CONGRUENCES ON ORTHODOX SEMIGROUPS WITH ASSOCIATE SUBGROUPS

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If S is a regular semigroup then an inverse transversal of S is an inverse subsemigroup T with the property that $|T \cap V(x)| = 1$ for every $x \in S$ where $V(x)$ denotes the set of inverses of $x \in S$. In a previous publication [**1**] we considered the similar concept of a subsemigroup T of S such that $|T \cap A(x)| = 1$ for every $x \in S$ where $A(x) = \{y \in S; xyx = x\}$ denotes the set of associates (or pre-inverses) of $x \in S$, and showed that such a subsemigroup T is necessarily a maximal subgroup H_α for some idempotent $\alpha \in S$. Throughout what follows, we shall assume that *S is orthodox and α is a middle unit* (in the sense that $x\alpha y = xy$ for all $x, y \in S$). Under these assumptions, we obtained in [**1**] a structure theorem which generalises that given in [**3**] for uniquely unit orthodox semigroups. Adopting the notation of [**1**], we let $T \cap A(x) = \{x^*\}$ and write the subgroup T as $H_\alpha = \{x^*; x \in S\}$, which we call an *associate subgroup* of S. For every $x \in S$ we therefore have $x^*\alpha = x^* = \alpha x^*$ and $x^*x^{**} = \alpha = x^{**}x^*$. As shown in [**1**, Theorems 4, 5] we also have $(xy)^* = y^*x^*$ for all $x, y \in S$, and $e^* = \alpha$ for every idempotent e.

Our objective here is to consider congruences on such a semigroup. Since the building bricks in the structure theorem [**1**] are the subgroup H_α and the sub-bands αE, $E\alpha$ of the band E of idempotents of S, these three subsemigroups will play an important rôle in what follows.

As we shall see, the study of congruences is intimately related to certain residuated mappings that arise naturally. We recall that if A, B are ordered sets then a mapping $f: A \to B$ is said to be *residuated* if the pre-image of every principal down-set of B is a principal down-set of A. For the general properties of residuated mappings we refer the reader to [**2**]. For our purposes here we require the fact that $f: A \to B$ is residuated if and only if it is isotone and there is a (necessarily unique) isotone mapping $f^+: B \to A$ such that $f^+ \circ f \geqslant \mathrm{id}_A$ and $f \circ f^+ \leqslant \mathrm{id}_B$.

Since, in the semigroups under consideration, the unary operation $x \mapsto x^*$ is significant, it is reasonable to expect that an important rôle will be played by the semigroup congruences ϑ such that

$$(x, y) \in \vartheta \Rightarrow (x^*, y^*) \in \vartheta,$$

i.e. the congruences on the algebra $(S, ., ^*)$ which we shall refer to as **-congruences*. We shall denote by $\mathrm{Con}\, S$ the complete lattice of (semigroup) congruences on S. It is easily seen that the set of *-congruences forms a complete sublattice of $\mathrm{Con}\, S$; we denote this by $\mathrm{Con}^*\, S$.

Definition. Let λ, π, μ be congruences on αE, H_α, $E\alpha$ respectively. We shall call the triple (λ, π, μ) *weighted* if there exists $\Theta \in \mathrm{Con}\, S$ such that

(a) $\Theta|_{H_\alpha} = \pi$;

(b) $\lambda|_{\alpha E\alpha} = \mu|_{\alpha E\alpha} = \Theta|_{\alpha E\alpha}$.

Also, for $e\alpha \in E\alpha$ we have

$$(e\alpha)^* = \alpha^* e^* = \alpha\alpha = \alpha, \qquad (e\alpha)^* e\alpha = \alpha e\alpha \in \alpha E\alpha, \qquad e\alpha(e\alpha)^* = e\alpha.$$

These equalities, together with (b), show that

$$(e\alpha, f\alpha) \in \Psi(\lambda, \pi, \mu) \Leftrightarrow (e\alpha, f\alpha) \in \mu$$

and that therefore $\Psi(\lambda, \pi, \mu)|_{E\alpha} = \mu$. Similarly, $\Psi(\lambda, \pi, \mu)|_{\alpha E} = \lambda$.

