Inverse semigroups generated by nilpotent transformations

John M. Howie
Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS

and

M. Paula O. Marques-Smith
Area de Matematica, Universidade do Minho, 4700 Braga, Portugal

Synopsis

Let X be a set with infinite cardinality $\mathbf{m}$ and let B be the Baer–Levi semigroup, consisting of all one-one mappings $\alpha: X \to X$ for which $|X \setminus X\alpha| = \mathbf{m}$. Let $K_{\mathbf{m}} = \langle B^{-1}B \rangle$, the inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}(X)$ generated by all products $\beta^{-1}\gamma$, with $\beta, \gamma \in B$. Then $K_{\mathbf{m}} = \langle N_2 \rangle$, where N_2 is the subset of $\mathscr{I}(X)$ consisting of all nilpotent elements of index 2. Moreover, $K_{\mathbf{m}}$ has 2-nilpotent-depth 3, in the sense that $N_2 \cup N_2^2 \subset K_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3$.

Let $P_{\mathbf{m}}$ be the ideal $\{\alpha \in K_{\mathbf{m}}: |\text{dom}\,\alpha| < \mathbf{m}\}$ in $K_{\mathbf{m}}$ and let $L_{\mathbf{m}}$ be the Rees quotient $K_{\mathbf{m}}/P_{\mathbf{m}}$. Then $L_{\mathbf{m}}$ is a 0-bisimple, 2-nilpotent-generated inverse semigroup with 2-nilpotent-depth 3. The minimum non-trivial homomorphic image $L_{\mathbf{m}}^*$ of $L_{\mathbf{m}}$ also has these properties and is congruence-free.

1. Introduction and background

In previous papers [**4, 7**], attention has been focused on certain subsemigroups of an infinite full transformation semigroup, and some interesting examples of (0-) bisimple idempotent-generated congruence-free semigroups have been obtained.

If one looks for an analogue in inverse semigroups, it is clear that 'idempotent-generated' is not an appropriate restriction, since in any inverse semigroup, the idempotents form a subsemigroup. However, a symmetric inverse semigroup $\mathscr{I}(X)$ contains a zero (the empty subset of $X \times X$, but usually denoted by 0) and it is reasonable to look instead at elements that are products of *nilpotent* elements. In this paper, we examine, for infinite $\mathscr{I}(X)$, the inverse subsemigroup generated by all elements α of $\mathscr{I}(X)$ which are *nilpotent of index* 2 (i.e. for which $\alpha \neq 0$, $\alpha^2 = 0$). In the end, we describe, for each infinite cardinal number $\mathbf{m}$, an inverse semigroup $L_{\mathbf{m}}^*$ of cardinality $2^{\mathbf{m}}$ which is 0-bisimple, congruence-free, and generated by the set N_2 consisting of its nilpotent elements of index 2. Moreover, $N_2 \cup N_2^2 \subset L_{\mathbf{m}}^* = N_2 \cup N_2^2 \cup N_2^3$.

2. Preliminaries

For undefined terms in semigroup theory see [**3**].

Let X be a set of infinite cardinality $\mathbf{m}$. For an element α of the symmetric inverse semigroup $\mathscr{I}(X)$, we denote the domain by dom α and the range by ran α.

It is convenient also to define the *gap* and the *defect* of α by

$$\text{gap } \alpha = X\backslash \text{dom } \alpha, \quad \text{def } \alpha = X\backslash \text{ran } \alpha.$$

Then the Baer-Levi semigroup B (of type $(\mathbf{m}, \mathbf{m})$) is defined as

$$B = \{\alpha \in \mathscr{I}(X) \colon \text{gap } \alpha = \varnothing, |\text{def } \alpha| = \mathbf{m}\}. \tag{2.1}$$

(See [**1**]; also [**2**, Section 8.1].) It is easy to see that B is a subsemigroup of $\mathscr{I}(X)$. It is not an inverse semigroup, however: indeed $\alpha \in B$ if and only if $\alpha^{-1} \in B^{-1}$, where

$$B^{-1} = \{\alpha \in \mathscr{I}(X) \colon |\text{gap } \alpha| = \mathbf{m}, \text{def } \alpha = \varnothing\}. \tag{2.2}$$

Then B^{-1} is also a subsemigroup (but not an inverse subsemigroup) of $\mathscr{I}(X)$. From [**2**], we know that B is right cancellative, right simple and has no idempotents. Hence, from the obvious anti-isomorphism $\alpha \mapsto \alpha^{-1}$ from B to B^{-1}, we have that B^{-1} is left cancellative, left simple and has no idempotents.

THEOREM 2.3. *If B, B^{-1} are as defined in* (2.1) *and* (2.2), *then* $BB^{-1} = \mathscr{I}(X)$.

