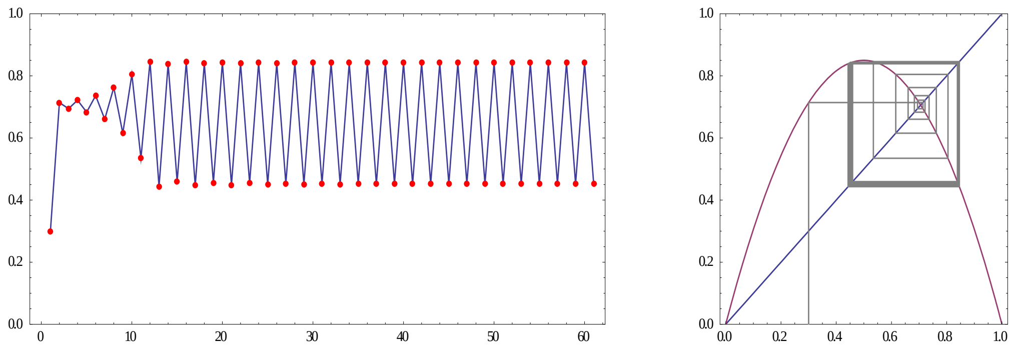

Figure 5. Two different graphical representation for a time series X(t).

Can we trust it?: It is no surprise for a population biologist the claim that, under certain circumstances, the number of individuals of a species can change periodically: it has been observed, both in laboratory and nature. But our model hides a huge surprise: if one chooses a parameter value over 3.57, the time series X(t) can exhibit a rather strange behaviour. Having fixed the parameter at A=3.825， the time series X(t), taken from a given initial choice X(0), shows a lot of fluctuations, perhaps corresponding to a periodic cycle of a very long period. But, more interesting than that, is what happens when we compare it with a second time series, X'(t), obtained from a different initial choice X'(0) as close as we want to X(0).

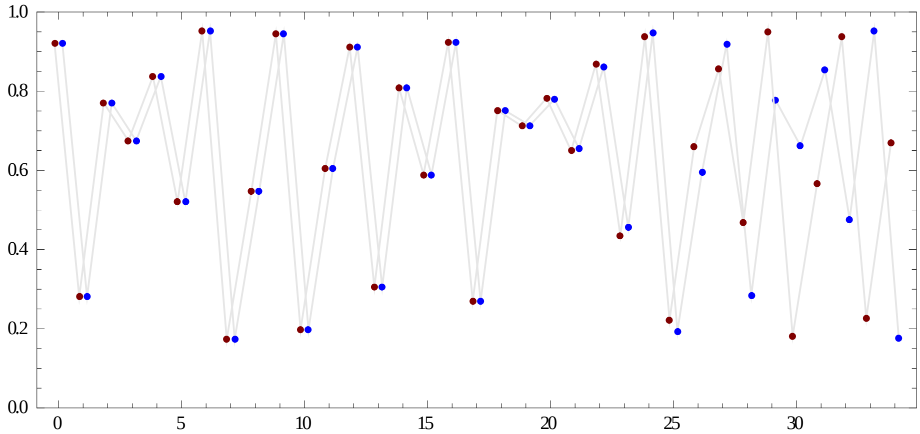

Figure 6. Graphical representation of two time series from very close choices for its initial condition.

Although both X(t) and X'(t) change similarly for the first 25 time steps, reflecting the closeness of X(0) and X'(0), then they seem to be changing in completely different ways, a clear evidence of the sensitivity of the model to the initial conditions' choice. Despite the fact that there does not exist yet an unequivocal example of this type of behaviour in a natural population, this intrinsic mathematical characteristic of the parabola has raised fundamental questions about the limits of predictability. The marvelous scientific and engineering achievements in this area of predictability led us all to an enthusiastic generalization which we now recognize as false: there are models, and simple ones, practically unpredictable.

Discussion

Following the idea that teaching and learning mathematics in context has enormous benefits, we propose a theme that, in our view, has four main advantages: first, it is transversal to both natural and social sciences, that will fit the wide range of interests of secondary school students; second, knowing that The American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics announced that the theme of Mathematics Awareness Month 2011 is Unraveling Complex Systems, we expect secondary school students to be able to reveal how the systems of the world that surrounds us evolve with time; third, being so graphical, there is a notorious growing interest of the media on complex studies. Finally, it is not a closed subject! We have chosen three topics about complexity in nature but there are more available information from which the teacher and the students can build their own topics.

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