OPTIMAL INVESTMENT DECISIONS FOR TWO POSITIONED FIRMS COMPETING IN A DUOPOLY MARKET WITH HIDDEN COMPETITORS*

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Abstract
This paper extends the literature dealing with the option to invest in a duopoly market for a leader-follower setting. A restrictive assumption embodied in the models in the current literature is that investment opportunities are semi-proprietary in that the two identified or positioned firms are guaranteed to hold at least the follower’s position. More competition is realistically captured in our model by introducing the concept of hidden rivals so that the places in the market can be taken not only by positioned firm but also by these hidden competitors. The value functions and the optimal triggers for the positioned firms differ materially in settings with(out) the presence of hidden rivals. Unlike existing models, our model allows for (a)symmetric market shares and investment costs for the leader and the follower. Cooperative entrance by the two positioned firms is also modeled.

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THE OPTIMAL DECISION TO INVEST FOR TWO POSITIONED FIRMS COMPETING IN A DUOPOLY MARKET WITH HIDDEN COMPETITORS

1. INTRODUCTION

One of the important economic drivers of share prices is the so-called competitive advantage period (CAP), which is defined as the time period over which a firm’s returns are expected to exceed its cost of capital (Mauboussin and Johnson, 1997). Various highly successful investors have used this concept for investment decision making. For example, according to the 1992 Annual Report of Berkshire Hathaway (p. 14), Warren Buffett buys businesses with "high returns on capital" (returns above the cost of capital) that have "deep and wide moats" (sustainable CAPs) and holds them "forever" (with the expectation that the CAPs will not change adversely). The notion also is implicit in more recent applications of the so-called residual income approach (RIM) of Feltham and Ohlson (1995) to estimate firm value, cost of capital and the equity risk premium (e.g., Claus and Thomas, 2001; Jiang and Lee, 2005; Ritter and Warr, 2002). Although the notion is fairly straightforward, exploiting corporate investment opportunities and their associated real options without suffering from overconfidence in order to maximize the value of the resulting CAPs presents a formidable challenge for operating financial managers.

In the less likely case where a real option to invest is entirely firm specific, the decision to invest is not influenced by the actions of potential competitors. Models to value such proprietary options are found in Dixit and Pindyck (1994). However, most real investment opportunities are available to other firms. Thus, models for evaluating investment merit need to incorporate this competitive dimension if they are to emit accurate investment signals.

Generally, the analysis of the option to invest in duopolies considers only the ordering of two firms in the market (e.g., Smets, 1991; Grenadier, 1996; Huisman and Kort, 1999; Paxson and Pinto, 2003; and

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1 In the economics literature, CAP is commonly referred to as the "value-growth duration" (Rappaport, 1992) and "T" (Miller and Modigliani, 1961; Stewart, 1991).
2 This is also implicit in papers that attempt to assess the contribution of ‘assets-in-place’ and ‘growth opportunities’ to the value of a firm’s shares (e.g., Danbolt et al., 2008).
3 Determining the magnitude and intertemporal behavior of CAPs is also a major challenge for investors (Shapiro, 1992) as is the avoidance of overconfidence (Malmeddier and Tate, 2005).
Tsekrekos, 2003). In such a setting, each firm needs to decide when to invest since the first mover (the leader) benefits from a temporary or permanent competitive advantage over the subsequent mover (the follower) by securing, for example, a higher market share (Paxson and Pinto, 2003; and Tsekrekos, 2003). These models treat the investment opportunities as semi-proprietary real options, since the remaining potential entrant receives a proprietary (and perpetual) option to enter the market as a follower after the leader’s entrance into the market.

A more realistic model is one where more than two firms potentially compete for the two places in the market. In this setting, the problem no longer is to define the roles between two firms that are explicitly competing for the two positions in the duopoly market (so-called positioned firms herein) but to define those roles in a context where one or both roles can be taken by some other rival firms (so-called hidden competitors herein). The hidden competitors are not yet revealed publicly as potential competitors but have the capacity to enter the market unexpectedly.

Business history is replete with numerous examples where “previously unknown competitors have appeared from nowhere to overtake established leaders with little apparent effort”. To illustrate, Michael Bloomberg, the current mayor of New York City who turned an initial order for 20 information terminals into a multi-billion dollar, multimedia financial communications conglomerate, stated that his greatest fear was “three guys in a garage right now doing the same thing to us that we did to Reuters and Dow Jones” (Ottoo, 2000).

The literature widely assumes that each positioned firm has full information about its competitor(s). As Lambrecht and Perraudin (2003, p. 621) argue, this full information assumption “is often unrealistic” and leads to the inference that firms should agree to split the surplus created by the real option to avoid the preemptive behavior associated with full information that generates substantial losses in value. Lambrecht and Perraudin develop a model with incomplete information about the costs of the rival firms that does not lead to such cooperative surplus splitting. The model developed herein allows for (in)complete information about competitors. Furthermore, the market roles of the leader and of the follower are obtained
endogenously for the two competing firms, conditional on the exogenous actions of the hidden competitors that are modeled as Poisson jumps.

To this end, we first study the decision to invest with hidden competitors assuming ex post market-share symmetry, where the leader’s competitive advantage expires at the end of the “monopolistic” period. We then relax this assumption to allow for ex post market-share asymmetry, where the leader’s competitive advantage is permanent. For reasons different from those in Pawlina and Kort (2002) or Huisman et al. (2004), this is followed by the introduction of asymmetric investment costs to capture the more plausible assumption that in securing a permanent market advantage, the leader will need more installed capacity at greater cost than that of the follower and/or incur greater initial investment costs per unit of capacity due to, for example, pioneering commissioning costs. Finally, our model is extended to allow for cooperative actions between the two positioned firms.

The remainder of the paper is organized as follows: The model for the decision to invest for two positioned companies facing hidden competition under ex post market-share symmetry is derived in the next section. The model is developed further to allow for a permanent competitive advantage (ex post market-share asymmetry) and (a)symmetric investment costs and illustrated using a numerical example in section 3. Section 4 concludes the paper. A model for cooperative actions between the two positioned firms is derived in an Appendix.

2. THE DECISION TO INVEST WITH HIDDEN COMPETITORS AND EX POST MARKET-SHARE SYMMETRY

Consider two risk-neutral firms, which are assumed to be identical (or symmetric) ex ante and well informed about each other, facing an opportunity to invest $K$ in a duopoly market. These two positioned firms are the only visible potential entrants into the market.

