Inverse semigroups generated by nilpotent transformations

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Synopsis
Let \( X \) be a set with infinite cardinality \( m \) and let \( B \) be the Baer-Levi semigroup, consisting of all one-one mappings \( \alpha: X \to X \) for which \( |X \setminus X_\alpha| = m \). Let \( K_m = (B^{-1}B) \), the inverse subsemigroup of the symmetric inverse semigroup \( \mathcal{S}(X) \) generated by all products \( \beta^{-1}\gamma \), with \( \beta, \gamma \in B \). Then \( K_m = (N_2) \), where \( N_2 \) is the subset of \( \mathcal{S}(X) \) consisting of all nilpotent elements of index 2. Moreover, \( K_m \) has 2-nilpotent-depth 3, in the sense that \( N_2 \cup N_2^2 \subseteq K_m = N_2 \cup N_2^2 \cup N_2^3 \).

Let \( P_m \) be the ideal \( \{ \alpha \in K_m \mid \text{dom} \alpha < m \} \) in \( K_m \) and let \( L_m \) be the Rees quotient \( K_m \setminus P_m \). Then \( L_m \) is a 0-bisimple, 2-nilpotent-generated inverse semigroup with 2-nilpotent-depth 3. The minimum non-trivial homomorphic image \( L_m^* \) of \( L_m \) also has these properties and is congruence-free.

1. Introduction and background

In previous papers [4, 7], attention has been focused on certain subsemigroups of an infinite full transformation semigroup, and some interesting examples of (0-)bisimple idempotent-generated congruence-free semigroups have been obtained.

If one looks for an analogue in inverse semigroups, it is clear that 'idempotent-generated' is not an appropriate restriction, since in any inverse semigroup, the idempotents form a subsemigroup. However, a symmetric inverse semigroup \( \mathcal{S}(X) \) contains a zero (the empty subset of \( X \times X \), but usually denoted by 0) and it is reasonable to look instead at elements that are products of nilpotent elements. In this paper, we examine, for infinite \( \mathcal{S}(X) \), the inverse subsemigroup generated by all elements \( \alpha \) of \( \mathcal{S}(X) \) which are nilpotent of index 2 (i.e. for which \( \alpha \neq 0, \alpha^2 = 0 \)). In the end, we describe, for each infinite cardinal number \( m \), an inverse semigroup \( L_m^* \) of cardinality \( 2^m \) which is 0-bisimple, congruence-free, and generated by the set \( N_2 \) consisting of its nilpotent elements of index 2. Moreover, \( N_2 \cup N_2^2 \subseteq L_m^* = N_2 \cup N_2^2 \cup N_2^3 \).

2. Preliminaries

For undefined terms in semigroup theory see [3].

Let \( X \) be a set of infinite cardinality \( m \). For an element \( \alpha \) of the symmetric inverse semigroup \( \mathcal{S}(X) \), we denote the domain by \( \text{dom} \alpha \) and the range by \( \text{ran} \alpha \).
It is convenient also to define the gap and the defect of \( \alpha \) by
\[
gap \alpha = X \setminus \text{dom } \alpha, \quad \text{def } \alpha = X \setminus \text{ran } \alpha.
\]
Then the Baer-Levi semigroup \( B \) (of type \( (m, m) \)) is defined as
\[
B = \{ \alpha \in \mathcal{J}(X) : \text{gap } \alpha = \emptyset, \text{def } \alpha = m \}.
\]  \hspace{1cm} (2.1)
(See [1]; also [2, Section 8.1].) It is easy to see that \( B \) is a subsemigroup of \( \mathcal{J}(X) \).
It is not an inverse semigroup, however: indeed \( \alpha \in B \) if and only if \( \alpha^{-1} \in B^{-1} \), where
\[
B^{-1} = \{ \alpha \in \mathcal{J}(X) : |\text{gap } \alpha| = m, \text{def } \alpha = \emptyset \}.
\]  \hspace{1cm} (2.2)
Then \( B^{-1} \) is also a subsemigroup (but not an inverse subsemigroup) of \( \mathcal{J}(X) \).
From [2], we know that \( B \) is right cancellative, right simple and has no idempotents. Hence, from the obvious anti-isomorphism \( \alpha \mapsto \alpha^{-1} \) from \( B \) to \( B^{-1} \), we have that \( B^{-1} \) is left cancellative, left simple and has no idempotents.

