A note on preservation of strong normalisation in the λ-calculus

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July 26, 2010

Abstract

An auxiliary notion of reduction ρ on the λ-terms preserves strong normalisation if all strongly normalising terms for β are also strongly normalising for $\beta \cup \rho$. We give a sufficient condition for ρ to preserve strong normalisation. As an example of application, we check easily the sufficient condition for Regnier's σ-reduction rules and the "assoc"-reduction rule inspired by calculi with let-expressions. This gives the simplest proof so far that the union of all these rules preserves strong normalisation.

1 Introduction

The study of auxiliary notions of reduction in the λ-calculus arises in different contexts and with diverse motivations (see e.g. [6]). A context where auxiliary notions of reduction are natural is in the study of translations from, or into, the λ-calculus. When the λ-calculus is the source of the translation, we may need to modify the equality generated by β, in order to characterize when two terms have the same image. This is the origin of the σ-rules of Regnier [10] (for a translation of the λ-calculus into proof nets), or the A-rules of Sabry-Felleisen [11] (for a CPS-translation), just to give two examples.

On the other hand, when the λ-calculus is the target of some translation, we may wish to simulate the reductions of the source calculus. For a number of related translations [2, 8, 4], based on the simple idea of translating as β-redexes a number of related constructors (let-expressions, generalised applications, explicit substitutions), a single set of auxiliary notions of reduction suffices in the target, in addition to β, for the purposes of simulation: it is the set consisting of rules named π_1 and π_2 in [4]. The first rule is nothing but one of the σ-rules,

*The author is supported by FCT via Centro de Matematica, Universidade do Minho.

named σ_1 here, and, simultaneously, a particular case of one of the A-rules of [11] - the rule named β_{lift}. On the other hand, rule π_2 (named θ_3 and β' in [6] and [2] resp.) is a mild generalisation of a rule sometimes called assoc [8, 7]; and the latter, in turn, is another particular case of β_{lift}, and also a mere translation into the ordinary syntax of the λ-calculus of the "associativity" of let-expressions, a rule of Moggi's computational λ-calculus [9].

Whether ρ is σ, or $\pi := \pi_1 \cup \pi_2$, or other auxiliary notion of reduction, it is often desirable that all the λ-terms strongly normalising for β remain so for $\beta \cup \rho$. When this happens we say that ρ *preserves strong normalisation*. For instance, if the translation $f : \mathcal{S} \to \lambda$ sends typable expressions of the system $\mathcal{S}$ to typable λ-terms, and if f sends reduction steps of the source to non-empty $\beta\rho$-reduction sequences in the λ-calculus, then preservation of strong normalisation by ρ entails that all typable expressions of $\mathcal{S}$ are strongly normalising.

In this note we prove a sufficient condition for an *arbitrary* notion of reduction ρ to preserve strong normalisation. Then, as an example of application, we check the sufficient condition for $\sigma \cup \pi$. The sufficient condition is the conjunction of three restrictions: (i) ρ is "substitutive" and "variable-preserving", which is a very mild requirement, trivial to check; (ii) ρ is itself strongly normalising, which is often known and/or easy; (iii) a certain property holds of weak head ρ-reduction. The proof that this conjunction of requirements is indeed sufficient relies on a single technical argument, showing roughly that once (iii) is true, the property mentioned in (iii) holds of full ρ-reduction. For the particular case of $\sigma \cup \pi$, (i) is immediate and (ii) is essentially known; it remains the verification of (iii), which is straightforward and short.

The rest of this note is organised as follows. Section 2 fixes notation and terminology. Section 3 proves the sufficient condition. Section 4 applies the sufficient condition to the notion of reduction $\sigma \cup \pi$. Section 5 reviews the literature and concludes.

2 Background

The set of λ-terms is denoted Λ, and ranged over by M, N, P, Q, L, R. $\overrightarrow{Q}$ ranges over (possibly empty) sequences of λ-terms. If, say, $\overrightarrow{Q} = N_1, N_2$, we denote by $M\overrightarrow{Q}$ the λ-term MN_1N_2. If $\overrightarrow{Q}$ is the empty sequence (denoted $-$) then $M\overrightarrow{Q}$ denotes M. $FV(M)$ denotes the of variables with free occurrences in M. Barendregt's variable convention is adopted. Substitution is written $[N/x]M$. The size of λ-term M, denoted $|M|$, is defined as follows: $|x| = 1$; $|\lambda x.M| = 1 + |M|$; $|MN| = 1 + |M| + |N|$.