Proof. Let $\alpha \in \mathscr{I}(X)$. Write $\text{dom } \alpha = P$, $\text{ran } \alpha = Q$, and let R_1, R_2, R_3 be pairwise disjoint subsets of x such that

$$|R_1| = |R_2| = |R_3| = \mathbf{m}, \quad X = R_1 \cup R_2 \cup R_3.$$

Certainly $|P| \leqq \mathbf{m}$ and $|X\backslash P| \leqq \mathbf{m}$; hence there are injections

$$\theta \colon P \to R_1, \quad \phi \colon X\backslash P \to R_2.$$

Define $\beta \in \mathscr{I}(X)$ by

$$x\beta = \begin{cases} x\theta & \text{if } x \in P, \\ x\phi & \text{if } x \in X\backslash P. \end{cases}$$

Then $\text{gap } \beta = \varnothing$, $\text{def } \beta \supseteq R_3$ and so $\beta \in B$.

Again, $|X\backslash Q| \leqq \mathbf{m}$ and so we have an injection $\psi \colon X\backslash Q \to R_3$. Define $\gamma \in \mathscr{I}(X)$ by

$$x\gamma = \begin{cases} x\theta^{-1}\alpha & \text{if } x \in P\theta, \\ x\psi^{-1} & \text{if } x \in (X\backslash Q)\psi. \end{cases}$$

Then $\text{ran } \gamma = X$ and $\text{gap } \gamma \supseteq R_2$; hence $\gamma \in B^{-1}$. It is now easy to verify that

$$\text{dom } \beta\gamma = (\text{ran } \beta \cap \text{dom } \gamma)\beta^{-1} = P = \text{dom } \alpha,$$
$$\text{ran } \beta\gamma = (\text{ran } \beta \cap \text{dom } \gamma)\gamma = Q = \text{ran } \alpha$$

and that $x\beta\gamma = x\alpha$ for all x in P. This completes the proof.

By contrast, the product $B^{-1}B$ is not even a subsemigroup of $\mathscr{I}(X)$. This will be easy to demonstrate when we have established

THEOREM 2.4. *If B, B^{-1} are as defined in* (2.1), (2.2), *then*

$$B^{-1}B = \{\alpha \in \mathscr{I}(X) \colon |\text{dom } \alpha| = |\text{ran } \alpha| = |\text{gap } \alpha| = |\text{def } \alpha| = \mathbf{m}\}.$$

Proof. Let $\beta \in B^{-1}$, $\gamma \in B$. Then $\text{ran } \beta = \text{dom } \gamma = X$ and so

$$\text{dom } \beta\gamma = (\text{ran } \beta \cap \text{dom } \gamma)\beta^{-1} = X\beta^{-1} = \text{dom } \beta.$$

Similarly,

$$\operatorname{ran} \beta\gamma = \operatorname{ran} \gamma.$$

It now follows that

$$|\operatorname{dom} \beta\gamma| = |\operatorname{dom} \beta| = \mathbf{m}, \quad |\operatorname{gap} \beta\gamma| = |\operatorname{gap} \beta| = \mathbf{m},$$
$$|\operatorname{ran} \beta\gamma| = |\operatorname{ran} \gamma| = \mathbf{m}, \quad |\operatorname{def} \beta\gamma| = |\operatorname{def} \gamma| = \mathbf{m}.$$

Conversely, let α be an element of $\mathscr{I}(X)$ such that

$$|\operatorname{dom} \alpha| = |\operatorname{ran} \alpha| = |\operatorname{gap} \alpha| = |\operatorname{def} \alpha| = \mathbf{m}$$

and let $\beta: \operatorname{dom} \alpha \to X$ be a bijection. Then define $\gamma: X \to \operatorname{ran} \alpha$ by the rule that

$$x\gamma = x\beta^{-1}\alpha \quad (x \in X).$$

Then $\beta \in B^{-1}$, $\gamma \in B$ and $\beta\gamma = \alpha$.

It is now clear that $B^{-1}B$ is not a subsemigroup of $\mathscr{I}(X)$: for example, if $X = Y \cup Z$, with $|Y| = |Z| = \mathbf{m}$ and $Y \cap Z = \varnothing$, then 1_Y, 1_Z are both in $B^{-1}B$, but $1_Y 1_Z = 0$. We can, however, fairly easily describe $\langle B^{-1}B\rangle$, the subsemigroup of $\mathscr{I}(X)$ generated by $B^{-1}B$.

THEOREM 2.5. *If* B, B^{-1} *are as defined in* (2.1), (2.2), *then*

$$\langle B^{-1}B\rangle = \{\alpha \in \mathscr{I}(X) : |\operatorname{gap} \alpha| = |\operatorname{def} \alpha| = \mathbf{m}\}.$$

Proof. Let us write

$$K_{\mathbf{m}} = \{\alpha \in \mathscr{I}(X) : |\operatorname{gap} \alpha| = |\operatorname{def} \alpha| = \mathbf{m}\}. \tag{2.6}$$

For all α, β in $\mathscr{I}(X)$, we have $\operatorname{dom} \alpha\beta \subseteq \operatorname{dom} \alpha$, $\operatorname{ran} \alpha\beta \subseteq \operatorname{ran} \beta$. Hence, $\operatorname{gap} \alpha\beta \supseteq \operatorname{gap} \alpha$, $\operatorname{def} \alpha\beta \supseteq \operatorname{def} \beta$. It now readily follows that if $\alpha, \beta \in K_{\mathbf{m}}$ then $\alpha\beta \in K_{\mathbf{m}}$. Thus $K_{\mathbf{m}}$, being a subsemigroup of $\mathscr{I}(X)$ containing $B^{-1}B$, must contain $\langle B^{-1}B\rangle$.