**Theorem 2.3.** If \( B, B^{-1} \) are as defined in (2.1) and (2.2), then \( BB^{-1} = \mathcal{J}(X) \).

**Proof.** Let \( \alpha \in \mathcal{J}(X) \). Write \( \text{dom } \alpha = P, \text{ran } \alpha = Q \), and let \( R_1, R_2, R_3 \) be pairwise disjoint subsets of \( X \) such that
\[
|R_1| = |R_2| = |R_3| = m, \quad X = R_1 \cup R_2 \cup R_3.
\]
Certainly \( |P| \leq m \) and \( |X \setminus P| \leq m \); hence there are injections
\[
\theta : P \to R_1, \quad \phi : X \setminus P \to R_2.
\]
Define \( \beta \in \mathcal{J}(X) \) by
\[
x \beta = \begin{cases} x \theta & \text{if } x \in P, \\ x \phi & \text{if } x \in X \setminus P. \end{cases}
\]
Then \( \text{gap } \beta = \emptyset, \text{def } \beta \supseteq R_3 \) and so \( \beta \in B \).
Again, \( |X \setminus Q| \leq m \) and so we have an injection \( \psi : X \setminus Q \to R_3 \). Define \( \gamma \in \mathcal{J}(X) \) by
\[
x \gamma = \begin{cases} x \theta^{-1} \alpha & \text{if } x \in P \theta, \\ x \psi^{-1} & \text{if } x \in (X \setminus Q) \psi. \end{cases}
\]
Then \( \text{ran } \gamma = X \) and \( \text{gap } \gamma \supseteq R_2 \); hence \( \gamma \in B^{-1} \). It is now easy to verify that
\[
\text{dom } \beta \gamma = (\text{ran } \beta \cap \text{dom } \gamma) \beta^{-1} = P = \text{dom } \alpha,
\]
\[
\text{ran } \beta \gamma = (\text{ran } \beta \cap \text{dom } \gamma) \gamma = Q = \text{ran } \alpha
\]
and that \( x \beta \gamma = x \alpha \) for all \( x \) in \( P \). This completes the proof.
By contrast, the product \( B^{-1}B \) is not even a subsemigroup of \( \mathcal{J}(X) \). This will be easy to demonstrate when we have established

**Theorem 2.4.** If \( B, B^{-1} \) are as defined in (2.1), (2.2), then
\[
B^{-1}B = \{ \alpha \in \mathcal{J}(X) : |\text{dom } \alpha| = |\text{ran } \alpha| = |\text{gap } \alpha| = |\text{def } \alpha| = m \}.
\]

**Proof.** Let \( \beta \in B^{-1}, \gamma \in B \). Then \( \text{ran } \beta = \text{dom } \gamma = X \) and so
\[
\text{dom } \beta \gamma = (\text{ran } \beta \cap \text{dom } \gamma) \beta^{-1} = X \beta^{-1} = \text{dom } \beta.
\]
Similarly, \( \text{ran } \beta \gamma = \text{ran } \gamma \).

It now follows that
\[
|\text{dom } \beta \gamma| = |\text{dom } \beta| = m, \quad |\text{gap } \beta \gamma| = |\text{gap } \beta| = m,
\]
\[
|\text{ran } \beta \gamma| = |\text{ran } \gamma| = m, \quad |\text{def } \beta \gamma| = |\text{def } \gamma| = m.
\]

Conversely, let \( \alpha \) be an element of \( \mathcal{F}(X) \) such that
\[
|\text{dom } \alpha| = |\text{ran } \alpha| = |\text{gap } \alpha| = |\text{def } \alpha| = m
\]
and let \( \beta : \text{dom } \alpha \to X \) be a bijection. Then define \( \gamma : X \to \text{ran } \alpha \) by the rule that
\[
x \gamma = x \beta^{-1} \alpha \quad (x \in X).
\]

Then \( \beta \in B^{-1}, \gamma \in B \) and \( \beta \gamma = \alpha \).

It is now clear that \( B^{-1}B \) is not a subsemigroup of \( \mathcal{F}(X) \): for example, if \( X = Y \cup Z \), with \( |Y| = |Z| = m \) and \( Y \cap Z = \emptyset \), then \( 1_Y, 1_Z \) are both in \( B^{-1}B \), but \( 1_Y 1_Z = 0 \). We can, however, fairly easily describe \( \langle B^{-1}B \rangle \), the subsemigroup of \( \mathcal{F}(X) \) generated by \( B^{-1}B \).