A notion of reduction, or reduction rule, ρ is a binary relation on Λ. $M \mapsto_\rho N$ (ρ-reduction at root position) means $(M, N) \in \rho$. For instance, β is the notion of reduction

$$(\lambda x.M)N \mapsto [N/x]M \ .$$

The other notions of reduction considered in this paper are:

$$
\begin{array}{rrcll}
(\pi_1/\sigma_1) & (\lambda x.M)NP & \mapsto & (\lambda x.MP)N & (x \notin FV(P)) \\
(\sigma_2) & (\lambda x.\lambda y.M)N & \mapsto & \lambda y.(\lambda x.M)N & (y \notin FV(N)) \\
(\pi_2) & M((\lambda x.P)N) & \mapsto & (\lambda x.MP)N & (x \notin FV(M))
\end{array}
$$

We allow two different names for the first rule. Let $\sigma = \sigma_1 \cup \sigma_2$ and $\pi = \pi_1 \cup \pi_2$. σ is introduced in [10], π is studied in [4] as a set of rules for "delaying" a "substitution" $(\lambda x.\_)N$. A particular case of π_2 is

$$
(\mathsf{assoc}) \quad (\lambda y.Q)((\lambda x.P)N) \ \mapsto \ (\lambda x.(\lambda y.Q)P)N \qquad (x \notin FV(Q))
$$

which is a translation of the "associativity" of let-expressions [9][1]

$$
\mathsf{let}\, y = (\mathsf{let}\, x = N \,\mathsf{in}\, P) \,\mathsf{in}\, Q \ \mapsto \ \mathsf{let}\, x = N \,\mathsf{in}\, (\mathsf{let}\, y = P \,\mathsf{in}\, Q) \qquad (x \notin FV(Q))
$$

Given ρ notion of reduction, $\rightarrow_\rho$ denotes ρ-reduction, that is, the compatible closure of ρ. $\rightarrow^n_\rho$ (resp. $\rightarrow^*_\rho$) denotes the n-fold self-composition (resp. the reflexive-transitive closure) of $\rightarrow_\rho$. It is an exercise to see that $\rightarrow^*_\rho$ is the same relation as the reflexive-transitive-compatible closure of ρ. $M \rightarrow_{wh\rho} N$ (M weak head ρ-reduces to N) is defined by: there are $L, R, \overrightarrow{Q} \in \Lambda$ such that $M = L\overrightarrow{Q}$, $N = R\overrightarrow{Q}$, and $L \mapsto_\rho R$. Given another notion of reduction ρ', we usually write $\rho\rho'$ instead of $\rho \cup \rho'$.

A reduction sequence $M = M_0 \rightarrow_\rho M_1 \rightarrow_\rho M_2 \rightarrow_\rho \cdots$ (finite or infinite) is said a ρ-reduction sequence from M. We say that M is strongly normalising for ρ (abbreviated M is ρ-SN, or $M \in \rho - SN$), if all ρ-reduction sequences from M are finite. We say that $\rightarrow_\rho$ is strongly normalising (abbreviated $\rightarrow_\rho$ is SN) if M is ρ-SN, for all M.

Let $||M||_\rho : \Lambda \rightarrow \omega + 1$ be defined by: $||M||_\rho$ is the length of the longest ρ-reduction sequence from the term M, if M is β-SN; and $||M||_\rho = \omega$, otherwise ($\omega + 1$ is the ordinal $\{0, 1, 2, \cdots, \omega\}$)[2].

Definition 1 *A notion of reduction* ρ preserves strong normalisation *if it holds that: M is β-SN iff M is $\beta\rho$-SN.*

Perpetual reduction $\rightarrow_B$ is the binary relation on Λ inductively defined by

[1] Here is a guide for the name of these rules in the literature:

Regnier [10]	Kfouri-Wells [6]	Lengrand [7]	David [3]	This paper
$\sigma(1)$	θ_1	-	γ	π_1/σ_1
$\sigma(2)$	γ	-	δ	σ_2
-	θ_3	-	assoc	π_2
-	-	assoc	-	assoc

Notice that this use of the name σ is inconsistent with its use in the explicit substitution literature, e.g. [4].