Now let $\alpha \in K_{\mathbf{m}}$, and write

$$\operatorname{gap} \alpha = Z \cup T, \quad \operatorname{def} \alpha = P \cup Q,$$

where $Z \cap T = P \cap Q = \varnothing$, $|Z| = |T| = |P| = |Q| = \mathbf{m}$. Let $\theta: Z \to P$ be a bijection, and define

$$\beta = \theta \cup \alpha : Z \cup \operatorname{dom} \alpha \to P \cup \operatorname{ran} \alpha.$$

Then $\operatorname{dom} \beta \supseteq Z$, $\operatorname{gap} \beta \supseteq T$, $\operatorname{ran} \beta \supseteq P$, $\operatorname{def} \beta \supseteq Q$ and so, by Theorem 2.4, $\beta \in B^{-1}B$. Equally, if γ is the identity mapping on $Q \cup \operatorname{ran} \alpha$ then $\gamma \in B^{-1}B$. Since $\operatorname{ran} \beta \cap \operatorname{dom} \gamma = \operatorname{ran} \alpha$, it is now clear that $\beta\gamma = \alpha$ and hence $\alpha \in (B^{-1}B)^2 \subseteq \langle B^{-1}B\rangle$.

Notice that in proving Theorem 2.5 we have proved incidentally that

$$\langle B^{-1}B\rangle = (B^{-1}B)^2.$$

Notice also that while $\langle B^{-1}B\rangle$ is defined merely as the *subsemigroup* generated by $B^{-1}B$, it does in fact turn out to be an *inverse subsemigroup*, as is clear from Theorem 2.5 and the observations that $\operatorname{gap} \alpha^{-1} = \operatorname{def} \alpha$, $\operatorname{def} \alpha^{-1} = \operatorname{gap} \alpha$.

3. A nilpotent-generated inverse semigroup

In this section, we examine some of the properties of the inverse semigroup $K_{\mathbf{m}}$ defined by (2.6). Notice that $0 \in K_{\mathbf{m}}$, since gap $0 = \operatorname{def} 0 = X$.

In a semigroup S with zero, let N_2 be the set of all nilpotent elements of index 2, i.e. the set of all non-zero s in S for which $s^2 = 0$. If $S = \langle N_2 \rangle$ then either the ascent

$$N_2 \subset N_2 \cup N_2^2 \subset N_2 \cup N_2^2 \cup N_2^3 \subset \ldots$$

continues infinitely, or there exists a least $k \geqq 1$ such that

$$N_2 \cup N_2^2 \cup \ldots \cup N_2^k = S.$$

In the former case we write $\Delta_2(S) = \infty$ and in the latter case we write $\Delta_2(S) = k$. We describe S as *2-nilpotent-generated* and refer to $\Delta_2(S)$ as its *2-nilpotent-depth*.

We note in passing that

$$N_2 \subseteq N_2^3;$$

for if $\alpha \in N_2$ then $\alpha^{-1} \in N_2$ and $\alpha = \alpha\alpha^{-1}\alpha$. It follows that

$$N_2 \cup N_2^2 \cup N_2^3 = N_2^2 \cup N_2^3.$$

It is clear that in $\mathscr{I}(X)$, an element α is nilpotent of index 2 if and only if

$$\operatorname{dom} \alpha \neq \varnothing, \quad \operatorname{dom} \alpha \cap \operatorname{ran} \alpha = \varnothing. \tag{3.1}$$

THEOREM 3.2. *Let $\mathscr{I}(X)$ be the symmetric inverse semigroup on a set X of cardinality $\mathbf{m}$, let $K_{\mathbf{m}}$ be as defined in (2.6), and let N_2 be the set of all nilpotent elements of index 2 in $\mathscr{I}(X)$. Then $K_{\mathbf{m}} = \langle N_2 \rangle$, and $\Delta_2(K_{\mathbf{m}}) = 3$.*

Proof. We begin by showing that $N_2 \subseteq K_{\mathbf{m}}$. Let $\alpha \in N_2$. Then α satisfies (3.1) and so $\operatorname{dom} \alpha \subseteq \operatorname{def} \alpha$, $\operatorname{ran} \alpha \subseteq \operatorname{gap} \alpha$. If $|\operatorname{dom} \alpha| = |\operatorname{ran} \alpha| = \mathbf{m}$ this implies that $|\operatorname{def} \alpha| = |\operatorname{gap} \alpha| = \mathbf{m}$, i.e. that $\alpha \in K_{\mathbf{m}}$.