**Theorem 2.5.** If \( B, B^{-1} \) are as defined in (2.1), (2.2), then
\[
\langle B^{-1}B \rangle = \{ \alpha \in \mathcal{F}(X) : |\text{gap } \alpha| = |\text{def } \alpha| = m \}.
\]

**Proof.** Let us write
\[
K_m = \{ \alpha \in \mathcal{F}(X) : |\text{gap } \alpha| = |\text{def } \alpha| = m \}.
\]

For all \( \alpha, \beta \in \mathcal{F}(X) \), we have \( \text{dom } \alpha \beta \subseteq \text{dom } \alpha, \text{ran } \alpha \beta \subseteq \text{ran } \beta \). Hence, \( \text{gap } \alpha \beta \geq \text{gap } \alpha, \text{def } \alpha \beta \geq \text{def } \beta \). It now readily follows that if \( \alpha, \beta \in K_m \) then \( \alpha \beta \in K_m \). Thus \( K_m \) being a subsemigroup of \( \mathcal{F}(X) \) containing \( B^{-1}B \), must contain \( \langle B^{-1}B \rangle \).

Now let \( \alpha \in K_m \), and write
\[
\text{gap } \alpha = Z \cup T, \quad \text{def } \alpha = P \cup Q,
\]
where \( Z \cap T = P \cap Q = \emptyset \), \( |Z| = |T| = |P| = |Q| = m \). Let \( \theta : Z \to P \) be a bijection, and define
\[
\beta = \theta \cup \alpha : Z \cup \text{dom } \alpha \to P \cup \text{ran } \alpha.
\]

Then \( \text{dom } \beta = Z, \text{gap } \beta = T, \text{ran } \beta = P, \text{def } \beta = Q \) and so, by Theorem 2.4, \( \beta \in B^{-1}B \). Equally, if \( \gamma \) is the identity mapping on \( Q \cup \text{ran } \alpha \) then \( \gamma \in B^{-1}B \). Since \( \text{ran } \beta \cap \text{dom } \gamma = \text{ran } \alpha \), it is now clear that \( \beta \gamma = \alpha \) and hence \( \alpha \in \langle B^{-1}B \rangle \).

Notice that in proving Theorem 2.5 we have proved incidentally that
\[
\langle B^{-1}B \rangle = \langle B^{-1}B \rangle^2.
\]

Notice also that while \( \langle B^{-1}B \rangle \) is defined merely as the *subsemigroup* generated by \( B^{-1}B \), it does in fact turn out to be an *inverse subsemigroup*, as is clear from Theorem 2.5 and the observations that \( \text{gap } \alpha^{-1} = \text{def } \alpha, \text{def } \alpha^{-1} = \text{gap } \alpha \).
3. A nilpotent-generated inverse semigroup

In this section, we examine some of the properties of the inverse semigroup \( K_m \) defined by (2.6). Notice that \( 0 \in K_m \), since \( \text{gap } 0 = \text{def } 0 = X \).

In a semigroup \( S \) with zero, let \( N_2 \) be the set of all nilpotent elements of index 2, i.e. the set of all non-zero \( s \) in \( S \) for which \( s^2 = 0 \). If \( S = \langle N_2 \rangle \) then either the ascent

\[ N_2 \subseteq N_2 \cup N_2^2 \subseteq N_2 \cup N_2^2 \cup N_2^3 \subseteq \ldots \]

continues infinitely, or there exists a least \( k \geq 1 \) such that

\[ N_2 \cup N_2^2 \cup \ldots \cup N_2^k = S. \]

In the former case we write \( \Delta_2(S) = \infty \) and in the latter case we write \( \Delta_2(S) = k \).

We describe \( S \) as 2-nilpotent-generated and refer to \( \Delta_2(S) \) as its 2-nilpotent-depth.

We note in passing that

\[ N_2 \subseteq N_2^2; \]

for if \( \alpha \in N_2 \) then \( \alpha^{-1} \in N_2 \) and \( \alpha = \alpha^{-1} \alpha. \) It follows that

\[ N_2 \cup N_2^2 \cup N_2^3 = N_2^2 \cup N_2^3. \]

It is clear that in \( \mathcal{F}(X) \), an element \( \alpha \) is nilpotent of index 2 if and only if

\[ \text{dom } \alpha \neq \emptyset, \quad \text{dom } \alpha \cap \text{ran } \alpha = \emptyset. \tag{3.1} \]

**Theorem 3.2.** Let \( \mathcal{F}(X) \) be the symmetric inverse semigroup on a set \( X \) of cardinality \( m \), let \( K_m \) be as defined in (2.6), and let \( N_2 \) be the set of all nilpotent elements of index 2 in \( \mathcal{F}(X) \). Then \( K_m = \langle N_2 \rangle \), and \( \Delta_2(K_m) = 3 \).

