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# A new special class of Petrov type D vacuum space-times in dimension five.

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**Abstract.** Using extensions of the Newman-Penrose and Geroch-Held-Penrose formalisms to five dimensions, we invariantly classify all Petrov type $D$ vacuum solutions for which the Riemann tensor is isotropic in a plane orthogonal to a pair of Weyl aligned null directions [1].

## 1. The role of the Newman-Penrose (NP) and Geroch-Held-Penrose (GHP) formalisms in General Relativity

The NP formalism [1] consists of an explicit expansion of the Ricci and Bianchi identities in a *null tetrad*, yielding a set of equations (henceforth called NP equations) for the *spin coefficients* and *curvature scalars*. Under certain restrictions (vacuum, algebraic type of the Weyl and/or Ricci tensors, etc) one can choose a null tetrad in which some curvature scalars vanish bringing a simplification to the NP equations. In some cases it is possible to perform a complete study of the consistency conditions of the constraints and their higher order differentials, thus enabling the classification of the family of solutions satisfying the restrictions imposed. The GHP formalism [2] further exploits the invariance of the null tetrad directions under the group of boost and spin transformations in order to work with those variables which are *covariant* under this group. This reduces the number of variables and simplifies the equations. From these considerations, we deduce that the GHP formalism is best suited for situations in which the boost-spin group is the natural invariance group in the geometric set-up. An important example of this can be found in the case of Petrov type D solutions, where the two principal null directions of the Weyl tensor are naturally invariant under boosts, and their orthogonal 2-plane is invariant under spins (see [3, 4] for further details).

## 2. A generalisation of the NP and GHP formalisms to dimension five

The techniques described in the previous paragraph have been developed for the case of a four-dimensional (4D) space-time, but the ideas are general for any dimension as explained in [5, 6, 7]. In our work we analyse these ideas for the particular case of a 5D space-time. In this particular

[1] We dedicate this work to the memory of Professor S. B. Edgar (1945-2010).

case, one can still rely on spinor calculus to introduce the essential quantities of the formalisms, in the same way as in the 4D case. The basics of spinor calculus in a 5D space-time can be found in [8] and we stress that the rationale behind many of our definitions lies in the use of spinors, even though this will not be made explicit in the present work.

We start by introducing a *semi-null pentad* defined as follows

$$N \equiv \{l^a, n^a, m^a, \bar{m}^a, u^a\}, \quad l^a n_a = -1\ , \quad m^a \bar{m}_a = 1\ , \quad u^a u_a = 1. \tag{1}$$

The *frame derivations* associated with each element of this frame are denoted as follows:

$$D \equiv l^a \nabla_a\ , \quad \Delta \equiv n^a \nabla_a\ , \quad \delta \equiv m^a \nabla_a\ , \quad \bar{\delta} \equiv \bar{m}^a \nabla_a\ , \quad \mathcal{D} \equiv u^a \nabla_a. \tag{2}$$

Next we need to define the *spin coefficients* and the *curvature scalars*. Some of them coincide with the quantities appearing in the standard 4D NP formalism (this is to be expected as we work in a frame obtained from the NP null tetrad by just adding the element $u^a$). Here we just enumerate the quantities.

- Twelve NP 4D spin coefficients: $\alpha$, $\beta$, $\gamma$, $\epsilon$, $\kappa$, $\lambda$, $\mu$, $\nu$, $\pi$, $\rho$, $\sigma$, $\tau$.
- Ten complex 5D spin coefficients: $\zeta$, $\eta$, $\theta$, $\chi$, $\omega$, $\phi$, $\xi$, $\upsilon$, $\psi$, $\varsigma$.
- Six real 5D spin coefficients: $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$, $\mathfrak{f}$.

$$2 \times 12 + 2 \times 10 + 6 = 50 \text{ real Ricci rotation coefficients.}$$

- Five complex 4D Weyl scalars: $\Psi_0$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $\Psi_4$,
- Eleven complex 5D Weyl scalars: ${}^*\Psi_0$, ${}^*\Psi_1$, $\Psi_1^*$, ${}^*\Psi_2$, $\Psi_2^*$, ${}^*\Psi_3$, $\Psi_3^*$, $\Psi_4^*$, $\Psi_{01}$, $\Psi_{02}$, $\Psi_{12}$.
- Three real 5D Weyl scalars: $\Psi_{00}$, $\Psi_{11}$, $\Psi_{22}$.
- 4D NP Ricci scalars: Real: $\Phi_{00}$, $\Phi_{11}$, $\Phi_{22}$, Complex: $\Phi_{01}$, $\Phi_{02}$, $\Phi_{12}$.
- 5D trace-free Ricci scalars: Real: $\Omega$, ${}^*\Phi_{01}$, ${}^*\Phi_{12}$, Complex: ${}^*\Phi_{02}$.
- Scalar curvature: $\Lambda$.