[2] In [4, 5], $||M||_\rho$ is defined only for ρ-SN terms.

$$\frac{N \to_B N'}{\lambda x.N \to_B \lambda x.N'}$$

$$\frac{N_i \to_B N_i'}{xN_1 \cdots N_{i-1} N_i \overrightarrow{Q} \to_B xN_1 \cdots N_{i-1} N_i' \overrightarrow{Q}} \; (i)$$

$$\frac{}{(\lambda x.M) N \overrightarrow{Q} \to_B ([N/x]M) \overrightarrow{Q}} \; (ii)$$

$$\frac{N \to_B N'}{(\lambda x.M) N \overrightarrow{Q} \to_B (\lambda x.M) N' \overrightarrow{Q}} \; (iii)$$

Provisos: (i) $i \geq 1$ and $\forall 1 \leq j < i \; N_j$ β-nf. (ii) $x \in FV(M)$ or N a β-nf. (iii) $x \notin FV(M)$ and N not a β-nf.

It is easy to see that $\to_B$ is actually a partial function, which we name B, such that $B(M)$ is undefined iff M is a β-nf. $||M||_B$ denotes the Barendregt's norm of M, that is, the length of the perpetual reduction sequence from M, if $M \in \beta - SN$; or ω, otherwise. $\to_B$ is important because of two properties: (i) M is β-SN iff the perpetual reduction from M is finite [1]; (ii) $||M||_\beta = ||M||_B$ [10, 12]. When this norm is meant, we may drop the subscript.

3 Sufficient condition for PSN

We say that:

- ρ is *substitutive* if $L \mapsto_\rho R$ implies, for all $N \in \Lambda$, $[N/x]L \mapsto_\rho [N/x]R$.
- ρ is *variable-preserving* if $M \mapsto_\rho N$ implies $FV(M) = FV(N)$.

It is routine to show that, for ρ substitutive, if $L \to_\rho R$ then, for all $N \in \Lambda$, $[N/x]L \to_\rho [N/x]R$ and $[L/x]N \to_\rho^* [R/x]N$. On the other hand, it is clear that, if ρ is variable-preserving, then $M \to_\rho N$ implies $FV(M) = FV(N)$.

We need the following abbreviation: $\phi(L, R, \overrightarrow{Q})$ iff there are $M \in \Lambda$ and natural numbers $m \geq n \geq 0$ such that $L\overrightarrow{Q} \to_B^m M$ and $R\overrightarrow{Q} \to_B^n M$. If $\overrightarrow{Q}$ is empty, we may write $\phi(L, R)$ instead of $\phi(L, R, -)$.

The crucial part of our sufficient condition for PSN is the following condition over ρ:

$$\forall L, R, \overrightarrow{Q} \in \Lambda \cdot (L\overrightarrow{Q} \in \beta - SN \,\&\, L \mapsto_\rho R) \Rightarrow \phi(L, R, \overrightarrow{Q}) \; . \tag{1}$$

This condition is equivalent to

$$(M \in \beta - SN \;\; \& \;\; M \to_{wh\rho} N) \Rightarrow \phi(M, N) \; ,$$

which obviously entails that weak head ρ-reduction does not increase the norm $||\_||$, that is

$$M \to_{wh\rho} N \Rightarrow ||M|| \geq ||N|| \ .$$

What is not so obvious is that, if condition (1) holds, then full ρ-reduction does not increase the norm $||\_||$:

$$M \to_{\rho} N \Rightarrow ||M|| \geq ||N|| \ .$$

Theorem 1 (Sufficient condition for PSN) *Let ρ be a substitutive, variable-preserving notion of reduction, satisfying condition (1). Then ρ-reduction does not increase $||\_||_\beta$; in addition, if $\to_\rho$ is SN, then ρ preserves strong normalisation.*

Proof: All there is to prove is that ρ satisfies

$$\forall L, R, \overrightarrow{Q} \in \Lambda \cdot (L\overrightarrow{Q} \in \beta - SN \,\&\, L \to_{\rho}^{*} R) \Rightarrow \phi(L, R, \overrightarrow{Q}) \ . \tag{2}$$

Indeed, from this it follows that ρ-reduction does not increase $||\_||_\beta$. In addition, if $\to_\rho$ is SN and M is β-SN, then we conclude, by induction on $(||M||_\beta, ||M||_\rho)$, that all $\beta\rho$-reduction sequences from M are finite (since ρ-reduction does not increase $||\_||_\beta$).