**Proof.** We begin by showing that \( N_2 \subseteq K_m \). Let \( \alpha \in N_2 \). Then \( \alpha \) satisfies (3.1) and so \( \text{dom } \alpha \subseteq \text{def } \alpha, \quad \text{ran } \alpha \subseteq \text{gap } \alpha. \) If \( |\text{dom } \alpha| = |\text{ran } \alpha| = m \) this implies that \( |\text{def } \alpha| = |\text{gap } \alpha| = m \), i.e. that \( \alpha \in K_m \).

Suppose therefore that \( |\text{dom } \alpha| = |\text{ran } \alpha| < m \). But then it is clear from the statements

\[ |\text{dom } \alpha| + |\text{gap } \alpha| = m, \quad |\text{ran } \alpha| + |\text{def } \alpha| = m \]

that \( |\text{gap } \alpha| = |\text{def } \alpha| = m \). Hence again \( \alpha \in K_m \). Thus \( N_2 \subseteq K_m \), from which it follows that

\[ N_2 \cup N_2^2 \cup N_2^3 \subseteq K_m. \]

Next, we notice that for an element \( \alpha \) of \( K_m \), we may have either

\[ |\text{gap } \alpha \cap \text{def } \alpha| = m \quad \text{or} \quad |\text{gap } \alpha \cap \text{def } \alpha| < m. \]

For an example of the first kind of element, consider \( \alpha = 1_Y \), where \( X = Y \cup Z \), \( Y \cap Z = \emptyset, \text{ with } |Y| = |Z| = m \); here \( \text{gap } \alpha = \text{def } \alpha = Z \). To find an example not in \( N_2 \) of the second kind of element, partition \( X \) into three mutually disjoint subsets \( Y, Z, T \), with \( |Y| = |Z| = |T| = m \). Let \( \theta: Z \to T \) be a bijection and let

\[ \alpha = \theta \cup 1_Y: Y \cup Z \to Y \cup T. \]

Then \( \alpha \notin N_2 \), \( \text{gap } \alpha = T \), \( \text{def } \alpha = Z \).
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The following lemma identifies those elements of $K_m$ that lie in $N_2^2$.

**Lemma 3.3.** Let $\alpha \in K_m$. Then $\alpha \in N_2^2$ if and only if $|\text{gap} \alpha \cap \text{def} \alpha| = m$.

**Proof.** Let $\alpha \in K_m$ be such that

$$|\text{gap} \alpha \cap \text{def} \alpha| = m.$$

Write $\text{gap} \alpha \cap \text{def} \alpha = Z \cup T$, with $|Z| = |T| = m$ and $Z \cap T = \emptyset$. Let $\beta: \text{dom} \alpha \to Z$ be an injection. Then $\beta \in N_2$ by (3.1). If $\gamma = \beta^{-1} \alpha$ then

$$\text{dom} \gamma = (\text{ran} \beta^{-1} \cap \text{dom} \alpha) \beta = \text{ran} \beta$$

(since $\text{ran} \beta^{-1} = \text{dom} \beta = \text{dom} \alpha$) and similarly

$$\text{ran} \gamma = \text{ran} \alpha.$$

Since $\beta \in Z \subseteq \text{gap} \alpha \cap \text{def} \alpha$, we have $\text{dom} \gamma \cap \text{ran} \gamma = \emptyset$ and so $\gamma \in N_2$. Also,

$$\beta \gamma = \beta \beta^{-1} \alpha = \alpha,$$

since $\text{dom} \beta = \text{dom} \alpha$, and so $\alpha \in N_2^2$.

Conversely, let $\alpha = \beta \gamma$, with $\beta, \gamma \in N_2$, and suppose first that $|\text{dom} \alpha| (= |\text{ran} \alpha|) = m$. Then,

$$m = |\text{ran} \alpha| = |\text{ran} \beta \cap \text{dom} \gamma \gamma| = |\text{ran} \beta \cap \text{dom} \gamma|.$$ (3.4)

Now $\text{dom} \alpha \subseteq \text{dom} \beta$ and so

$$(\text{ran} \beta \cap \text{dom} \gamma) \cap \text{dom} \alpha \subseteq \text{ran} \beta \cap \text{dom} \beta = \emptyset$$

(since $\beta \in N_2$); hence,

$$\text{ran} \beta \cap \text{dom} \gamma \subseteq \text{gap} \alpha.$$

Similarly, since $\text{ran} \alpha \subseteq \text{ran} \gamma$, we have

$$(\text{ran} \beta \cap \text{dom} \gamma) \cap \text{ran} \alpha \subseteq \text{dom} \gamma \cap \text{ran} \gamma = \emptyset$$

and so

$$\text{ran} \beta \cap \text{dom} \gamma \subseteq \text{def} \alpha.$$

Thus,

$$|\text{gap} \alpha \cap \text{def} \alpha| \geq |\text{ran} \beta \cap \text{dom} \gamma|$$

and so by (3.4), $|\text{gap} \alpha \cap \text{def} \alpha| = m$.