It follows from these observations that

$$\Psi^+\Psi(\lambda, \pi, \mu) = (\lambda, \pi, \mu). \tag{1}$$

Moreover, using the identity $x = xx^*x^{**}x^*x$ and the fact that $\vartheta|_{H_\alpha}$ is a group congruence, we have that

$$(x, y) \in \Psi\Psi^+(\vartheta) \Rightarrow (x^*x, y^*y) \in \vartheta|_{\alpha E}, (x^*, y^*) \in \vartheta|_{H_\alpha}, (xx^*, yy^*) \in \vartheta|_{E\alpha}$$
$$\Rightarrow (x, y) = (xx^*x^{**}x^*x, yy^*y^{**}y^*y) \in \vartheta,$$

and therefore

$$\Psi\Psi^+(\vartheta) \subseteq \vartheta. \tag{2}$$

Since both Ψ and Ψ^+ are isotone, it follows from (1) and (2) that Ψ is injective and residuated, with residual Ψ^+.

By Theorem 1 we have that $\operatorname{Im}\Psi \subseteq \operatorname{Con}^* S$. Conversely, if $\vartheta \in \operatorname{Con}^* S$ then $(x, y) \in \vartheta$ gives $(x^*x, y^*y) \in \vartheta$, $(x^*, y^*) \in \vartheta$, $(xx^*, yy^*) \in \vartheta$ whence $(x, y) \in \Psi\Psi^+(\vartheta)$. Thus $\vartheta \subseteq \Psi\Psi^+(\vartheta)$ and it follows from (2) that $\vartheta = \Psi\Psi^+(\vartheta) \in \operatorname{Im}\Psi$. Consequently, $\operatorname{Im}\Psi = \operatorname{Con}^* S$. Now since Ψ is isotone and injective it induces an isotone bijection $\Psi_*: \mathrm{WT}(S) \to \operatorname{Im}\Psi$. It follows from the above that the restriction of Ψ^+ to $\operatorname{Con}^* S$ is isotone and is the inverse of Ψ_*. Consequently, $\mathrm{WT}(S) \simeq \operatorname{Im}\Psi$. It follows from these observations that $\mathrm{WT}(S)$ is isomorphic to the lattice $\operatorname{Con}^* S$.

COROLLARY 1. *The relation* $\equiv$ *defined on* Con S *by*

$$\vartheta \equiv \varphi \Leftrightarrow \vartheta|_{\alpha E} = \varphi|_{\alpha E}, \qquad \vartheta|_{H_\alpha} = \varphi|_{H_\alpha}, \qquad \vartheta|_{E\alpha} = \varphi|_{E\alpha}$$

is a dual closure equivalence. The smallest element in the $\equiv$*-class of* ϑ *is* $\Psi\Psi^+(\vartheta)$.

Proof. Observe that, since Ψ is residuated, $\Psi\Psi^+$ is a dual closure; and, since $\Psi^+\Psi\Psi^+ = \Psi^+$,

$$\vartheta \equiv \varphi \Leftrightarrow \Psi^+(\vartheta) = \Psi^+(\varphi) \Leftrightarrow \Psi\Psi^+(\vartheta) = \Psi\Psi^+(\varphi).$$

COROLLARY 2. *There is a lattice isomorphism* $\operatorname{Con}^* S \simeq (\operatorname{Con} S)/\equiv$.

COROLLARY 3. *If* $(\lambda, \pi, \mu), (\lambda', \pi', \mu') \in \mathrm{WT}(S)$ *then*

$$\Psi(\lambda, \pi, \mu) \cap \Psi(\lambda', \pi', \mu') = \Psi(\lambda \cap \lambda', \pi \cap \pi', \mu \cap \mu');$$
$$\Psi(\lambda, \pi, \mu) \vee \Psi(\lambda', \pi', \mu') = \Psi(\lambda \vee \lambda', \pi \vee \pi', \mu \vee \mu').$$

Proof. This follows immediately from the fact that $(\lambda, \pi, \mu) \mapsto \Psi(\lambda, \pi, \mu)$ is a lattice isomorphism from $\mathrm{WT}(S)$ to $\operatorname{Con}^* S$.