So we finish by proving (2). Suppose $L\overrightarrow{Q} \in \beta - SN$ and $L \to_\rho^* R$. We prove $\phi(L, R, \overrightarrow{Q})$, that is, we want to exhibit $M \in \Lambda$ and natural numbers $m \geq n \geq 0$ such that $L\overrightarrow{Q} \to_B^m M$ and $R\overrightarrow{Q} \to_B^n M$. The proof is by induction on $||L\overrightarrow{Q}||$ and sub-induction on $L \to_\rho^* R$. Cases according to the last closure rule used in deriving $L \to_\rho^* R$.

First case: $L \mapsto_\rho R$. $\phi(L, R, \overrightarrow{Q})$ follows from (1).

Second case: $L = R$. $\phi(L, R, \overrightarrow{Q})$ is proved by taking $M \equiv L\overrightarrow{Q} \equiv R\overrightarrow{Q}$ and $m = n = 0$.

Third case: $L \equiv L_0 Q_0 \to_\rho^* R_0 Q_0 \equiv R$, with $L_0 \to_\rho^* R_0$. $||L\overrightarrow{Q}|| = ||L_0 Q_0 \overrightarrow{Q}||$. By sub-IH, one obtains $\phi(L_0, R_0, Q_0\overrightarrow{Q})$, which is equivalent to $\phi(L, R, \overrightarrow{Q})$.

Fourth case: $L \equiv \lambda x.L_0 \to_\rho^* \lambda x.R_0 \equiv R$, with $L_0 \to_\rho^* R_0$. There are two sub-cases.

- First subcase: $\overrightarrow{Q}$ is empty. $||\lambda x.L_0|| = ||L_0||$. By sub-IH, $\phi(L_0, R_0, -)$, that is, there are $M_0 \in \Lambda$ and $m \geq n \geq 0$ such that $L_0 \to_B^m M_0$ and $R_0 \to_B^n M_0$. But then $\lambda x.L_0 \to_B^m \lambda x.M_0$ and $\lambda x.R_0 \to_B^n \lambda x.M_0$. So it suffices to take $M = \lambda x.M_0$.
- Second sub-case: $\overrightarrow{Q} = Q_0\overrightarrow{P}$, say. By variable-preservation, $FV(L_0) = FV(R_0)$. Then

$$(\lambda x.L_0)Q_0\overrightarrow{P} \to_B^k ([Q_0'/x]L_0)\overrightarrow{P} \text{ and } (\lambda x.R_0)Q_0\overrightarrow{P} \to_B^k ([Q_0'/x]R_0)\overrightarrow{P} \quad (*)$$

where k is 1 and $Q_0' = Q_0$ (resp. k is $1 + ||Q_0||$ and Q_0' is the β-nf of Q_0) if $x \in FV(L_0)$ (resp. $x \notin FV(L_0)$). In addition, $||(\lambda x.L_0)Q_0\overrightarrow{P}|| >$

$||([Q'_0/x]L_0)\overrightarrow{P}||$, and $([Q'_0/x]L_0)\overrightarrow{P} \rightarrow^*_\rho ([Q'_0/x]R_0)\overrightarrow{P}$ by substitutivity. By IH, $\phi(([Q'_0/x]L_0), ([Q'_0/x]R_0), \overrightarrow{P})$ holds. From this fact and $(*)$ it follows that $\phi(\lambda x.L_0, \lambda x.R_0, Q_0\overrightarrow{P})$

Fifth Case: $L \equiv PL_0 \rightarrow^*_\rho PR_0 \equiv R$, with $L_0 \rightarrow^*_\rho R_0$. There are two subcases.