Suppose now that $|\text{dom} \alpha| (= |\text{ran} \alpha|) < m$. Then $|\text{dom} \alpha \cup \text{ran} \alpha| < m$ and so

$$|\text{gap} \alpha \cap \text{def} \alpha| = |X \setminus (\text{dom} \alpha \cup \text{ran} \alpha)| = m.$$ (3.5)

Thus Lemma 3.3 is proved. It follows from the existence of elements $\alpha \notin N_2$ such that $|\text{gap} \alpha \cap \text{def} \alpha| < m$ that $N_2 \cup N_2^2 \neq K_m$.

To complete the proof of Theorem 3.2, we establish

**Lemma 3.5.** Let $\alpha \in K_m$ be such that $|\text{gap} \alpha \cap \text{def} \alpha| < m$; then $\alpha \in N_2^2$.

**Proof.** Since $\alpha \in K_m$, we have $|\text{gap} \alpha| = |\text{def} \alpha| = m$. Hence, denoting
gap $\alpha \cap \text{def } \alpha$ by $Y$, we see that
\[ |\text{gap } \alpha \setminus Y| = |\text{def } \alpha \setminus Y| = m. \]
Since
\[ \text{gap } \alpha \setminus Y \subseteq \text{ran } \alpha \quad \text{and} \quad \text{def } \alpha \setminus Y \subseteq \text{dom } \alpha, \]
we must therefore have that
\[ |\text{dom } \alpha| = |\text{ran } \alpha| = m. \]
Let $\beta : \text{dom } \alpha \to \text{gap } \alpha \setminus Y$, $\gamma : \text{gap } \alpha \setminus Y \to \text{def } \alpha \setminus Y$ be bijections. Then $\beta, \gamma \in N_2$. Let
\[ \delta = \gamma^{-1} \beta^{-1} \alpha : \text{def } \alpha \setminus Y \to \text{ran } \alpha. \]
Then again $\delta \in N_2$, and it is clear that $\alpha = \beta \gamma \delta$. This completes the proof of Lemma 3.5, and thereby the proof of Theorem 3.2.

4. A class of nilpotent-generated congruence-free inverse semigroups

It is a routine matter to verify that for each cardinal number $k \leq m$, the set
\[ P_k = \{ \alpha \in K_m : |\text{dom } \alpha| (= |\text{ran } \alpha|) < k \} \quad (4.1) \]
is a two-sided ideal of $K_m$. We define
\[ L_m = K_m / P_m, \quad (4.2) \]
the Rees quotient of $K_m$, by the ideal $P_m$. By Theorem 2.4, we have that $K_m / P_m = B^{-1}B$, and we frequently find it convenient to identify $L_m$ with $B^{-1}B \cup \{0\}$.

**Theorem 4.3.** The inverse semigroup $L_m$ defined by (4.2) is 0-bisimple. It is 2-nilpotent generated, and $\Delta_2(L_m) = 3$.

**Proof.** Since $L_m$ is a homomorphic image of $K_m$, it follows from Theorem 3.2 that $L_m$ is a 2-nilpotent-generated inverse semigroup and that $\Delta_2(L_m) \leq 3$. To complete the proof, we require to show first that $L_m$ is 0-bisimple and then that $\Delta_2(L_m) > 2$.

Since $L_m$ is regular, it will follow by [3, Proposition II.3.2] that $L_m$ is 0-bisimple if we show that $(\varepsilon, \eta) \in \mathcal{D}$ for every pair $\varepsilon, \eta$ of idempotents in $K_m / P_m$. Accordingly, let
\[ \varepsilon = 1_A, \quad \eta = 1_B, \]
where $|A| = |B| = |X \setminus A| = |X \setminus B| = m$, be idempotent in $K_m / P_m$. Then if $\alpha : A \to B$ is a bijection, it is clear that $\alpha \in K_m / P_m$ and that $(\varepsilon, \alpha) \in \mathcal{R}$, $(\alpha, \eta) \in \mathcal{L}$. Thus $(\varepsilon, \eta) \in \mathcal{D}$ as required.