We now consider the extension of congruences on αE, H_α, $E\alpha$ to $*$-congruences on

Similarly, $\mu \in \operatorname{Con} E\alpha$ will be called *special* if, for all $e, f \in E\alpha$,

$$(e,f) \in \mu \Rightarrow (\forall x \in S) \qquad (xex^*, xfx^*) \in \mu.$$

In what follows, given $\lambda \in \operatorname{Con} \alpha E$ we shall be interested in the relation $\bar{\lambda}$ defined on S by

$$(a,b) \in \bar{\lambda} \Leftrightarrow (\forall e \in R) \qquad (a^*ea, b^*eb) \in \lambda;$$

and dually, for $\mu \in \operatorname{Con} E\alpha$, the relation $\bar{\mu}$ defined on S by

$$(a,b) \in \bar{\mu} \Leftrightarrow (\forall e \in E) \qquad (aea^*, beb^*) \in \mu.$$

Note that $\bar{\lambda}$ is a left congruence on S; for if $(a,b) \in \bar{\lambda}$ then, since $x^*ex \in E$ for every $x \in S$, we have $(a^*x^*exa, b^*x^*exb) \in \lambda$, so that $(xa, xb) \in \bar{\lambda}$. Similarly, $\bar{\mu}$ is a right congruence.

THEOREM 5. *For $\lambda \in \operatorname{Con} \alpha E$ the following statements are equivalent*:

(1) *λ is special*;

(2) *$\lambda = \vartheta|_{\alpha E}$ for some $\vartheta \in \operatorname{Con} S$.*

Proof. (1)$\Rightarrow$(2). Suppose that λ is special. Then we have $\bar{\lambda} \in \operatorname{Con} S$. For, if $(a,b) \in \bar{\lambda}$ then, since λ is special and $a^*ea \in \alpha E$, for every $x \in S$ we have $(x^*a^*eax, x^*b^*ebx) \in \lambda$, so that $(ax, bx) \in \bar{\lambda}$, whence $\bar{\lambda}$ is a right congruence. As observed above, $\bar{\lambda}$ is a left congruence.

If now $f, g \in \alpha E$ are such that $(f,g) \in \bar{\lambda}$ then since $f^* = g^* = \alpha$ we have $(\alpha ef, \alpha eg) \in \lambda$ for every $e \in E$. Taking $e = \alpha$ we obtain $(f,g) \in \lambda$. Conversely, if $(f,g) \in \lambda$ then $(\alpha ef, \alpha eg) \in \lambda$ gives $(f^*ef, g^*eg) \in \lambda$ whence $(f,g) \in \bar{\lambda}$. Hence $\bar{\lambda}|_{\alpha E} = \lambda$, so (2) holds with $\vartheta = \bar{\lambda}$.

(2)$\Rightarrow$(1). If $\vartheta \in \operatorname{Con} S$ then for $e, f \in \alpha E$ we have

$$(e,f) \in \vartheta|_{\alpha E} \Rightarrow (e,f) \in \vartheta \Rightarrow (\forall x \in S)\ (x^*ex, x^*fx) \in \vartheta.$$

Since $x^*ex \in \alpha E$, it follows that $\vartheta|_{\alpha E}$ is special.