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4 These include access to capital markets, organizational flexibilities, and regulations (Huisman et al., 2004).
5 Paxson and Pinto (2003) and Tsekrekos (2003) assume identical investment costs for the leader and follower, although a permanent competitive advantage is assumed for the leader.
6 Both firms have the same well-informed expectations about each other and about the cash flows and investment costs of the project.
Let $x$ be the net cash flow for the whole market, which evolves stochastically according to a geometric Brownian motion as follows:

$$dx = \alpha x dt + \sigma xdZ$$  \hspace{1cm} (1)

where $x > 0$, $\alpha \in [0, r)$ and $\sigma$ are the drift parameter and instantaneous volatility respectively, $r$ is the risk-free rate, and $dZ$ is the increment of the Wiener process. The total net cash flow for an operating firm depends on $x$ and on the number of firms already in the market. Let $D(C) \leq 1$ be a deterministic parameter which when multiplied by $x$ gives the firm’s total net cash flow. In our model, $D(C)$ represents the market share of a firm.$^7$

In our duopoly market, $C \in \{1, 2\}$; $D(1)$ is the market share for the leader if alone in the market, and $D(2)$ is the equal market share of the leader and follower after the entrance of the later. If an entrant is in a monopoly position, its total net cash flow is $xD(1)$. After the entrance of the follower, the first installed firm must share the market with the newly entered firm so the net cash flow decreases to an equal $xD(2)$ for both firms.$^8$ In order to guarantee a first mover advantage, $D(1) > D(2)$.

Let us now introduce the core aspect of our approach. Instead of competition-based ordering of only two positioned firms in the conventional duopoly market, we assume a non-zero probability for the existence of some hidden competitors, which may enter the market before one or both positioned firms. The two positioned firms both compete for the remaining place (follower) in the market if one hidden competitor enters first. If the hidden competitor enters after the entrance of one of the two positioned firms, then the second positioned firm loses the chance to invest. Thus, in contrast to the existing literature, the non-entered positioned firm does not have a proprietary right to be a follower, nor do both positioned firms have a guarantee of at least occupying the follower’s position. In our model, the entrance of a hidden competitor is assumed to be an exogenous event, corresponding to a Poisson jump with the same expected intensity $\lambda$ for both positioned firms.

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$^7$ Alternatively, $D(C)$ can capture, for example, the monopolistic “excess price” for the goods during the period when the leader is alone in the market.

$^8$ Remember that we are assuming a temporary first-mover advantage. This will be relaxed later in this paper.
As with the existing models, the leader is assumed to have a competitive advantage over the follower, which means that both firms compete for the position to preempt its rival over some interval of the state variable. In this section, this competitive advantage is assumed to be temporary, meaning that the advantage disappears after the follower’s entrance, and both firms share the market equally.  

2.1 The Value Function and Trigger for the Follower

This problem is solved backwards, starting with the follower and assuming that the leader is already in the market. However, unlike existing real duopoly models, we have to analyze two different solutions; namely, the situation where the leader is and is not one of the two positioned firms.

2.1.1 Situation One: A Positioned Firm is Already the Market Leader

Let both the diffusion of $x$ and $D(C)$ be as previously defined. The follower’s value function, $F(x)$, given that the leader has already entered the market, must satisfy the following ordinary differential equation (ODE) during the period in which it is not yet optimal to invest:

$$\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} + \alpha x \frac{\partial F(x)}{\partial x} - (r + \lambda)F(x) = 0$$  \hspace{2cm} (2)

subject to the boundary conditions:

$$\lim_{x \to 0} F(x) = 0$$  \hspace{2cm} (3)

$$\lim_{x \to x_p} F(x) = \frac{x_p D(2)}{r - \alpha} - K$$  \hspace{2cm} (4)

$$\lim_{x \to x_p} \frac{\partial F(x)}{\partial x} = \frac{D(2)}{r - \alpha}$$  \hspace{2cm} (5)

where $x_p$ is the trigger value for the follower (i.e., the value of $x$ at which it is optimal for the follower to invest); and $\lambda \in [0,1)$ is the mean arrival rate of a hidden competitor. During an infinitesimal period of time $dt$, the probability for entrance is given by $\lambda dt$.

After considering the boundary condition (3), the solution for equation (2) takes the form:

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9 Subsequently, the model is derived assuming ex post market-share asymmetry between the leader and the follower after the entrance of the later.

10 This guarantees the existence of the leader’s trigger.
\[ F(x) = Ax^\beta, \] 

where \( A = \frac{D(2)}{\beta x_F^{\beta-1}(r-\alpha)} \), and \( \beta \) is the positive root of the fundamental quadratic equation

\[ 0.5\sigma^2 \beta (\beta - 1) + \alpha \beta - (r + \lambda) = 0; \text{ or:} \]

\[ \beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r + \lambda)}{\sigma^2}} > 1 \] 

(7)

Using the boundary conditions, we obtain:

\[ x_F = \frac{\beta}{\beta - 1} \frac{(r-\alpha)}{D(2)} K \] 

(8)

**Proposition 1.** In the presence of hidden competition and one positioned firm as leader, the follower invests optimally when the state variable \( x \) hits \( x_F \). Thus, the optimal timing for the follower to invest is:

\[ T_F = \inf \left\{ t \geq T_L : x(t) \geq x_F = \frac{\beta}{\beta - 1} \frac{(r-\alpha)}{D(2)} K \right\}. \]

The value of \( \beta \) is increased by the existence of a non-zero (and, of course, positive) \( \lambda \). In turn, this decreases the factor \( \beta/(\beta-1) \), and decreases the trigger value for the follower. Thus, the threshold for the follower is lower (i.e., the follower is a more impatient investor) the higher the probability of entrance of a hidden competitor. Also, the follower wants to preempt the hidden competitor since the entrance of the later eliminates the option to invest, but this interest in preemption depends upon the probability of entrance of the hidden competitor.

As a consequence, the follower’s optimal strategy is to invest immediately when \( x \geq x_F \) by paying \( K \) and receiving \( xD(2)/(r-\alpha) \). Until this action is triggered, the firm has a non-proprietary American option to invest, which can suddenly disappear with the entrance of a hidden competitor. The value of this option is \( K \left( x \right)^\beta / \beta - 1 ( x_F ) \). Thus, the closed-form solution for \( F(x) \) is:

\[ F(x) = \frac{K \left( x \right)^\beta}{\beta - 1 ( x_F )}. \]

\[ 11 \text{ Note that from equation (8), } D(2) = \frac{\beta(r-\alpha)K}{(\beta - 1)x_F} \text{ so that } F(x) = Ax^\beta = \frac{D(2)}{\beta x_F^{\beta-1}(r-\alpha)} x^\beta = \frac{K}{\beta - 1 ( x_F )}. \]
2.1.2 Situation Two: A Hidden Competitor is Already the Market Leader

Since the first firm to enter the market is not a positioned firm, the two positioned firms compete now for the last available place in the market as the follower. Both positioned firms also must consider the possible actions of other hidden competitors.

Recalling our assumption that the two positioned firms are identical and well informed about each other, both firms have \( x_F \) as an optimal trigger for investing. However, precisely because of this, one firm will want to invest a little bit sooner than the optimal trigger, say \( x_F - \varepsilon \), in order to preempt its rival. But anticipating this behavior, the other firm will act even sooner, say at \( x_F - 2\varepsilon \), to not be preempted.