- First subcase: the β-nf of P is $xN_1 \cdots N_q$, with $q \geq 0$ and each N_i β-nf. Then, for some k,

  $$PL_0\overrightarrow{Q} \rightarrow^k_B xN_1 \cdots N_q L_0 \overrightarrow{Q} \text{ and } PR_0\overrightarrow{Q} \rightarrow^k_B xN_1 \cdots N_q R_0 \overrightarrow{Q} \quad (*)$$

  Notice that $||PL_0\overrightarrow{Q}|| \geq ||L_0||$. So, by IH or sub-IH, $\phi(L_0, R_0, -)$ holds, whence $\phi(xN_1 \cdots N_q L_0, xN_1 \cdots N_q R_0, \overrightarrow{Q})$. From this fact and $(*)$ follows $\phi(PL_0, PR_0, \overrightarrow{Q})$.

- Second subcase: the β-nf of P is an abstraction. Then, for some k, x, and P_0,

  $$PL_0\overrightarrow{Q} \rightarrow^k_B (\lambda x.P_0)L_0\overrightarrow{Q} \text{ and } PR_0\overrightarrow{Q} \rightarrow^k_B (\lambda x.P_0)R_0\overrightarrow{Q} \quad (**)$$

  Next we face a further, and last, bifurcation.

  (i) $x \notin FV(P_0)$. Similarly to the first sub-case, we conclude, from IH or sub-IH, that $\phi(L_0, R_0, -)$ holds. Then $\phi((\lambda x.P_0)L_0, (\lambda x.P_0)R_0, \overrightarrow{Q})$. From this fact and $(**)$ follows $\phi(PL_0, PR_0, \overrightarrow{Q})$.

  (ii) $x \in FV(P_0)$. From $(**)$ we get

  $$PL_0\overrightarrow{Q} \rightarrow^{k+1}_B ([L_0/x]P_0)\overrightarrow{Q} \text{ and } PR_0\overrightarrow{Q} \rightarrow^{k+1}_B ([R_0/x]P_0)\overrightarrow{Q} \quad (***)$$

  Now $||PL_0\overrightarrow{Q}|| > ||([L_0/x]P_0)\overrightarrow{Q}||$ and $([L_0/x]P_0)\overrightarrow{Q} \rightarrow^*_\rho ([R_0/x]P_0)\overrightarrow{Q}$ by substitutivity. So, by IH, $\phi(([L_0/x]P_0), ([R_0/x]P_0), \overrightarrow{Q})$ holds. From this fact and $(***)$ follows $\phi(PL_0, PR_0, \overrightarrow{Q})$.

Sixth, and last, case: there is P such that $L \rightarrow^*_\rho P$ and $P \rightarrow^*_\rho R$. By sub-IH, there are $M_1 \in \Lambda$ and natural numbers $m_1 \geq n_1 \geq 0$ such that $L\overrightarrow{Q} \rightarrow^{m_1}_B M_1$ and $P\overrightarrow{Q} \rightarrow^{n_1}_B M_1$. So $P\overrightarrow{Q}$ is β-SN and $||L\overrightarrow{Q}|| \geq ||P\overrightarrow{Q}||$. Hence, by IH or sub-IH, there are $M_2 \in \Lambda$ and natural numbers $m_2 \geq n_2 \geq 0$ such that $P\overrightarrow{Q} \rightarrow^{m_2}_B M_2$ and $R\overrightarrow{Q} \rightarrow^{n_2}_B M_2$. From $P\overrightarrow{Q} \rightarrow^{n_1}_B M_1$ and $P\overrightarrow{Q} \rightarrow^{m_2}_B M_2$ and the fact that $\rightarrow_B$ is a function, we see that there are three subcases:

- First subcase: $n_1 > m_2$ and M_2 is a term in the reduction sequence $P\overrightarrow{Q} \rightarrow^{n_1}_B M_1$. Take $M = M_1$, $m = m_1$ and $n = n_2 + (n_1 - m_2)$. Then $m_1 \geq n_1 = m_2 + (n_1 - m_2) \geq n_2 + (n_1 - m_2) = n$.

- Second subcase: $n_1 = m_2$ and $M_1 = M_2$. Take $M = M_1 = M_2$, $m = m_1$, and $n = n_2$. Then $m = m_1 \geq n_1 = m_2 \geq n_2 = n$.
- Third subcase: $n_1 < m_2$ and M_1 is a term in the reduction sequence $P\overrightarrow{Q} \rightarrow_B^{m_2} M_2$. Take $M = M_2$, $m = m_1 + (m_2 - n_1)$, and $n = n_2$. Then $m = m_1 + (m_2 - n_1) \geq n_1 + (m_2 - n_1) = m_2 \geq n_2 = n$. ∎

4 Example

Here we exemplify the use of Theorem 1 for $\sigma \cup \pi$.