$$(2 \times 16 + 3) + (2 \times 4 + 6) + 1 = 50 \text{ independent real Riemann tensor components.}$$

The procedure is now to expand the Ricci and Bianchi identities in the semi-null pentad (1) and write them out in terms of the scalar quantities just introduced. The resulting equations are referred to as the 5D NP equations and they can be regarded as an *extension* of the well-known 4D NP equations. One can also compute the commutation relations of the operators defined in (2) in terms of the spin coefficients and again a set of equations is obtained which extends to dimension five the known commutation relations of the NP formalism.

Under the action of the boost-spin group the semi-null pentad transforms according to

$$l^a \mapsto z\bar{z}\, l^a\ , \quad n^a \mapsto \frac{n^a}{z\bar{z}}\ , \quad m^a \mapsto \frac{z}{\bar{z}}\, m^a\ , \quad \bar{m}^a \mapsto \frac{\bar{z}}{z}\, \bar{m}^a\ , \quad u^a \mapsto u^a \tag{3}$$

for some complex number $z \in \mathbb{C}$. A scalar quantity $Q$ is said to be *weighted* if under (3) it transforms as

$$Q \mapsto z^p \bar{z}^q Q, \tag{4}$$

for some integers $p$ and $q$ (the pair $(p, q)$ being called the weight of $Q$). It is not difficult to check that all the curvature scalars are weighted quantities whereas only a subset of the spin coefficients is weighted. Now, we can follow a procedure similar to the one presented in [2] and introduce new derivations *covariant* under boosts and spins. These new derivations turn out to be uniquely defined and are given by [8]

$$\begin{aligned} &þQ \equiv (D - p\epsilon - q\bar{\epsilon})Q\ , \quad þ'Q \equiv (\Delta - p\gamma - q\bar{\gamma})Q\ , \\ &ðQ \equiv (\delta - p\beta - q\bar{\alpha})Q\ , \quad ð'Q \equiv (\bar{\delta} - p\alpha - q\bar{\beta})Q\ , \quad \widehat{\mathcal{D}}Q \equiv (\mathcal{D} - p\theta - q\bar{\theta})Q\ , \end{aligned} \tag{5}$$

where $Q$ is a quantity with weight $(p, q)$ as in (4). They are referred to as *GHP operators*. Regarding the covariance under a boost-spin transformation (3) one specifically has

$$\begin{aligned} &þQ \mapsto z^{1+p}\bar{z}^{1+q}þQ\ , \quad þ'Q \mapsto z^{p-1}\bar{z}^{q-1}þQ\ , \\ &ðQ \mapsto z^{p-1}\bar{z}^{q+1}ðQ\ , \quad ð'Q \mapsto z^{p+1}\bar{z}^{q-1}ð'Q\ , \quad \widehat{\mathcal{D}}Q \mapsto \widehat{\mathcal{D}}Q\ . \end{aligned} \tag{6}$$

One can now extract the part of the NP equations which only involves derivatives of weighted quantities and obtain a set of equations which we call the 5D GHP equations. The remaining NP equations are absorbed in the commutator relations of the GHP operators.

## 3. The $\mathcal{A}$-class and its invariant classification

The *null alignment theory* [9] provides an algebraic classification of any tensor in a Lorentzian vector space of arbitrary dimension. For the Weyl tensor this implies a generalization of the 4D *Petrov types*. These are characterised by the *Weyl aligned null directions* (WANDS) and their *alignment order*. The WANDs generalise the notion of *principal null direction* of a 4D Weyl tensor. An important difference lies in the fact that the number of WANDs might be infinite or zero (the latter being in fact the generic case) in dimensions higher than four, whereas it is exactly four (counting multiplicity) in the 4D case. We refer the reader to [9] for further details.