Let us denote the set of nilpotent elements of index 2 in $L_m$ by $T$. It is important to realise that in passing from $K_m$ to the Rees quotient $L_m$, we introduce many 'new' nilpotent elements. To be more precise, if we denote by $\alpha \mapsto \tilde{\alpha}$ the Rees homomorphism from $K_m$ onto $L_m = K_m / P_m$, then certainly $\alpha^2 = 0$ in $K_m$ implies $\tilde{\alpha}^2 = 0$ in $L_m$. However, $\tilde{\alpha}^2 = 0$ in $L_m$ implies only that $\alpha^2 \in P_m$ in
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Thus, to show that \( T \cup T^2 \) is properly contained in \( I_m \), we must show that
there exists \( \alpha \) in \( K_m \setminus P_m \) such that \( \alpha^2 \in K_m \setminus P_m \) and
which cannot be expressed as a product \( \beta \gamma \) with \( \beta, \gamma \in K_m \setminus P_m \), \( \beta^2, \gamma^2 \in P_m \). Now to say that \( \beta^2, \gamma^2 \in P_m \) is to say that

\[
|\text{dom } \beta \cap \text{ran } \beta| < m, \quad |\text{dom } \gamma \cap \text{ran } \gamma| < m.
\]

Thus, in view of the existence of elements \( \alpha \) in \( K_m \setminus P_m \), for which \( |\text{gap } \alpha \cap \text{def } \alpha| < m \), the following lemma completes the proof of the theorem. It is a generalisation of part of Lemma 3.3.

**Lemma 4.4.** Let \( \alpha, \alpha^2 \in K_m \setminus P_m \) and suppose that \( \alpha = \beta \gamma \), where \( \beta, \gamma \in K_m \setminus P_m \) and

\[
|\text{dom } \beta \cap \text{ran } \beta| < m, \quad |\text{dom } \gamma \cap \text{ran } \gamma| < m.
\]

Then \( |\text{gap } \alpha \cap \text{def } \alpha| = m \).

**Proof.** Suppose, by way of contradiction, that

\[
|\text{gap } \alpha \cap \text{def } \alpha| < m.
\]

By assumption, since \( \alpha^2 \in K_m \setminus P_m \), we have

\[
|\text{dom } \alpha \cap \text{ran } \alpha| = m.
\]

<table>
<thead>
<tr>
<th>dom ( \alpha )</th>
<th>gap ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ran ( \alpha )</td>
<td>( A_{11} )</td>
</tr>
<tr>
<td>def ( \alpha )</td>
<td>( A_{21} )</td>
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Diagrammatically, we have

\[
|A_{22}| < m, \quad |A_{11}| = m. \quad (4.5)
\]

Since \( \text{def } \alpha = |A_{21} \cup A_{22}| = m \) and \( \text{gap } \alpha = |A_{12} \cup A_{22}| = m \), we must have

\[
|A_{12}| = |A_{21}| = m.
\]

Now from \( \beta \gamma = \alpha \), we deduce that \( \text{dom } \alpha \subseteq \text{dom } \beta \). Hence,

\[
|\text{dom } \alpha \cap \text{ran } \beta| = |\text{dom } \beta \cap \text{ran } \beta| < m. \quad (4.6)
\]

Now write

\[
Q = (\text{dom } \alpha)\beta = (\text{ran } \alpha)\gamma^{-1} = \text{ran } \beta \cap \text{dom } \gamma.
\]

Then \( |Q| = m \). Since

\[
Q = (Q \cap \text{dom } \alpha) \cup (Q \cap A_{12}) \cup (Q \cap A_{22}),
\]

it now follows from (4.5) and (4.6) that

\[
|Q \cap A_{12}| = m.
\]

Hence certainly \( |\text{dom } \gamma \cap A_{12}| = m \). But

\[
A_{12} \subseteq \text{ran } \alpha \subseteq \text{ran } \gamma
\]
and so it now follows that
\[ |\text{dom } \gamma \cap \text{ran } \gamma| = m, \]
contrary to hypothesis. This completes the proof of Lemma 4.4 and hence also of Theorem 4.3.