Suppose therefore that $|\operatorname{dom} \alpha| = |\operatorname{ran} \alpha| < \mathbf{m}$. But then it is clear from the statements

$$|\operatorname{dom} \alpha| + |\operatorname{gap} \alpha| = \mathbf{m}, \quad |\operatorname{ran} \alpha| + |\operatorname{def} \alpha| = \mathbf{m}$$

that $|\operatorname{gap} \alpha| = |\operatorname{def} \alpha| = \mathbf{m}$. Hence again $\alpha \in K_{\mathbf{m}}$. Thus $N_2 \subseteq K_{\mathbf{m}}$, from which it follows that

$$N_2 \cup N_2^2 \cup N_2^3 \subseteq K_{\mathbf{m}}.$$

Next, we notice that for an element α of $K_{\mathbf{m}}$, we may have either

$$|\operatorname{gap} \alpha \cap \operatorname{def} \alpha| = \mathbf{m} \quad \text{or} \quad |\operatorname{gap} \alpha \cap \operatorname{def} \alpha| < \mathbf{m}.$$

For an example of the first kind of element, consider $\alpha = 1_Y$, where $X = Y \cup Z$, $Y \cap Z = \varnothing$, $|Y| = |Z| = \mathbf{m}$; here gap $\alpha = \operatorname{def} \alpha = Z$. To find an example not in N_2 of the second kind of element, partition X into three mutually disjoint subsets Y, Z, T, with $|Y| = |Z| = |T| = \mathbf{m}$. Let $\theta: Z \to T$ be a bijection and let

$$\alpha = \theta \cup 1_Y : Y \cup Z \to Y \cup T.$$

Then $\alpha \notin N_2$, gap $\alpha = T$, def $\alpha = Z$.

The following lemma identifies those elements of $K_{\mathbf{m}}$ that lie in N_2^2.

LEMMA 3.3. *Let* $\alpha \in K_{\mathbf{m}}$. *Then* $\alpha \in N_2^2$ *if and only if* $|\text{gap } \alpha \cap \text{def } \alpha| = \mathbf{m}$.

Proof. Let $\alpha \in K_{\mathbf{m}}$ be such that

$$|\text{gap } \alpha \cap \text{def } \alpha| = \mathbf{m}.$$

Write $\text{gap } \alpha \cap \text{def } \alpha = Z \cup T$, with $|Z| = |T| = \mathbf{m}$ and $Z \cap T = \varnothing$. Let $\beta\colon \text{dom } \alpha \to Z$ be an injection. Then $\beta \in N_2$ by (3.1). If $\gamma = \beta^{-1}\alpha$ then

$$\text{dom } \gamma = (\text{ran } \beta^{-1} \cap \text{dom } \alpha)\beta = \text{ran } \beta$$

(since $\text{ran } \beta^{-1} = \text{dom } \beta = \text{dom } \alpha$) and similarly

$$\text{ran } \gamma = \text{ran } \alpha.$$

Since $\text{ran } \beta \subseteq Z \subset \text{gap } \alpha \cap \text{def } \alpha$, we have $\text{dom } \gamma \cap \text{ran } \gamma = \varnothing$ and so $\gamma \in N_2$. Also,

$$\beta\gamma = \beta\beta^{-1}\alpha = \alpha,$$

since $\text{dom } \beta = \text{dom } \alpha$, and so $\alpha \in N_2^2$.

Conversely, let $\alpha = \beta\gamma$, with $\beta, \gamma \in N_2$, and suppose first that $|\text{dom } \alpha|\ (= |\text{ran } \alpha|) = \mathbf{m}$. Then,

$$\mathbf{m} = |\text{ran } \alpha| = |(\text{ran } \beta \cap \text{dom } \gamma)\gamma| = |\text{ran } \beta \cap \text{dom } \gamma|. \tag{3.4}$$

Now $\text{dom } \alpha \subseteq \text{dom } \beta$ and so

$$(\text{ran } \beta \cap \text{dom } \gamma) \cap \text{dom } \alpha \subseteq \text{ran } \beta \cap \text{dom } \beta = \varnothing$$

(since $\beta \in N_2$); hence,

$$\text{ran } \beta \cap \text{dom } \gamma \subseteq \text{gap } \alpha.$$

Similarly, since $\text{ran } \alpha \subseteq \text{ran } \gamma$, we have

$$(\text{ran } \beta \cap \text{dom } \gamma) \cap \text{ran } \alpha \subseteq \text{dom } \gamma \cap \text{ran } \gamma = \varnothing$$

and so

$$\text{ran } \beta \cap \text{dom } \gamma \subseteq \text{def } \alpha.$$

Thus,

$$\text{gap } \alpha \cap \text{def } \alpha \supseteq \text{ran } \beta \cap \text{dom } \gamma$$

and so by (3.4), $|\text{gap } \alpha \cap \text{def } \alpha| = \mathbf{m}$.

Suppose now that $|\text{dom } \alpha|\ (= |\text{ran } \alpha|) < \mathbf{m}$. Then $|\text{dom } \alpha \cup \text{ran } \alpha| < \mathbf{m}$ and so

$$|\text{gap } \alpha \cap \text{def } \alpha| = |X \backslash (\text{dom } \alpha \cup \text{ran } \alpha)| = \mathbf{m}.$$

Thus Lemma 3.3 is proved. It follows from the existence of elements $\alpha \notin N_2$ such that $|\text{gap } \alpha \cap \text{def } \alpha| < \mathbf{m}$ that $N_2 \cup N_2^2 \neq K_{\mathbf{m}}$.

To complete the proof of Theorem 3.2, we establish

LEMMA 3.5. *Let* $\alpha \in K_{\mathbf{m}}$ *be such that* $|\text{gap } \alpha \cap \text{def } \alpha| < \mathbf{m}$; *then* $\alpha \in N_2^3$.