This fear of preemption leads to full preemption, which means that the game only stops when an additional \( \varepsilon \) makes the project worthless. So the new trigger is simply the value for \( x \) which implies a zero-NPV (i.e., the traditional Marshallian trigger) given by:  

\[
F(x) = \begin{cases} 
    \frac{K}{\beta - 1} \left( \frac{x}{x_F} \right)^\beta & \text{for } x < x_F \\
    \frac{xD(2)}{r - \alpha} - K & \text{for } x \geq x_F 
\end{cases}
\]  

(9)

Proposition 2. If the leader is a hidden competitor and both positioned firms are identical ex ante, the competition between them will completely erode the value of the option to defer. Thus, when the leader is a hidden competitor, the optimal timing trigger to invest for both positioned firms is Marshallian; or:

\[
x'_F = \frac{(r - \alpha)}{D(2)} K < x_F
\]

(10)

After observing the leader’s position occupied by a hidden competitor, both positioned firms will decide to invest simultaneously for any \( x \geq x'_F \). However, only one of them will effectively enter the market given an equal or 0.5 probability of success.

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\[\text{After observing the leader’s position occupied by a hidden competitor, both positioned firms will decide to invest simultaneously for any } x \geq x'_F. \text{ However, only one of them will effectively enter the market given an equal or 0.5 probability of success.}\]
The equilibrium presented in proposition 2 may not arise if our assumption that both firms are identical ex-ante does not hold. In other words, the firms may not “share” the same Marshallian trigger if they have, for example, different investment costs for the same position in the market, or different expectations about the future prospects of the project, or different expected operating costs. While one of the firms randomly becomes the leader in the case of symmetry, the advantaged firm enters first to become the leader in the case of asymmetry, which results in a partial erosion of the option to defer.

2.2 The Value Function and the Trigger for the Leader

If the leader has already exercised its option to invest, its value function, \( L(x) \), prior to the entrance of the follower, must satisfy the following non-homogeneous ODE:

\[
\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 L(x)}{\partial x^2} + \alpha x \frac{\partial L(x)}{\partial x} - rL(x) + xD(1) + \lambda [\Phi(x) - L(x)] = 0
\]  

Although this equation is similar to those that appear in some existing models, it has an additional term: \( \lambda [\Phi(x) - L(x)] \), where \( \Phi(x) = xD(2)/(r-\alpha) \) corresponds to the leader’s value function after the entrance of a second firm into the market. This term captures the effect of the expected loss on the leader’s value function from the market entrance of a hidden competitor as a follower when \( x \) has not yet activated the trigger \( x_F \) (i.e., when it is not yet optimal for the positioned follower to invest). If that happens, the leader is no longer alone in the market, and its temporary monopoly advantage disappears sooner than expected.

Two boundary conditions must be placed on this value function; namely:

\[
\lim_{x \to 0} L(x) = 0
\]  

\[
\lim_{x \to x_F} L(x) = \frac{x_F D(2)}{r-\alpha}
\]

After considering the first boundary condition (12), the solution for this non-homogeneous ODE is:

\[
Bx^\beta + \frac{xD(1)}{r-\alpha + \lambda} + \frac{\lambda}{r-\alpha + \lambda} \frac{xD(2)}{r-\alpha}
\]

where \( \beta \) is as presented in equation (7) and

\[\text{13} \text{ Unlike the follower’s ODE, we only need two boundaries here, because we only have two unknowns coming from the solution to the homogeneous part of this ODE. The typical third unknown, the trigger value for the leader, is obtained by indifference between the value functions for the leader and for the follower, rather than by optimization.}\]
\[ B = \frac{x_F [D(2) - D(1)]}{x_F^\beta (r - \alpha + \lambda)}. \] (15)

Since \( x_F = \frac{\beta}{\beta - 1} \frac{(r - \alpha)}{D(2)} K \), then \( B \) can be rearranged as:

\[ B = \frac{\beta}{\beta - 1} \frac{r - \alpha}{r - \alpha + \lambda} \left[ 1 - \frac{D(1)}{D(2)} \right] K \frac{1}{x_F^\beta}. \] (16)

Accordingly, the solution for \( L(x) \) can be expressed as:\(^{14}\)

\[
L(x) = \begin{cases} 
\frac{x_D(1)}{r - \alpha + \lambda} + \frac{\lambda}{r - \alpha + \lambda} \frac{x_D(2)}{r - \alpha} + \frac{\beta}{\beta - 1} \frac{r - \alpha}{r - \alpha + \lambda} \left[ 1 - \frac{D(1)}{D(2)} \right] K \left( \frac{x}{x_F} \right)^\beta, & \text{for } x < x_F \\
\frac{x_D(2)}{r - \alpha}, & \text{for } x \geq x_F
\end{cases}
\] (17)

The trigger for the leader, \( x_L < x_F \), must be such that it will be indifferent for both positioned firms to be a leader or a follower for that value of \( x \). This happens when the value function of the leader minus the investment cost equals the value function of the follower. Specifically:

\[ L(x_L) - K = F(x_L) \] (18)

**Proposition 3.** A unique point \( x_L \in (0, x_F) \) which has the following properties:

\[ L(x_L) - K = F(x_L) \]

\[ L(x) - K < F(x), \quad \text{for } x < x_L \]

\[ L(x) - K \geq F(x), \quad \text{for } x > x_L. \]

**Proof.** See Appendix A■

The condition \( L(x) - K < F(x) \) for \( x < x_L \) means that both firms prefer to be a follower for \( x < x_L \). The condition \( L(x) - K \geq F(x) \) for \( x > x_L \) ensures that the leader’s value function is above that of the follower until

\(^{14}\) The first term in the leader’s value function for \( x < x_F \) represents the present value of the monopolistic cash flows for the leader, which can suddenly disappear with the entrance of a hidden rival. Thus, increasing the discount rate by \( \lambda \) (the instantaneous probability of entrance) accommodates this “risk”. When entrance of a hidden rival occurs, the leader’s cash flow drops to \( x_D(2) \). Thus, the second term captures the expected present value of the cash flows for the leader when sharing the market with a previously hidden follower. Finally, the last term in the leader’s value function is a negative increment that captures the loss that occurs with the entrance of a second positioned firm (as a follower) at \( T_F \).
\( x \) hits \( x_F \) [i.e., \( L(x) - K > F(x) \) for \( x_L < x < x_F \)], and then both functions permanently meet [i.e., \( L(x) - K = F(x) \) for \( x \geq x_F \)].\(^{15,16}\)

**Corollary 1.** The optimal strategy for the leader is to invest at the first moment that \( x \) hits \( x_L \), as obtained in Proposition 3. Thus, the optimal timing for the leader to invest is:

\[
T_L = \inf\{ t \geq 0; x(t) \geq x_L \}. \quad \blacksquare
\]

### 2.3 The Equilibria

If no hidden competitor has already entered the market, two types of strategic equilibria can be considered. Since both firms have the incentive to become the leader if \( x_0 \in (x_L, x_F) \), they invest sequentially with one preemption the other.\(^{17}\) This leads to a preemption equilibrium. None of the positioned firms enter the market if \( x_0 \in (0, x_L) \), because they both prefer to be a follower for that level of \( x \). This means that they will wait until \( x \) hits \( x_L \). In turn, this leads to sequential entrances and preemption equilibrium. For \( x_0 \geq x_F \), both firms are interested in investing immediately, which leads to a simultaneous equilibrium. As shown below, these two types of equilibria can be strongly influenced by the presence of hidden competition.