The semigroup \( L_m \), being 0-bisimple, has no proper ideals. It is not, however, congruence-free. To see this, we invoke a result due to Liber [5], which implies that for each cardinal \( p < m \), the relation
\[ \delta_p = \{(\alpha, \beta) \in K_m \times K_m : (|\alpha \setminus \beta| \cup (\beta \setminus \alpha| < p)\} \] (4.7)
is a congruence on \( K_m \). Here we are interested only in \( \delta_m \) and remark that \( (\alpha, \beta) \in \delta_m \) if and only if
\[ |\text{dom } \alpha \setminus \text{dom } \beta| < m, \]
and
\[ |\text{dom } \beta \setminus \text{dom } \alpha| < m \]
and
\[ |D(\alpha, \beta)| < m, \] (4.8)
where
\[ D(\alpha, \beta) = \{x \in \text{dom } \alpha \cap \text{dom } \beta : x\alpha \neq x\beta\} \]
From our point of view, the crucial (and easily verified) property of \( \delta_m \) is that \( 0\delta_m = P_m \); i.e., any \( \alpha \in K_m \) satisfies \( (\alpha, 0) \in \delta_m \) if and only if \( |\text{dom } \alpha| < m \). Thus, if we denote the Rees congruence
\[ (P_m \times P_m) \cup \{(\alpha, \alpha) : \alpha \in K_m \setminus P_m\} \]
by \( \sigma \), we have \( \sigma \in \delta_m \) and so there is a commutative diagram
\[
\begin{array}{ccc}
K_m & \xrightarrow{\pi} & K_m/\delta_m \\
\sigma \downarrow & & \downarrow \\
L_m & & \\
\end{array}
\]
showing that \( K_m/\delta_m \) is a homomorphic image of \( L_m \). We denote it by \( L^*_m \). It is clear that \( L^*_m \) is a 0-bisimple inverse semigroup.

Lemma 4.9. \( L^*_m \) is congruence free.

Proof. From [9] and [10], recall that a regular 0-simple semigroup \( S \) is congruence-free if and only if the congruence
\[ \sigma = \{(a, b) \in S \times S : s(Y, t \in S^1) sat = 0 \Leftrightarrow stb = 0\} \]
is trivial. Thus what we must show is that if \( \alpha, \beta \in K_m \setminus P_m \) are such that \( (\alpha, \beta) \notin \delta_m \), then there exist \( \lambda, \mu \in K_m \setminus P_m \) such that \( \lambda \alpha \mu \in K_m \setminus P_m \), \( \lambda \beta \mu \in P_m \) (or \( \lambda \alpha \mu \in P_m \), \( \lambda \beta \mu \in K_m \setminus P_m \)).

Suppose therefore that \( (\alpha, \beta) \notin \delta_m \). Then at least one of \(|\text{dom } \alpha \setminus \text{dom } \beta|\), \(|\text{dom } \beta \setminus \text{dom } \alpha|\) and \( D(\alpha, \beta) \) (see (4.8)) has cardinality \( m \). Suppose first that \( |D(\alpha, \beta)| = m \). Then by [6, Lemma 2] there is a subset \( Y \) of \( D(\alpha, \beta) \) such that
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Let \( |Y| = m \) and \( Y \cap Y^\beta = \emptyset \). Let \( \lambda = 1_Y \), \( \mu = 1_{Y^\alpha} \). Then \( \lambda, \mu \in K_m \setminus P_m \), \( \lambda \mu = \alpha \) \( Y \in K_m \setminus P_m \), \( \lambda \beta \mu = 0 \in P_m \).

To complete the proof of Lemma 4.9, it will suffice to consider the case where \( \text{dom} \alpha \setminus \text{dom} \beta \mid = m \). The other case, where \( \text{dom} \beta \setminus \text{dom} \alpha \mid = m \), is identical. Suppose therefore that \( \text{dom} \alpha \setminus \text{dom} \beta \mid = m \) and let \( \lambda, \mu \) be respectively the identity mappings on \( \text{dom} \alpha \setminus \text{dom} \beta \) and on \( \text{dom} \alpha \setminus \text{dom} \beta \alpha \). Then \( \lambda, \mu \in K_m \setminus P_m \).

\[
\lambda \alpha \mu = \alpha \mid (\text{dom} \alpha \setminus \text{dom} \beta) \in K_m \setminus P_m,
\lambda \beta \mu = 0 \in P_m.
\]

This completes the proof of Lemma 4.9.

We have in fact proved most of the following theorem.

**Theorem 4.10.** Let \( L_m^* = K_m / \Delta_m \), where \( K_m \) is defined by (2.6) and \( \Delta_m \) by (4.7). Then \( L_m^* \) is a 0-bisimple, congruence-free 2-nilpotent-generated inverse semigroup, and \( \Delta_2(L_m^*) = 3 \).