Proposition 1 (Sufficient condition for $\sigma \cup \pi$)

1. *$\sigma \cup \pi$ is substitutive and variable preserving.*
2. *$\rightarrow_{\sigma\pi}$ is SN.*
3. *$\sigma \cup \pi$ satisfies condition (1).*

Proof: 1. Immediate.

2. We extend to $\sigma\pi = \sigma\pi_2$ the proof for $\pi = \pi_1\pi_2$ in [4]. The argument is the same, let us repeat it. Strong normalisation of $\rightarrow_\sigma$ is in [10]. Define $w(M)$, the *weight* of a λ-term M, as follows: $w(x) = 0$; $w(\lambda x.M) = w(M)$; $w(MN) = |N| + w(M) + w(N)$. It holds that, if $M \rightarrow_\sigma N$, then $w(M) = w(N)$ and $|M| = |N|$; and that, if $M \rightarrow_{\pi_2} N$, then $w(M) > w(N)$ and $|M| = |N|$. The proofs are by induction on $M \rightarrow_\sigma N$ and $M \rightarrow_{\pi_2} N$, respectively (the statements about size are induction loading). Finally, one proves that all $\sigma\pi_2$-reduction sequences from M are finite by induction on $(w(M), ||M||_\sigma)$.

3. Let us prove the condition for each of σ_1, σ_2, and π_2.

Case σ_1: Let $Q_0 = L\overrightarrow{Q} = (\lambda x.M)NP\overrightarrow{Q}$ be β-SN and $Q_1 = R\overrightarrow{Q} = (\lambda x.MP)N\overrightarrow{Q}$. Let k be either $||N||$, if $x \notin FV(M)$; or 0, otherwise. Then $Q_i \rightarrow_B^{k+1} ([N/x]M)P\overrightarrow{Q}$, $i = 0, 1$. So $\phi(L, R, \overrightarrow{Q})$.

Case σ_2: Let $Q_0 = L\overrightarrow{Q} = (\lambda x.\lambda y.M)N\overrightarrow{Q}$ be β-SN and $Q_1 = R\overrightarrow{Q} = (\lambda y.(\lambda x.M)N)\overrightarrow{Q}$. There are two subcases.

First sub-case: $\overrightarrow{Q} = -$. Let k be either $||N||$, if $x \notin FV(M)$; or 0, otherwise. Then $Q_i \rightarrow_B^{k+1} (\lambda y.[N/x]M)$, $i = 0, 1$. So $\phi(L, R)$.

Second sub-case: $\overrightarrow{Q} = Q\overrightarrow{P}$, say. Let k and l be defined as follows. If $x \notin FV(M)$, then $k = ||N||$; otherwise, $k = 0$. If $y \notin FV(M)$, then $l = ||Q||$; otherwise, $l = 0$. $\phi(L, R, \overrightarrow{Q})$ is verified as follows:

$$\begin{array}{rcl} Q_0 & = & (\lambda x.\lambda y.M)NQ\overrightarrow{P} \\ & \rightarrow_B^{k+1} & (\lambda y.[N/x]M)Q\overrightarrow{P} \\ & \rightarrow_B^{l+1} & ([Q/y][N/x]M)\overrightarrow{P} \\ & = & ([N/x][Q/y]M)\overrightarrow{P} \;, \end{array}$$

where the last equality is by substitution lemma and $y \notin FV(N)$.

$$
\begin{array}{rcl}
Q_1 & = & (\lambda y.(\lambda x.M)N)Q\overrightarrow{P} \\
& \rightarrow_B^{l+1} & (\lambda x.[Q/y]M)N\overrightarrow{P} \\
& \rightarrow_B^{k+1} & ([N/x][Q/y]M)\overrightarrow{P} \ .
\end{array}
$$