#### 2.3.1 The Preemption Equilibrium with No One in the Market

The preemption equilibrium occurs when the initial level of the state variable is lower than that of the follower’s trigger; i.e., when \( x_0 < x_F \). In this case, the optimal action for both positioned firms is to preempt each other. Consequently, they both decide to invest immediately if \( x_0 \in (x_L, x_F) \), or as soon as \( x \) hits \( x_L \) if \( x_0 \in (0, x_L) \).

As in some existing models, it is assumed herein that only one of the firms can win the leader’s position. As a result, one positioned firm becomes the leader, and the other has the chance to be the follower. We assume that this happens randomly based on equal odds for both positioned firms.

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\(^{15}\) Note that the assumption that firms are ex post market-share symmetric is relaxed in the next section.  
\(^{16}\) Another way to interpret these properties is to examine the numerical example presented in section 2.4, and depicted in Figure 3.  
\(^{17}\) In this case, both firms act in order to become the leader, although only one randomly achieves this objective.
After not obtaining the leader’s position, the other positioned firm waits to invest until the state variable hits \( x_F \). Unique to our models, the follower’s position is not proprietary (i.e., guaranteed), since there is a non-zero probability for the entrance of a hidden competitor, which is incorporated into \( x_F \) (see Proposition 1).

2.3.2 *The Simultaneous Equilibrium with no Current Market Entrants*

If the initial level of the state variable is higher than or equal to the trigger (i.e., \( x_0 \geq x_F \)), then the optimal strategic action for both positioned companies is to invest immediately. Under this simultaneous equilibrium given ex post market-share symmetry, the roles of the two positioned firms are irrelevant because they have the same value.\(^{18}\)

2.3.3 *The Impact of the Hidden Competitors on the Equilibria*

We now show that movement by a hidden competitor has a major impact on the previously presented equilibria. Suppose that no one is in the market and \( x_0 \in (0, x_L) \). If a hidden competitor enters the market, the preemption equilibrium no longer holds. Since the leader’s position is occupied by a hidden competitor, the two positioned firms compete for the follower’s position. As shown previously, this competition completely erodes the value of the option to defer and the follower now has a classic Marshallian trigger (\( x_F^M \)). Since both positioned firms compete for the follower’s position, only one randomly achieves this objective since each firm has a 0.5 probability to first enter the market as a follower. There also is an additional possibility that a second hidden competitor enters the market before \( x \) hits \( x_F^M \). This eliminates the option to invest for both positioned firms. Neither positioned firm is interested in investing if \( x \) is lower than \( x_F^M \), since that results in a negative NPV.

If \( x_0 \in [x_L, x_F) \) and the leader is one of the two positioned firms, then the other positioned firm optimally waits until \( x \) hits \( x_F \). While waiting, this firm may face the undesired entrance of a hidden competitor, which has a catastrophic impact on its option to invest. Thus, the trigger \( x_F \) must incorporate the probability for this possible occurrence. Figure 1 shows the impact of the probability of entrance of a

\(^{18}\) As we will see later, this will not be the case under asymmetry in ex post market shares and investment costs.
hidden competitor on the follower’s trigger. As expected from equation (8), the trigger $x_F$ is lower for greater $\lambda$, and $x_F$ decreases rapidly for changes at lower levels of $\lambda$. This “risk” has a major impact on the option to wait since firms will want to invest much sooner even if the probability of an exogenous entrance is lower. In contrast, the leader’s trigger increases as $\lambda$ increases, approaching, but not hitting, the follower’s trigger (see Figure 2).

**Figure 1.** The trigger for the follower.
This figure shows the impact of the probability of entrance of a hidden competitor on the follower’s trigger. The parameters are: $D(1) = 1; D(2) = 0.5; K = 80; r = 0.06; \alpha = 0.02; \sigma = 0.2$; and $\lambda$ from 0 to 1.

**Figure 2.** The triggers for the leader and the follower.
This figure shows the impact of the probability of entrance of a hidden competitor on the triggers of the follower and leader. The parameters are: $D(1) = 1; D(2) = 0.5; K = 80; r = 0.06; \alpha = 0.02; \sigma = 0.2$; and $\lambda$ from 0 to 1.
The explanation is straightforward. The leader’s earlier investment is justified because it leads to a temporary market advantage that corresponds to the monopoly period. Since the expected time span over which the leader has a (temporary) competitive advantage decreases as \( \lambda \) increases, the leader will be less interested in investing earlier. Additionally, the trigger values are more sensitive to changes in \( \lambda \) when \( \lambda \) takes on lower values, and the impact on the triggers is marginally less significant as \( \lambda \) increases.

### 2.4 Numerical Example

We present a hypothetical example to illustrate the implementation of the model. Let the inputs be: 

\[
D(1) = 1; \quad D(2) = 0.5; \quad K = 80; \quad r = 0.06; \quad \alpha = 0.02; \quad \sigma = 0.2; \quad \text{and} \quad \lambda = 0.15.
\]

If no hidden competitor is in the market, the triggers for the leader and the follower are an \( x_L \) and \( x_F \) of 6.639 and 9.257, respectively.

If the state variable \( x \) is below 6.639, neither of the positioned firms invests because both prefer to be followers, since the follower’s payoff is higher than that of the leader. Since \( F(V) \) dominates \( L(x) - K \) for \( x \in [0, x_L] \), both firms wait until \( x \) hits 6.639. A preemption equilibrium occurs if \( x \) is at least 6.639 and below 9.257. One of the positioned firms enters as a leader, and the other waits to invest until \( x_F \) is achieved. If \( x \) is above 9.257, then both firms invest simultaneously. The value functions of the leader and follower are plotted in Figure 3.

If a hidden competitor occupies the leader’s position, then the two positioned companies compete for the last available place in the market. The fear of preemption leads to full preemption. In this case, the new trigger is \( x^{\lambda H}_F \), or the value of \( x \) which gives a zero-NPV project \( (x^{\lambda H}_F = 6.4) \). This value is well below \( x_F \) (see Figure 4).