**Proof.** Since \( L_m^\ast \) is a homomorphic image of \( L_m \), it is immediate from Theorem 4.3 that it is a 0-bisimple inverse semigroup, that it is 2-nilpotent-generated, and that \( \Delta_2(L_m^*) \leq 3 \). We have already seen that it is congruence-free, and so all that remains is to show that \( \Delta_2(L_m^*) > 2 \).

To investigate this, notice first that an element \( \beta \delta_m \) of \( L_m^* \) (with \( \mu \in K_m \setminus P_m \)) is nilpotent of index 2 in \( L_m^* \) if and only if \( \beta^2 \in 0 \delta_m = P_m \), i.e. if and only if

\[
|\text{dom} \beta \cap \text{ran} \beta| < m.
\]

What we require, therefore, is the following generalization of Lemma 4.4:

**Lemma 4.11.** Let \( \alpha, \alpha^2 \in K_m \setminus P_m \) and suppose that \( (\alpha, \beta \gamma) \in \delta_m \), where \( \beta, \gamma \in K_m \setminus P_m \) and

\[
|\text{dom} \beta \cap \text{ran} \beta| < m, \quad |\text{dom} \gamma \cap \text{ran} \gamma| < m.
\]

Then \( |\text{gap} \alpha \cap \text{def} \alpha| = m \).

**Proof.** Notice that the hypotheses of the lemma give that

\[
|\text{dom} \alpha \setminus \text{dom} \beta \gamma|, \quad |\text{dom} \beta \gamma \setminus \text{dom} \alpha|,
|\text{ran} \alpha \setminus \text{ran} \beta \gamma|, \quad |\text{ran} \beta \gamma \setminus \text{ran} \alpha|.
\]

are all strictly less than \( m \). We use the same notation as in the proof of Lemma 4.4 and assume by way of contradiction that

\[
|A_2| = |\text{gap} \alpha \cap \text{def} \alpha| < m.
\]

Since

\[
\text{dom} \alpha \subseteq \text{dom} \beta \gamma \cup (\text{dom} \alpha \setminus \text{dom} \beta \gamma),
\]

we deduce that

\[
|\text{dom} \alpha \cap \text{ran} \beta| \leq |\text{dom} \beta \gamma \cap \text{ran} \beta| + |(\text{dom} \alpha \setminus \text{dom} \beta \gamma) \cap \text{ran} \beta| \\
\leq |\text{dom} \beta \cap \text{ran} \beta| + |(\text{dom} \alpha \setminus \text{dom} \beta \gamma) \cap \text{ran} \beta| = m.
\]
Again, as in the proof of Lemma 4.4, let
\[ Q = \text{ran}\, \beta \cap \text{dom}\, \gamma. \]
Then, since \( \beta \gamma \notin P_m \), we have \( |Q| = m \). Since
\[ Q = (Q \cap \text{dom}\, \alpha) \cup (Q \cap A_{12}) \cup (Q \cap A_{22}), \]
it follows from (4.13) and (4.14) that
\[ |Q \cap A_{12}| = m; \]
hence certainly \( |\text{dom}\, \gamma \cap \text{ran}\, \alpha| = m \).
Now, if we write \( Z = \text{ran}\, \alpha \cap \text{ran}\, \beta \gamma \), we see that
\[ \text{dom}\, \gamma \cap \text{ran}\, \alpha = (\text{dom}\, \gamma \cap Z) \cup (\text{dom}\, \gamma \cap (\text{ran}\, \alpha \setminus \text{ran}\, \beta \gamma)). \]
Hence, \( |\text{dom}\, \gamma \cap Z| = m \) and so, by (4.12), since
\[ Z \subseteq \text{ran}\, \beta \gamma \subseteq \text{ran}\, \gamma, \]
we have
\[ |\text{dom}\, \gamma \cap \text{ran}\, \gamma| = m, \]
contrary to hypothesis.

Since, as already remarked, we can find elements \( \alpha \) in \( K_m \) which are not nilpotent of index 2 and for which \( |\text{gap}\, \alpha \cap \text{def}\, \alpha| < m \), it follows that there are elements of \( L_m \) that cannot be expressed as products of nilpotents of index 2.
Hence Theorem 4.10 is proved.
We remark finally that
\[ |K_m| = |L_m| = |L_m^m| = 2^m. \]
The proofs, which are not all quite obvious, may be found in [8].

References


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