Proof. Since $\alpha \in K_{\mathbf{m}}$, we have $|\text{gap } \alpha| = |\text{def } \alpha| = \mathbf{m}$. Hence, denoting gap $\alpha \cap$ def α by Y, we see that

$$|\text{gap } \alpha \setminus Y| = |\text{def } \alpha \setminus Y| = \mathbf{m}.$$

Since

$$\text{gap } \alpha \setminus Y \subseteq \text{ran } \alpha \quad \text{and} \quad \text{def } \alpha \setminus Y \subseteq \text{dom } \alpha,$$

we must therefore have that

$$|\text{dom } \alpha| = |\text{ran } \alpha| = \mathbf{m}.$$

Let β: dom $\alpha \to$ gap $\alpha \setminus Y$, γ: gap $\alpha \setminus Y \to$ def $\alpha \setminus Y$ be bijections. Then $\beta, \gamma \in N_2$. Let

$$\delta = \gamma^{-1}\beta^{-1}\alpha : \text{def } \alpha \setminus Y \to \text{ran } \alpha.$$

Then again $\delta \in N_2$, and it is clear that $\alpha = \beta\gamma\delta$. This completes the proof of Lemma 3.5, and thereby the proof of Theorem 3.2.

4. A class of nilpotent-generated congruence-free inverse semigroups

It is a routine matter to verify that for each cardinal number $\mathbf{k} \leqq \mathbf{m}$, the set

$$P_{\mathbf{k}} = \{\alpha \in K_{\mathbf{m}} : |\text{dom } \alpha| \ (= |\text{ran } \alpha|) < \mathbf{k}\} \tag{4.1}$$

is a two-sided ideal of $K_{\mathbf{m}}$. We define

$$L_{\mathbf{m}} = K_{\mathbf{m}}/P_{\mathbf{m}}, \tag{4.2}$$

the Rees quotient of $K_{\mathbf{m}}$, by the ideal $P_{\mathbf{m}}$. By Theorem 2.4, we have that $K_{\mathbf{m}} \setminus P_{\mathbf{m}} = B^{-1}B$, and we frequently find it convenient to identify $L_{\mathbf{m}}$ with $B^{-1}B \cup \{0\}$.

Theorem 4.3. *The inverse semigroup $L_{\mathbf{m}}$ defined by* (4.2) *is 0-bisimple. It is 2-nilpotent generated, and $\Delta_2(L_{\mathbf{m}}) = 3$.*

Proof. Since $L_{\mathbf{m}}$ is a homomorphic image of $K_{\mathbf{m}}$, it follows from Theorem 3.2 that $L_{\mathbf{m}}$ is a 2-nilpotent-generated inverse semigroup and that $\Delta_2(L_{\mathbf{m}}) \leqslant 3$. To complete the proof, we require to show first that $L_{\mathbf{m}}$ is 0-bisimple and then that $\Delta_2(L_{\mathbf{m}}) > 2$.

Since $L_{\mathbf{m}}$ is regular, it will follow by [**3**, Proposition II.3.2] that $L_{\mathbf{m}}$ is 0-bisimple if we show that $(\varepsilon, \eta) \in \mathscr{D}$ for every pair ε, η of idempotents in $K_{\mathbf{m}} \setminus P_{\mathbf{m}}$. Accordingly, let

$$\varepsilon = 1_A, \quad \eta = 1_B,$$

where $|A| = |B| = |X \setminus A| = |X \setminus B| = \mathbf{m}$, be idempotent in $K_{\mathbf{m}} \setminus P_{\mathbf{m}}$. Then if $\alpha : A \to B$ is a bijection, it is clear that $\alpha \in K_{\mathbf{m}} \setminus P_{\mathbf{m}}$ and that $(\varepsilon, \alpha) \in \mathscr{R}$, $(\alpha, \eta) \in \mathscr{L}$. Thus $(\varepsilon, \eta) \in \mathscr{D}$ as required.

Let us denote the set of nilpotent elements of index 2 in $L_{\mathbf{m}}$ by T. It is important to realise that in passing from $K_{\mathbf{m}}$ to the Rees quotient $L_{\mathbf{m}}$, we introduce many 'new' nilpotent elements. To be more precise, if we denote by $\alpha \mapsto \bar{\alpha}$ the Rees homomorphism from $K_{\mathbf{m}}$ onto $L_{\mathbf{m}} = K_{\mathbf{m}} \setminus P_{\mathbf{m}}$, then certainly $\alpha^2 = 0$ in $K_{\mathbf{m}}$ implies $\bar{\alpha}^2 = \bar{0}$ in $L_{\mathbf{m}}$. However, $\bar{\alpha}^2 = \bar{0}$ in $L_{\mathbf{m}}$ implies only that $\alpha^2 \in P_{\mathbf{m}}$ in

$K_{\mathbf{m}}$. Thus, to show that $T \cup T^2$ is properly contained in $L_{\mathbf{m}}$, we must show that there exists α in $K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ such that $\alpha^2 \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ and which cannot be expressed as a product $\beta\gamma$ with $\beta, \gamma \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$, $\beta^2, \gamma^2 \in P_{\mathbf{m}}$. Now to say that $\beta^2, \gamma^2 \in P_{\mathbf{m}}$ is to say that

$$|\mathrm{dom}\, \beta \cap \mathrm{ran}\, \beta| < \mathbf{m}, \quad |\mathrm{dom}\, \gamma \cap \mathrm{ran}\, \gamma| < \mathbf{m}.$$

Thus, in view of the existence of elements α in $K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ for which $|\mathrm{gap}\, \alpha \cap \mathrm{def}\, \alpha| < \mathbf{m}$, the following lemma completes the proof of the theorem. It is a generalisation of part of Lemma 3.3.