If no hidden competition is assumed (i.e., \( \lambda = 0 \)), then the investment opportunity is semi-proprietorial, and the triggers of the leader \( (x_L) \) and the follower \( (x_F) \) are significantly lower and higher at 4.880 and 15.143, respectively. We can easily verify the effect of the hidden competition on both triggers by examining Figure 5 where \( L(x) \) and \( F(x) \) with \( x_L \) and \( x_F \) when \( \lambda = 0.15 \), and \( x'_L \) and \( x'_F \) when \( \lambda = 0 \).
Fig. 3. The value functions for the leader and for the follower. This figure shows the value functions and the triggers for the leader and the follower. \( L(x) - K \) and \( F(x) \) are the payoff functions for the leader and follower, respectively. The parameters are: \( D(1) = 1; D(2) = 0.5; K = 80; r = 0.06; \alpha = 0.02; \sigma = 0.2; \) and \( \lambda = 0.15 \).

Fig. 4. The value functions for the leader and the follower. This figure shows the triggers for both positioned companies if a hidden competitor enters the market as a leader. \( L(x) - K \) and \( F(x) \) are the payoff functions for the leader and follower, respectively. \( NPV = V - K \).
This figure shows the value functions and triggers for the leader and follower. \( L(x) - K \) and \( F(x) \) are the payoff functions for the leader and follower, respectively. The parameters are: \( D(1) = 1; D(2) = 0.5; K = 80; r = 0.06; \alpha = 0.02; \sigma = 0.2; \) and \( \lambda \) equal to 0 and 0.15.

3. THE DECISION TO INVEST WITH HIDDEN COMPETITORS AND EX POST MARKET-SHARE ASYMMETRY

The assumption that the competitive first mover advantage is temporary is relaxed in this section in order to incorporate the possibility of a permanent market-share advantage for the leader. This advantage may come, for example, from the longer relation between the leader and market, which may result in a higher market share for the leader, even after the entrance of the follower.

Let \( x, D(C) \) and \( xD(C) \) be as previously defined. To incorporate ex post market-share asymmetry, let \( xD(1) \) be the net cash flow for the leader in a monopolistic market, \( xD'(2) \) be the net cash flow for the leader after the entrance of the follower, and \( xD'(2) \) be the net cash flow for the follower. Within the context of this new framework, the first mover advantage is such that \( D(1) > D'(2) > D''(2) \).

Unlike the literature,\(^{19}\) we also assume that the leader has investment costs that are \( k(\%) \) higher than those of the follower.\(^{20}\) As noted earlier, two possible reasons for the higher investment costs are: firstly, the leader may have installed capacity to respond to total market demand during the monopolist period, and secondly, the leader may sell more than that of the follower after the entrance of the follower. This seems to

\(^{19}\) See footnote 2.
\(^{20}\) The model is flexible enough so that \( k(\%) \) can be equal to zero.
be more realistic than just assuming that both firms invest $K$, especially when they are ex post market-share asymmetric.

3.1 The Value Function and the Trigger for the Follower

Starting with the follower, we assume that the leader has already entered the market. As in section 2.1, we must distinguish between two possibilities: firstly, the leader is one of the two positioned firms, and secondly, the leader is a hidden competitor.

If the leader is one of the two positioned firms, then the other positioned firm has the non-proprietary option to be the follower. The value function $F(x)$ must satisfy the ODE (2) subject to the same type of boundaries, but (4) and (5) should read now:

$$\lim_{x \to x_F} F(x) = \frac{x_F D^F(2)}{r - \alpha} - K \quad (19)$$

$$\lim_{x \to x_F} \frac{\partial F(x)}{\partial x} = \frac{D^F(2)}{r - \alpha} \quad (20)$$

Using the same procedures, we find that the follower’s trigger, and its value function are as follows:

$$x_F = \frac{\beta}{\beta - 1} \left( \frac{r - \alpha}{D^F(2)} \right) K \quad (21)$$

$$F(x) = \begin{cases} 
K \left( \frac{x}{x_F} \right)^{\beta} & \text{for } x < x_F \\
\frac{x D^F(2)}{r - \alpha} - K & \text{for } x \geq x_F
\end{cases} \quad (22)$$

Proposition 4. In the presence of hidden competition and a permanent competitive advantage for a positioned leader firm, the follower invests optimally when the state variable $x$ hits $x_F$. So the optimal timing for the follower is:

$$T_F = \inf \left\{ t \geq T_L : x(t) \geq x_F = \frac{\beta}{\beta - 1} \left( \frac{r - \alpha}{D^F(2)} \right) K \right\}. \quad \blacksquare$$

In contrast, if the leader’s position is occupied by a hidden competitor, then the two positioned firms compete to be the follower. The fear of preemption completely erodes the options to defer project
implementation for both positioned firms. This means that both firms will decide to invest immediately when \( x \geq x_F^M = \frac{(r-\alpha)}{D^f(2)} K \), where \( x_F^M \) is the Marshallian trigger or the value for \( x \) that leads to a zero NPV.

**Proposition 5.** If the leader is a hidden competitor and both positioned firms are identical ex ante, their competition will completely erode the value of their options to defer, so that the Marshallian trigger is valid under these circumstances. The optimal timing trigger to invest for both positioned firms becomes:

\[
T_F^M = \inf \left\{ t \geq 0 : x(t) \geq x_F^M = \frac{(r-\alpha)}{D^f(2)} K \right\}. 
\]

3.2 The Value Function and the Trigger for the Leader

The value function for the leader, \( L(x) \), must satisfy the following nonhomogeneous ODE:

\[
\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 L(x)}{\partial x^2} + \alpha x \frac{\partial L(x)}{\partial x} - rL(x) + xD(1) + \lambda \left( x \Omega(x) - L(x) \right) = 0
\]

(23)

This equation is similar to equation (11) except for the last term on the left-hand side of the equation; namely, \( \Omega(x) = xD^f(2)/(r-\alpha) \). With the market entrance of a hidden rival, the leader’s value function is no longer \( L(x) \) (for \( x < x_F \)) since the leader shares the market with that rival while maintaining the first-mover advantage. The boundary condition (12) remains unchanged and the other boundary condition becomes:

\[
\lim_{x \to x_F} L(x) = x_F D^f(2) / (r-\alpha).
\]

(24)

Solving this ODE while considering the boundaries leads to the following solution:

\[
L(x) = \begin{cases} 
\frac{x D(1)}{r - \alpha} + \frac{\lambda}{r - \alpha + \lambda} \frac{x D^f(2)}{r - \alpha} + \frac{\beta}{\beta - 1} \frac{r - \alpha}{r - \alpha + \lambda} \left[ \frac{D^f(2) - D(1)}{D^f(2)} \right] K \left( \frac{x}{x_F} \right)^\rho & \text{for } x < x_F \\
\frac{x D^f(2)}{r - \alpha} & \text{for } x \geq x_F
\end{cases}
\]

(25)

As found previously, the leader’s trigger exists and is obtained by indifference between the value functions for the leader and for the follower. The trigger is the unique value below \( x_F \) where the value functions (net of the investment cost) of the leader and follower intersect. Specifically:

\[
L(x_L) - K_L = F(x_L)
\]

(26)
where $K_L = (1+k)K$. The factor $(1+k)$ reflects the additional price for being the leader instead of the follower.