Case π_2: Let $Q_0 = M((\lambda x.P)N)\overrightarrow{Q}$ be β-SN and $Q_1 = (\lambda x.MP)N\overrightarrow{Q}$. The goal is to exhibit M_0 and $m \geq n \geq 0$ such that $Q_0 \rightarrow_B^m M_0$ and $Q_1 \rightarrow_B^n M_0$. Q_0 and Q_1 have a common reduct, namely $M([N/x]P)\overrightarrow{Q}$ ($=MP\overrightarrow{Q}$, if $x \notin FV(P)$), a common reduct to which Q_1 reduces by perpetual reduction. Since the common reduct is a reduct of Q_0, it is β-SN. So, the perpetual reduction of Q_1 terminates and Q_1 is β-SN as well. In addition, Q_0 and Q_1 have the same β-nf. Let M_0 be this β-nf. Take $m = ||Q_0||$ and $n = ||Q_1||$. Hence $Q_0 \rightarrow_B^m M_0$ and $Q_1 \rightarrow_B^n M_0$. We show $m \geq n$. If $x \notin FV(P)$, then $Q_0 \rightarrow_\beta^{k+1} MP\overrightarrow{Q}$ and $Q_1 \rightarrow_B^{k+1} MP\overrightarrow{Q}$, where $k = ||N||$; so, $||Q_0|| \geq k + 1 + ||MP\overrightarrow{Q}|| = ||Q_1||$. If, on the other hand, $x \in FV(P)$, then $Q_0 \rightarrow_\beta M([N/x]P)\overrightarrow{Q}$ and $Q_1 \rightarrow_B M([N/x]P)\overrightarrow{Q}$; so, $||Q_0|| \geq 1 + ||M([N/x]P)\overrightarrow{Q}|| = ||Q_1||$. ∎

Theorem 2 (PSN for $\sigma \cup \pi$)

1. $M \rightarrow_{\sigma\pi} N \Rightarrow ||M|| \geq ||N||$.
2. $M \in \beta\sigma\pi - SN \Leftrightarrow M \in \beta - SN$.

Proof: From the previous Proposition and Theorem 1.[3] ∎

5 Final remarks

Preservation of strong normalisation was addressed in several papers [2, 8, 4, 7, 3]. The rule π_2 was considered in [2], but only the finiteness of developments for $\beta \cup \pi_2$ was proved. In [4] preservation of strong normalisation by π is stated, but the proof of one auxiliary result is incomplete [4]. In [8, 7] preservation of strong normalisation by $\mathsf{assoc} \subset \pi_2$ is considered, but only [7] gives a full proof, by refining the idea of postponing assoc-steps. [3] proves that all λ-terms typable in the well known intersection type system $\mathcal{D}$ are strongly normalising for $\beta \cup \sigma \cup \pi$, from which preservation of strong normalisation by $\sigma \cup \pi$ follows.

[3] A side remark on σ-reduction. Regnier [10] observes that

$$M \rightarrow_\sigma N \Rightarrow ||M|| \leq ||N||$$

is an immediate consequence of the commutation between σ and β (Corollary 3.5 in [10]), but obtains the other inequality only after a quite complex argument. The other inequality is contained in the first statement of this theorem.

[4] That's Proposition 6 on page 173, saying that π-reduction does not increase the norm $||\_||_\beta$. The author thanks Stéphane Lengrand for pointing this out to him. The present paper, in particular, closes this gap. A preliminary version of the present paper was made publicly available in the author's web page [5]. The author also thanks Ralph Matthes for comments on [5].

In this paper we offer a generic method for proving preservation of strong normalisation, so that, whenever confronted with a particular reduction rule ρ, all there is to do is to verify the sufficient condition. It is not discussed in [7, 3] whether the methods of these papers are extensible to other reduction rules. What is clear though is that, for the particular case of $\rho = \sigma \cup \pi$, the effort of checking the sufficient condition is much smaller than the effort in [3]. The same remark applies to $\rho = \mathsf{assoc}$ and [7].

A notable example of notion of reduction which does not satisfy condition (1) is η: just observe that $\phi(\lambda x.yx, y, -)$ is false. Yet, η preserves strong normalisation. This follows easily from termination of $\to_\eta$ and a result of postponement: if $M \to_\eta N \to_\beta P$, then there is Q such that $M \to_\beta^+ Q \to_\eta^* P$; the latter, in turn, is easily proved by induction on $M \to_\eta N$. Incidentally, we have just seen that our condition for PSN, albeit sufficient, is not necessary.

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