LEMMA 4.4. *Let $\alpha, \alpha^2 \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ and suppose that $\alpha = \beta\gamma$, where $\beta, \gamma \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$* and

$$|\mathrm{dom}\, \beta \cap \mathrm{ran}\, \beta| < \mathbf{m}, \quad |\mathrm{dom}\, \gamma \cap \mathrm{ran}\, \gamma| < \mathbf{m}.$$

Then $|\mathrm{gap}\, \alpha \cap \mathrm{def}\, \alpha| = \mathbf{m}$.

Proof. Suppose, by way of contradiction, that

$$|\mathrm{gap}\, \alpha \cap \mathrm{def}\, \alpha| < \mathbf{m}.$$

By assumption, since $\alpha^2 \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$, we have

$$|\mathrm{dom}\, \alpha \cap \mathrm{ran}\, \alpha| = \mathbf{m}.$$

	dom α	gap α
ran α	A_{11}	A_{12}
def α	A_{21}	A_{22}

Diagrammatically, we have

$$|A_{22}| < \mathbf{m}, \quad |A_{11}| = \mathbf{m}. \tag{4.5}$$

Since def $\alpha = |A_{21} \cup A_{22}| = \mathbf{m}$ and gap $\alpha = |A_{12} \cup A_{22}| = \mathbf{m}$, we must have

$$|A_{12}| = |A_{21}| = \mathbf{m}.$$

Now from $\beta\gamma = \alpha$, we deduce that dom $\alpha \subseteq$ dom β. Hence,

$$|\mathrm{dom}\, \alpha \cap \mathrm{ran}\, \beta| \leqslant |\mathrm{dom}\, \beta \cap \mathrm{ran}\, \beta| < \mathbf{m}. \tag{4.6}$$

Now write

$$Q = (\mathrm{dom}\, \alpha)\beta = (\mathrm{ran}\, \alpha)\gamma^{-1} = \mathrm{ran}\, \beta \cap \mathrm{dom}\, \gamma.$$

Then $|Q| = \mathbf{m}$. Since

$$Q = (Q \cap \mathrm{dom}\, \alpha) \cup (Q \cap A_{12}) \cup (Q \cap A_{22}),$$

it now follows from (4.5) and (4.6) that

$$|Q \cap A_{12}| = \mathbf{m}.$$

Hence certainly $|\mathrm{dom}\, \gamma \cap A_{12}| = \mathbf{m}$. But

$$A_{12} \subseteq \mathrm{ran}\, \alpha \subseteq \mathrm{ran}\, \gamma$$

and so it now follows that

$$|\text{dom}\,\gamma \cap \text{ran}\,\gamma| = \mathbf{m},$$

contrary to hypothesis. This completes the proof of Lemma 4.4 and hence also of Theorem 4.3.

The semigroup $L_{\mathbf{m}}$, being 0-bisimple, has no proper ideals. It is not, however, congruence-free. To see this, we invoke a result due to Liber [**5**], which implies that for each cardinal $\mathbf{p} < \mathbf{m}$, the relation

$$\delta_{\mathbf{p}} = \{(\alpha, \beta) \in K_{\mathbf{m}} \times K_{\mathbf{m}} \colon |(\alpha \backslash \beta) \cup (\beta \backslash \alpha)| < \mathbf{p}\} \tag{4.7}$$

is a congruence on $K_{\mathbf{m}}$. Here we are interested only in $\delta_{\mathbf{m}}$, and remark that $(\alpha, \beta) \in \delta_{\mathbf{m}}$ if and only if

$$|\text{dom}\,\alpha \backslash \text{dom}\,\beta| < \mathbf{m}, \quad |\text{dom}\,\beta \backslash \text{dom}\,\alpha| < \mathbf{m}$$

and

$$|D(\alpha, \beta)| < \mathbf{m},$$

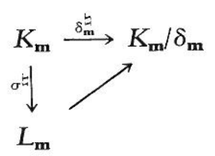
where

$$D(\alpha, \beta) = \{x \in \text{dom}\,\alpha \cap \text{dom}\,\beta \colon x\alpha \neq x\beta\} \tag{4.8}$$

From our point of view, the crucial (and easily verified) property of $\delta_{\mathbf{m}}$ is that $0\delta_{\mathbf{m}} = P_{\mathbf{m}}$; i.e. any α in $K_{\mathbf{m}}$ satisfies $(\alpha, 0) \in \delta_{\mathbf{m}}$ if and only if $|\text{dom}\,\alpha| < \mathbf{m}$. Thus, if we denote the Rees congruence

$$(P_{\mathbf{m}} \times P_{\mathbf{m}}) \cup \{(\alpha, \alpha) \colon \alpha \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}\}$$

by σ, we have $\sigma \subset \delta_{\mathbf{m}}$ and so there is a commutative diagram



showing that $K_{\mathbf{m}}/\delta_{\mathbf{m}}$ is a homomorphic image of $L_{\mathbf{m}}$. We denote it by $L^*_{\mathbf{m}}$. It is clear that $L^*_{\mathbf{m}}$ is a 0-bisimple inverse semigroup.