**Proposition 6.** There exists a unique trigger value $x_L \in (0, x_F)$, which has the following properties:

$$L(x_L) - K_L = F(x_L)$$

$L(x) - K_L < F(x)$, for $x < x_L$

$L(x) - K_L > F(x)$, for $x > x_L$

**Proof.** See Appendix B ■

Thus, if the state variable is below $x_L$ both firms prefer to be the follower $[L(x) - K_L < F(x)]$, and the leader’s value function is above the follower’s for all $x > x_L$. This ensures the permanent competitive advantage for the leader $[i.e., L(x) - K_L > F(x)]$. In order not to violate the latter property, a restriction must be imposed on $kK$, which is the additional amount of money that a firm must spend to enter the market as a leader.

**Proposition 7.** In order to respect the property $L(x) - K_L > F(x)$ for $x > x_L$, the following restriction must be imposed on $k$:

$$k \in [0,k^\ast)$$

where

$$k^\ast = \frac{x_F[D^L(2) - D^F(2)]}{(r-\alpha)K}$$

**Proof.** At $x = x_F$, the property becomes $L(x_F) - K_L > F(x_F) \iff x_F[D^L(2) - D^F(2)] \frac{r-\alpha}{r} - (1+k)K > x_F[D^F(2)] \frac{r-\alpha}{r} - K$. It follows that the additional investment for being the leader must be such that:

$$kkK < \frac{x_F[D^L(2) - D^F(2)]}{(r-\alpha)K}, \quad (27)$$

or, in terms of $k$:

$$k < \frac{x_F[D^L(2) - D^F(2)]}{(r-\alpha)K}. \quad (28)$$

Thus, $k \in \left[0, \frac{x_F[D^L(2) - D^F(2)]}{(r-\alpha)K}\right]$ ■
**Corollary 2.** The optimal strategy for the leader is to invest as soon as \( x \) hits \( x_L \), as was obtained in Proposition 6 for the restrictions presented in Proposition 7. So, the optimal time for the leader to invest is \( T_L = \inf \{ t \geq 0 : x(t) \geq x_L \} \).

### 3.3 The Equilibria

The derivation of the equilibria is similar to the one presented in section 2.3. If no hidden competitor has become the leader, the equilibrium can be either a preemption or a simultaneous equilibrium, depending upon the initial level of the state variable. Since the leader’s permanent competitive advantage (\( k \)) and the hidden competition have an impact on the trigger, they also have an impact on the equilibrium.

With regard to the preemption equilibrium, none of the positioned firms will be interested in immediately exercising the option to invest if \( x_0 < x_L \), since they both prefer to be followers. They will not act until \( x \) hits \( x_L \). As we can see from Figures 6 and 7, \( x_L \) increases as \( k \) increases, and decreases as the permanent competitive advantage increases, ceteris paribus. This means that the existence of some additional costs for being the leader induce the firms to invest later. Also, higher levels of permanent competitive advantage for the leader induce the firms to invest sooner.

![Fig. 6. The trigger for the leader.](image)

This figure shows the impact of the parameter \( k \) on the leader’s trigger. The parameters are: \( D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 80; r = 0.06; \alpha = 0.02; \sigma = 0.2; \lambda = 0.15; \) and \( k \) from 0 to 0.7.
This figure shows the impact of the permanent competitive advantage on the leader’s trigger. The advantage comes in the form of the ratio \( \frac{D^r(2)}{D^F(2)} \) with higher ratios indicating higher competitive advantages. The parameters are: \( K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2; k = 0.2; \) and \( \frac{D^r(2)}{D^F(2)} \) from 1.1 to 2.0.

Both positioned firms want to enter the market as the leader if \( x_0 ∈ [x_L, x_F) \), but only one of them randomly achieves this objective. After the entrance of the leader, the other positioned firm receives a non-proprietary option to be the follower. The later acts optimally by waiting until \( x \) hits \( x_F \). As in the previous section, the follower’s trigger, \( x_F \), incorporates the probability that a hidden competitor will enter to occupy the follower’s position. The higher this probability, the lower is \( x_F \). As a result, the second positioned company will be less willing to defer the implementation of the project.

The simultaneous equilibrium occurs for a \( x_0 ≥ x_F \). In this case, both positioned firms have a strong preemptive incentive. Since \( L(x) > F(x) \) for \( x > x_F > x_L \), they will both invest immediately. However, in contrast to the ex-post market-share symmetry situation, the roles of the positioned firms will be important. One of them will secure the leader’s position with its higher market share and higher investment cost. The other becomes the follower with a lower market share and lower investment cost.

This model is more realistic than some other existing models. For example, ex post market-share asymmetry is present in Paxson and Pinto (2003) and Tsekrekos (2003), even under simultaneous equilibrium (whenever \( x_0 > x_F \)). However, if both firms invest simultaneously in identical projects with the same investment costs, no apparent rationale exists for the leader’s competitive advantage. In our model, the market-share advantage occurs because one of the firms invests more.
We now analyze the impact of hidden competition on the equilibria. To avoid repeating the same arguments, we note that the analysis presented in section 2.3 also is valid here: $x_L$ (leader’s trigger) increases as $\lambda$ increases but due to the permanent competitive advantage enjoyed by the leader, and $x_L$ remains below $x_F$ (see Figure 8). The entrance of a hidden competitor when $x_F$ has not yet been reached now has a more muted impact on the leader’s value function. Thus, in order to guarantee this permanent competitive advantage, the leader is still interested in investing much earlier, even for higher values of $\lambda$.

![Fig. 8. The triggers for the leader and the follower.](image)

This figure shows a simultaneous analysis of the impact of the probability of entrance of a hidden competitor on the triggers. The parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 80; k = 0.2; r = 0.06; \alpha = 0.02; \sigma = 0.2; \lambda$ from 0 and 1.

### 3.4 Numerical Example

We now implement the model using a numerical example. Let the inputs be: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 80; k = 0.2; r = 0.06; \alpha = 0.02; \sigma = 0.2;$ and $\lambda = 0.15$. For these parameters, the leader’s and follower’s value functions and respective triggers are presented in Figure 9.
Fig. 9. The value functions for the leader and the follower.
This figure shows the value functions and triggers for the leader and follower. The parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 80; k = 0.2; r = 0.06; \alpha = 0.02; \sigma = 0.2; \text{ and } \lambda = 0.15$. $L(x) - K_L$ and $F(x)$ are the payoffs to the leader and follower, respectively, given ex post market-share asymmetry.

Unlike the situation of a temporary competitive advantage for the leader (i.e., during the monopolistic period), the advantage here is permanent, which can be verified by examining the leader’s value function. In fact, $L(x)$ is always above $F(x)$ for all $x \geq x_L$. The triggers for the leader and follower are, respectively, 6.039 and 11.571. In the absence of hidden competition, these triggers would be 5.052 for the leader (lower than when $\lambda = 0.15$) and 18.928 for the follower (significantly higher than when $\lambda = 0.15$).