In view of the complete classification of the 4D Petrov type D vacuum space-times (containing the well-known vacuum black hole solutions) [10, 11, 12, 13, 3] it is natural to endeavour the classification of 5D Petrov type D vacua (or *Einstein spaces*). By definition, these are space-times (possibly admitting a cosmological constant $\Lambda$) with vanishing trace-free Ricci tensor and exhibiting a pair of double WANDs spanned by $l^a$ and $n^a$ at each point. Taking these as the first two vectors of a semi-null pentad, the only surviving components of the Weyl tensor are the zero boost-weight Weyl scalars

$$\Psi_{11}\ , \quad \Psi_2\ , \quad \Psi_{02}\ , \quad \Psi_2^*\ , \quad {}^*\Psi_2. \tag{7}$$

It turns out that the resulting GHP equations and their consequences are rather unmanageable, although some general properties (even in general dimensions) were deduced in [14, 15]. In order to simplify the computations further, we restricted ourselves to the class $\mathcal{A}$, defined by the property that the Riemann (or Weyl) tensor is isotropic in a plane orthogonal to $l^a$ and $n^a$. Taking the pentad vectors $m^a$ and $\bar{m}^a$ along the complex null directions of this plane and $u^a$ along its normal, only the $(0, 0)$-weighted Weyl scalars $\Psi_{11}$ and $\Psi_2$ are possibly non-zero.

The procedure followed consists of imposing the conditions just described on the extended GHP equations and analyse the corresponding consistency conditions. The details of the integrability analysis will be presented in [16]. Here the possible cases are summarised in table 1. Each case is characterised by algebraic relations fulfilled by the quantities $\Psi_{11}$ and $\Psi_2$. The case $\Psi_2 = 2\Psi_{11}$ has been completely integrated in [15] and only here is the pair of double WANDs not unique. The case $\Psi_{11} = 0$ is a direct generalization of the 4D solutions and can be fully integrated, just as the families characterized by the conditions $h_1$ and $\Psi_2 = -2\Psi_{11}$. All cases only depend on a number of invariantly defined continuous parameters ('global charges'). It is possible to obtain invariant information (even without performing the actual integrations) in the same spirit as e.g. [3]. The Karlhede bound refers to the number of covariant derivatives of the Riemann tensor needed in the invariant classification algorithm [13]. The symbol $s$ ($r$) denotes the dimension of the isotropy (isometry) group.

**Table 1.** Characterizing relations for the subclasses of $\mathcal{A}$. The symbol $h_1$ denotes the condition $\Psi_2 = \bar{\Psi}_2$, $\Psi_2\bar{\Psi}_2 \neq 4\Psi_{11}^2 \neq 0$ and the symbol $h_2$ stands for the condition $\Psi_2\bar{\Psi}_2 = 4\Psi_{11}^2$, $\bar{\Psi}_2 \neq \Psi_2$.

| Subclass | $\Psi_{11} = 0$ | $h_1$ | $h_2$ | $\Psi_2 = 2\Psi_{11}$ | $\Psi_2 = -2\Psi_{11}$ |
|---|---|---|---|---|---|
| Karlhede bound | $\leq 2$ | 1 | $\geq 1$ | $\leq 2$ | $\leq 2$ |
| Global charges | $\leq 4$ | 2 | $\leq 3$ | $\leq 1$ | $\leq 1$ |
| $s$ | $\leq 2$ | 2 | $\leq 2$ | 3 or 4 | 3 or 4 |
| $r$ | $\geq 2$ | 6 | $\geq 3$ | 7 or 9 | 7 or 9 |

## 4. Conclusion and further research

We have performed the invariant classification of a particular class $\mathcal{A}$ of five-dimensional vacuum metrics of Weyl-Petrov type D, possibly admitting a cosmological constant. Some issues have been left open in this work. Perhaps the most important one is a complete integration of the $h_2$ case in table 1, which would enable us to carry out a detailed study of the properties of the corresponding solutions. This would require the further development of integration techniques for the differential equations arising in the 5D GHP equations, in the line of those largely explored by Brian Edgar in the 4D case (see e.g. [17, 18, 3]). Another interesting avenue is to explore other Petrov type D subtypes different to the $\mathcal{A}$-class (the generalisation of our analysis to obtain the generic five-dimensional type D vacuum solution seems to be rather involved).

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