Lemma 4.9. *$L^*_{\mathbf{m}}$ is congruence free.*

Proof. From [**9**] and [**10**], recall that a regular 0-simple semigroup S is congruence-free if and only if the congruence

$$\sigma = \{(a, b) \in S \times S \colon s(\forall, t \in S^1) sat = 0 \Leftrightarrow sbt = 0\}$$

is trivial. Thus what we must show is that if α, β in $K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ are such that $(\alpha, \beta) \notin \delta_{\mathbf{m}}$, then there exist λ, μ in $K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ such that $\lambda\alpha\mu \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$, $\lambda\beta\mu \in P_{\mathbf{m}}$ (or $\lambda\alpha\mu \in P_{\mathbf{m}}$, $\lambda\beta\mu \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$).

Suppose therefore that $(\alpha, \beta) \notin \delta_{\mathbf{m}}$. Then at least one of $|\text{dom}\,\alpha \backslash \text{dom}\,\beta|$, $|\text{dom}\,\beta \backslash \text{dom}\,\alpha|$ and $D(\alpha, \beta)$ (see (4.8)) has cardinality $\mathbf{m}$. Suppose first that $|D(\alpha, \beta)| = \mathbf{m}$. Then by [**6**, Lemma 2] there is a subset Y of $D(\alpha, \beta)$ such that

$|Y|=\mathbf{m}$ $Y\alpha \cap Y\beta = \varnothing$. Let $\lambda = 1_Y$, $\mu = 1_{Y\alpha}$. Then $\lambda, \mu \in K_m \backslash P_m$, $\lambda\alpha\mu = \alpha \mid Y \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$, $\lambda\beta\mu = 0 \in P_{\mathbf{m}}$.

To complete the proof of Lemma 4.9, it will suffice to consider the case where $|\text{dom}\,\alpha \backslash \text{dom}\,\beta| = \mathbf{m}$. The other case, where $|\text{dom}\,\beta \backslash \text{dom}\,\alpha| = \mathbf{m}$, is identical. Suppose therefore that $|\text{dom}\,\alpha \backslash \text{dom}\,\beta| = \mathbf{m}$ and let λ, μ be respectively the identity mappings on $\text{dom}\,\alpha \backslash \text{dom}\,\beta$ and on $(\text{dom}\,\alpha \backslash \text{dom}\,\beta)\alpha$. Then $\lambda, \mu \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$,

$$\lambda\alpha\mu = \alpha \mid (\text{dom}\,\alpha \backslash \text{dom}\,\beta) \in K_{\mathbf{m}} \backslash P_{\mathbf{m}},$$
$$\lambda\beta\mu = 0 \in P_{\mathbf{m}}.$$

This completes the proof of Lemma 4.9.

We have in fact proved most of the following theorem.

THEOREM 4.10. *Let $L^*_{\mathbf{m}} = K_{\mathbf{m}}/\delta_{\mathbf{m}}$, where $K_{\mathbf{m}}$ is defined by* (2.6) *and $\delta_{\mathbf{m}}$ by* (4.7). *Then $L^*_{\mathbf{m}}$ is a 0-bisimple, congruence-free 2-nilpotent-generated inverse semigroup, and $\Delta_2(L^*_{\mathbf{m}}) = 3$.*

Proof. Since $L^*_{\mathbf{m}}$ is a homomorphic image of $L_{\mathbf{m}}$, it is immediate from Theorem 4.3 that it is a 0-bisimple inverse semigroup, that it is 2-nilpotent-generated, and that $\Delta_2(L^*_{\mathbf{m}}) \leqq 3$. We have already seen that it is congruence-free, and so all that remains is to show that $\Delta_2(L^*_{\mathbf{m}}) > 2$.

To investigate this, notice first that an element $\beta\delta_{\mathbf{m}}$ of $L^*_{\mathbf{m}}$ (with $\nu \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$) is nilpotent of index 2 in $L^*_{\mathbf{m}}$ if and only if $\beta^2 \in 0\delta_{\mathbf{m}} = P_{\mathbf{m}}$, i.e. if and only if

$$|\text{dom}\,\beta \cap \text{ran}\,\beta| < \mathbf{m}.$$

What we require, therefore, is the following generalization of Lemma 4.4:

LEMMA 4.11. *Let $\alpha, \alpha^2 \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ and suppose that $(\alpha, \beta\gamma) \in \delta_{\mathbf{m}}$, where $\beta, \gamma \in K_{\mathbf{m}} \backslash P_{\mathbf{m}}$ and*

$$|\text{dom}\,\beta \cap \text{ran}\,\beta| < \mathbf{m}, \quad |\text{dom}\,\gamma \cap \text{ran}\,\gamma| < \mathbf{m}.$$

Then $|\text{gap}\,\alpha \cap \text{def}\,\alpha| = \mathbf{m}$.