The impact of $k$ on the value function and trigger of the leader are depicted in Figure 10. A higher cost for the leader results in a lower value function and to a higher $x$ level at which it is optimal to enter the market. Table 1 presents the triggers for the leader for different levels of $\lambda$. According to condition (30) and for these parameters, $k$ must be lower than 0.7232.
This figure shows the impact of $k$ on the value function and on the trigger of the leader. The parameters are: $D(1) = 1$; $D(2) = 0.6$; $D'(2) = 0.4$; $K = 80$; $r = 0.06$; $\alpha = 0.02$; $\sigma = 0.2$; $\lambda = 0.15$; and $k = 0$ and 0.2. $L(x)$ and $F(x)$ are the payoffs to the leader and follower, respectively, given ex post market-share asymmetry.

Table 1

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<th>$k$</th>
<th>4.893</th>
<th>6.039</th>
<th>7.357</th>
<th>9.089</th>
<th>11.520</th>
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</thead>
<tbody>
<tr>
<td>$x_L$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.5 An Extension for a Cooperative Game between the Two Positioned Firms

Unlike the previously presented situations where the companies fully compete, the positioned firms can act cooperatively. This type of market behavior can be rational when information is complete, as is the case of the positioned firms, and when some preemptive moves generate substantial losses in value for the competing firms (cfr.: Lambrecht and Perraudin, 2003). In Appendix C, we develop a model where the positioned firms agree to invest together in order to benefit from a lower cooperative investment cost. We show the impact of this economy-of-scale on the triggers, the positive impact of acting cooperatively, and a possible rational deviation from the agreement if the cooperative investment cost is not sufficiently low.
4. CONCLUDING REMARKS

This paper developed an approach to value real options in a duopoly setting when the pool of entrants consists of two known or positioned firms and one or more possible hidden competitors. This approach appears to be the first to include both full and incomplete information environments about competitors. The additional competition has a major impact on the decision to invest (both on the positioned firms’ value functions and triggers), and can completely erode the value of options to defer in some cases. The approach also allows for (a)symmetry in ex post market-share and investment costs for the leader and the follower. Some of the presented equilibria may not arise if the positioned firms are not identical ex ante as assumed herein. Thus, an interesting extension would be to examine the nature of the various equilibria given ex ante asymmetry in terms of the positioned firms.
APPENDICES

APPENDIX A: THE PROOF OF THE UNIQUENESS OF $x_L$ UNDER EX POST MARKET-SHARE SYMMETRY

In this appendix, we prove the uniqueness of $x_L \in (0, x_F)$ under ex post market-share symmetry. Let $H(x) = L(x) - K - F(x)$. Then:

$$H(x) = \frac{xD(1)}{r - \alpha + \lambda} + \frac{\lambda}{r - \alpha + \lambda} + \frac{\beta}{\beta - 1} \frac{r - \alpha}{r - \alpha + \lambda} \left[ D(1) - D(2) \right] K \left( \frac{x}{x_F} \right)^\beta - K \frac{K}{\beta - 1} \left( \frac{x}{x_F} \right)^\beta \quad (A1)$$

Calculating $H(x)$ at $x = 0$ and at $x = x_F$, we obtain $H(0) = -K$ and $H(x_F) = 0$. The derivative of $H(x)$ at $x_F$ is:

$$\frac{dH(x)}{dx} \bigg|_{x=x_F} = \frac{(\beta - 1)\left[D(1) - D(2)\right]}{r - \alpha + \lambda} < 0 \quad (A2)$$

This means that $H(x)$ must have at least one root in the interval $(0, x_F)$.

To prove uniqueness, we only need to demonstrate strict concavity of $H(x)$ over the interval $(0, x_F)$. The second derivative of $H(x)$ is:

$$\frac{\beta K}{x^\beta} \left[ -\left( \frac{x}{x_F} \right)^\beta + \frac{\left( \frac{x}{x_F} \right)^\beta (r - \alpha) \beta \left[ 1 - \frac{D(1)}{D(2)} \right]}{(r - \alpha + \lambda)} \right] < 0 \quad (A3)$$

Thus, since $\lambda$ is nonnegative, $r > \alpha$, $\beta > 1$ and $D(1) > D(2)$, $x_L$ is unique over the interval $(0, x_F)$.

APPENDIX B: THE PROOF OF THE UNIQUENESS OF $x_L$ UNDER EX POST MARKET-SHARE ASYMMETRY

In this appendix, we prove the uniqueness of $x_L \in [0, x_F)$ given ex post market-share asymmetry. Let $H(x) = L(x) - K_L - F(x) = L(x) - (1 + k)K - F(x)$. Then:

$$\frac{xD(1)}{r - \alpha + \lambda} + \frac{\lambda}{r - \alpha + \lambda} + \frac{\beta}{\beta - 1} \frac{r - \alpha}{r - \alpha + \lambda} \left[ D^+(1) - D^+(2) \right] K \left( \frac{x}{x_F} \right)^\beta -(1 + k)K \frac{K}{\beta - 1} \left( \frac{x}{x_F} \right)^\beta \quad (B1)$$

Calculating $H(x)$ at $x = 0$ and at $x = x_F$ yields:
\[ H(0) = -(1 + k)K \]  
\[ H(x_F) = K \left[ \frac{\beta}{\beta - 1} \left( \frac{D^F(2)}{D^F(2)} - 1 \right) - k \right] > 0 \text{ for } k \in [0, k^*), \]  
where \( k^* = \frac{\beta}{\beta - 1} \frac{D^F(2)}{D^F(2)} \) (see also Proposition 7). Thus, \( H(x) \) must have at least one root over the interval \((0, x_L)\).

To prove uniqueness, we only need to demonstrate strict concavity of \( H(x) \) over the interval \((0, x_F)\). The second derivative of \( H(x) \) is:

\[ \frac{\beta K}{x^2} \left[ -\left( \frac{x}{x_F} \right)^\beta + \frac{x}{x_F} \frac{D^F(2)}{D^F(2)} \left( r - \alpha \right) \frac{D^F(2)}{D^F(2)} \right] < 0 \]  

Since \( \lambda \) is nonnegative, \( r > \alpha, \beta > 1 \) and \( D(1) > D^F(2) \), \( x_L \) is unique over the interval \((0, x_F)\) for \( k \in [0, k^*). \)

**APPENDIX C: A COOPERATIVE GAME OPTION BETWEEN THE POSITIONED FIRMS**

In this appendix, we analyze the option based on cooperative action between the two positioned firms. This cooperative option can be a rational decision when the information is complete, as is the case of the positioned firms, and when a preemptive action generates substantial losses in value for the competing firms (cfr.: Lambrecht and Perraudin, 2003).

In our illustrative cooperative game, the two positioned firms agree to invest together, reducing the joint investment cost due to an economy-of-scale effect. A further major benefit from cooperation is the elimination of the fear of preemption for each of the two positioned firms from each other’s action. However, note that competitive pressures remain since the cooperating positioned firms can both be preempted by hidden rivals.