Proof. Notice that the hypotheses of the lemma give that

$$|\text{dom}\,\alpha \backslash \text{dom}\,\beta\gamma|, \quad |\text{dom}\,\beta\gamma \backslash \text{dom}\,\alpha|,$$
$$|\text{ran}\,\alpha \backslash \text{ran}\,\beta\gamma|, \quad |\text{ran}\,\beta\gamma \backslash \text{ran}\,\alpha| \tag{4.12}$$

are all strictly less than $\mathbf{m}$. We use the same notation as in the proof of Lemma 4.4 and assume by way of contradiction that

$$|A_{22}| = |\text{gap}\,\alpha \cap \text{def}\,\alpha| < \mathbf{m}. \tag{4.13}$$

Since

$$\text{dom}\,\alpha \subseteq \text{dom}\,\beta\gamma \cup (\text{dom}\,\alpha \backslash \text{dom}\,\beta\gamma),$$

we deduce that

$$\begin{aligned}|\text{dom}\,\alpha \cap \text{ran}\,\beta| &\leqq |\text{dom}\,\beta\gamma \cap \text{ran}\,\beta| + |(\text{dom}\,\alpha \backslash \text{dom}\,\beta\gamma) \cap \text{ran}\,\beta| \\ &\leqq |\text{dom}\,\beta \cap \text{ran}\,\beta| + |(\text{dom}\,\alpha \backslash \text{dom}\,\beta\gamma) \cap \text{ran}\,\beta| < \mathbf{m}.\end{aligned} \tag{4.14}$$

Again, as in the proof of Lemma 4.4, let

$$Q = \operatorname{ran} \beta \cap \operatorname{dom} \gamma.$$

Then, since $\beta\gamma \notin P_{\mathbf{m}}$, we have $|Q| = \mathbf{m}$. Since

$$Q = (Q \cap \operatorname{dom} \alpha) \cup (Q \cap A_{12}) \cup (Q \cap A_{22}),$$

it follows from (4.13) and (4.14) that

$$|Q \cap A_{12}| = \mathbf{m};$$

hence certainly $|\operatorname{dom} \gamma \cap \operatorname{ran} \alpha| = \mathbf{m}$.

Now, if we write $Z = \operatorname{ran} \alpha \cap \operatorname{ran} \beta\gamma$, we see that

$$\operatorname{dom} \gamma \cap \operatorname{ran} \alpha = (\operatorname{dom} \gamma \cap Z) \cup (\operatorname{dom} \gamma \cap (\operatorname{ran} \alpha \backslash \operatorname{ran} \beta\gamma)).$$

Hence, $|\operatorname{dom} \gamma \cap Z| = \mathbf{m}$ and so, by (4.12), since

$$Z \subseteq \operatorname{ran} \beta\gamma \subseteq \operatorname{ran} \gamma,$$

we have

$$|\operatorname{dom} \gamma \cap \operatorname{ran} \gamma| = \mathbf{m},$$

contrary to hypothesis.

Since, as already remarked, we can find elements α in $K_{\mathbf{m}}$ which are not nilpotent of index 2 and for which $|\operatorname{gap} \alpha \cap \operatorname{def} \alpha| < \mathbf{m}$, it follows that there are elements of $L^*_{\mathbf{m}}$ that cannot be expressed as products of nilpotents of index 2. Hence Theorem 4.10 is proved.

We remark finally that

$$|K_{\mathbf{m}}| = |L_{\mathbf{m}}| = |L^*_{\mathbf{m}}| = 2^{\mathbf{m}}.$$

The proofs, which are not all quite obvious, may be found in [**8**].

References

1 R. Baer and F. Levi. Vollständige irreduzibele Systeme von Gruppenaxiomen. (Beitrage zur Algebra, No. 18) *Sitzungsber. Heidelb. Akad. Wiss. Abh.* **2** (1932), 1–12.

2 A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups*, vols. 1 and 2 (Providence, R.I.: American Math. Soc., 1961 and 1967).

3 J. M. Howie. *An introduction to semigroup theory* (London: Academic Press, 1976).

4 J. M. Howie. A class of bisimple idempotent-generated congruence-free semigroups. *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), 169–184.

5 A. E. Liber. On symmetric generalized groups. *Mat. Sb.* **33** (**75**) (1953), 531–544 (in Russian).

6 D. Lindsey and B. Madison. The lattice of congruences on a Baer-Levi semigroup. *Semigroup Forum* **12** (1976), 63–70.

7 M. Paula O. Marques. A congruence-free semigroup associated with an infinite cardinal number. *Proc. Roy. Soc. Edinburgh Sect. A* **93** (1983), 245–257.

8 M. Paula O. Marques. *Infinite transformation semigroups* (Ph.D. Thesis, University of St Andrews, 1983).

9 B.M. Schein. Homomorphisms and subdirect decomposition of semigroups. *Pacific J. Math.* **17** (1966), 529–547.

10 P. G. Trotter. Congruence-free regular semigroups with zero. *Semigroup Forum* **12** (1976), 1–5.

(*Issued* 17 *January* 1985)