Accordingly, consider an agreement where the two positioned firms accept to invest simultaneously in order to dominate the entire market. The total cost of this cooperative investment is \( K_C, \)
where $K_c \in (K; 2K)$. Assume also that this agreement is of the “all or nothing” type, so that the options to enter the market disappear because the positioned firms do not invest if a hidden rival enters. Technically, this means that the project matures for the positioned firms at the first random entrance of a hidden rival.

Let $C(x)$ be the value function for the positioned firms when acting cooperatively. By the usual arguments, $C(x)$ must satisfy the following ODE, where $\lambda$ is the probability for the entrance of a hidden rival:

$$\frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C(x)}{\partial x^2} + \alpha x \frac{\partial C(x)}{\partial x} - (r + \lambda)C(x) = 0 \quad (C1)$$

subject to the following boundary conditions:

$$\lim_{x \to 0} C(x) = 0 \quad (C2)$$

$$\lim_{x \to x_c} C(x) = \frac{x_c}{r - \alpha} - K_c \quad (C3)$$

$$\lim_{x \to x_c} \frac{\partial C(x)}{\partial x} = \frac{1}{r - \alpha} \quad (C4)$$

Thus, the solution for $C(x)$ takes the form:

$$C(x) = \begin{cases} 
\frac{1}{\beta x_c^\beta} \alpha (r - \alpha) x^\beta & \text{for } x < x_c \\
\frac{x}{r - \alpha} - K_c & \text{for } x \geq x_c
\end{cases} \quad (C5)$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\alpha}{\sigma^2})^2 + \frac{2(r + \lambda)}{\sigma^2}} > 1$, and $x_c$ is the optimal trigger for investing given by:

$$x_c = \frac{\beta}{\beta - 1} (r - \alpha) K_c \quad (C6)$$

**Proposition C1.** If the positioned firms cooperate by agreeing to invest simultaneously with the aim of capturing the whole market under an “all or nothing” scenario by investing $K_c \in (K; 2K)$, they enter the

---

21 $K$ is the investment cost for a single firm, so $K_c$ should be higher than $K$ and lower than $2K$ (combined single-firm cost), due to the economy-of-scale effect.

22 It represents the probability for the project to become worthless in this context.
market optimally when the state variable $x$ hits $x_C$, given the possibility of hidden competition. The optimal timing trigger for the cooperating positioned firms to invest is:

$$T_c = \inf \left\{ t \geq T_c : x(t) \geq x_C = \frac{\beta}{\beta - 1} (r - \alpha) K_c \right\}.$$ 

In this context, an economy-of-scale effect is necessary to promote cooperation among the positioned firms. Since the value of the joint investment is the same as that of the follower in ex post market-share symmetry in the absence of an economy of scale effect, the positioned firms have no incentive to enter into an agreement to cooperate. It is easily shown that equations (9) and (C5) are both similar, after rescaling the investment cost and the cash flow in (C5). In fact, when $K_c = 2K$, the value function is $F(x) = \frac{1}{2} C(x)$ and the trigger is $x_c = x_F$.

**Proposition C2.** If $K_c \rightarrow 2K$, then $x_c \rightarrow x_F$. Thus, if the cooperative investment cost tends to the sum of the individual investments, then the optimal trigger for the cooperating firms tends to the trigger of the ex post market-share symmetric follower, given earlier by equation (8).

**Proof.** We need to prove that $x_c = x_F$ in the limit when $K_c = 2K$ under ex post market symmetry, which is straightforward. After substituting and accounting for (8) and (C6), the following obtains for the ex post case of market-share symmetry:

$$\frac{\beta}{\beta - 1} (r - \alpha) 2K = \frac{\beta}{\beta - 1} \frac{(r - \alpha)}{D(2)} K,$$

if: $2(r - \alpha) = \frac{(r - \alpha)}{D(2)} \Leftrightarrow D(2) = 0.5$.

Figure C1 depicts the cooperative value functions for different levels of $K_c$. We can see that the value of the investment opportunity increases with a decrease in the cooperative investment cost, and that firms will intend to invest sooner with lower cooperative investment costs, as reflected by the lower trigger values.
Fig. C1. The value functions for the firms.
This figure shows the value functions and triggers for the positioned firms acting cooperatively for different levels of $K_C$ in terms of $K$. The parameters are: $K = 80$; $r = 0.06$; $α = 0.02$; $σ = 0.2$; and $λ = 0.10$. The trigger $x_F$ is given by (8).

An interesting finding is that we can find a range of $x$ for a $K_C$ not sufficiently low where it would be more valuable for a firm to deviate from the cooperative action and enter the market immediately as leader, instead of waiting for the optimal timing for the joint investment. This is illustrated in Figure C2.

![Figure C1](image1.png)

Fig. C2. The value functions for the leader and the follower.
This figure shows the value functions for the leader, the follower and for a firm acting cooperatively. The parameters are: $K_C = 1.9K$; $K = 80$; $r = 0.06$; $α = 0.02$; $σ = 0.2$; and $λ = 0.10$. $F(x)$, $L(x)$ and $C(x)$ are given by equations (9), (19) and (C5), respectively.

For a cash flow between $x_1$ and $x_2$, the leader’s value function, $L(x)$, dominates the cooperative position.
In this case, a positioned firm has some incentive to enter the market as leader, and not respect the
agreement by waiting for \( x_C \) (where \( x_C > x_L \)). This incentive increases in relevance if one firm believes that this could be the behavior of the other firm. If this is the case, one company will try not to be preempted by deciding to enter the market before the other firm.

This potential problem can be prevented by establishing some contractual penalty for the firm that does not respect the agreement. Since this penalty must act as a disincentive against not entering into the cooperative investment, the penalty should be proportional to the benefit that one positioned firm could obtain from not respecting the agreement.

For sufficiently large economy-of-scale benefits, this problem may not appear since the cooperative value function dominates the other positions for all \( x \). This is illustrated in Figure C3.

![Figure C3](image_url)

**Fig. C3.** The value functions for the leader and the follower.
This figure shows the value functions for the leader, the follower and for a firm acting cooperatively. The parameters are: \( K_C = 1.5K; \ K = 80; \ r = 0.06; \ \alpha = 0.02; \ \sigma = 0.2; \) and \( \lambda = 0.10. \ F(x), L(x) \) and \( C(x) \) are given by equations (9), (19) and (C5), respectively.

Although we explored the hypothesis for a cooperative game between two positioned firms based on an economy-of-scale effect from joint investment, other reasons that could justify firm collaboration include a tacit agreement for prices, an agreement concerning regional market splitting, and an agreement to acquire business know-how. These can be analyzed in a similar way as is done in this appendix for the economy-of-scale effect from joint investment.

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23 To enter the market optimally, the cooperating positioned firms must wait until \( x \) hits \( x_C \). For these parameters, this occurs when \( x_C = 9.405. \)
REFERENCES