COROLLARY 1. *For every special congruence λ on αE there is a biggest *-congruence ϑ on S such that $\vartheta|_{\alpha E} = \lambda$, namely $\vartheta = \Psi\Psi^+(\bar{\lambda})$.*

Proof. By Theorem 2, $\Psi\Psi^+(\bar{\lambda}) \in \operatorname{Con}^* S$ with $\Psi\Psi^+(\bar{\lambda})|_{\alpha E} = \bar{\lambda}|_{\alpha E} = \lambda$. If now $\zeta \in \operatorname{Con}^* S$ is such that $\zeta|_{\alpha E} = \lambda$ then

$$(a,b) \in \zeta \Rightarrow (\forall e \in E)\ (a^*ea, b^*eb) \in \zeta|_{\alpha E} = \lambda \Rightarrow (a,b) \in \bar{\lambda},$$

so $\zeta \subseteq \bar{\lambda}$ and consequently $\zeta = \Psi\Psi^+(\zeta) \subseteq \Psi\Psi^+(\bar{\lambda})$.

COROLLARY 2. *The following statements concerning $\lambda \in \operatorname{Con} \alpha E$ are equivalent*:

(1) *λ is special*;

(2) *there is a weighted triple whose first component is λ.*

Proof. (1)$\Rightarrow$(2). If (1) holds then the weighted triple associated with $\Psi\Psi^+(\bar{\lambda})$ has first component λ.

(2)$\Rightarrow$(1). If there is a weighted triple of the form $(\lambda, -, -)$ then by Theorem 2 we have

$$(\lambda, -, -) = \Psi^+\Psi(\lambda, -, -)$$

whence $\lambda = \Psi(\lambda, -, -)|_{\alpha E}$ and so λ is special.

If $(a^*, b^*) \in \pi$ then we have $(a^{**}, b^{**}) \in \pi$ whence $(a^*, b^*) \in \Theta$ and $(a^{**}, b^{**}) \in \Theta$, from which it follows that

$$(\forall e \in E) \quad (a^*ea^{**}, b^*eb^{**}) \in \Theta|_{\alpha E\alpha} = \lambda|_{\alpha E\alpha}$$

and therefore $(a^{**}, b^{**}) \in \bar{\lambda}$.

Since $\lambda = \bar{\lambda}|_{\alpha E}$ and since, as seen above, $\mu \subseteq \bar{\lambda}|_{E\alpha}$ it follows that

$$\begin{aligned}(a, b) \in P &\Rightarrow (a^*a, b^*b) \in \bar{\lambda}, (a^{**}, b^{**}) \in \bar{\lambda}, (aa^*, bb^*) \in \bar{\lambda}\\ &\Rightarrow (a, b) \in \bar{\lambda}\end{aligned}$$

so $P \subseteq \bar{\lambda}$ and therefore $P = \Psi\Psi^+(P) \subseteq \Psi\Psi^+(\bar{\lambda})$. Arguing similarly with μ, we obtain

$$P \subseteq \Psi\Psi^+(\bar{\lambda}) \cap \Psi\Psi^+(\bar{\mu})$$

from which the result follows.

COROLLARY 2. *The biggest weighted triple of the form* $(\lambda, -, \mu)$ *has middle component* $\pi_{\lambda,\mu} = \bar{\lambda}|_{H_\alpha} = \bar{\mu}|_{H_\alpha}$.

Given $\lambda \in \operatorname{SpCon} \alpha E$ and $\mu \in \operatorname{SpCon} E\alpha$, let $T_{\lambda,\mu}$ be the set of weighted triples with first component λ and third component μ. As in the proof of Theorem 7, let $\pi_{\lambda,\mu} = \bar{\lambda}|_{H_\alpha} = \bar{\mu}|_{H_\alpha}$.

THEOREM 8. $T_{\lambda,\mu}$ *is a sublattice of* WT(S), *isomorphic to the interval* $[\omega, \pi_{\lambda,\mu}]$ *of* Con H_α.

Proof. Clearly, $T_{\lambda,\mu}$ is a sublattice of WT(S). If now $(\lambda, \vartheta, \mu) \in T_{\lambda,\mu}$ then by Corollary 2 of Theorem 7 we have $\vartheta \subseteq \pi_{\lambda,\mu}$. Consider the mapping $\zeta: T_{\lambda,\mu} \to [\omega, \pi_{\lambda,\mu}]$ given by $\zeta(\lambda, \vartheta, \mu) = \vartheta$. Clearly, ζ is an injective $\cap$-morphism. It suffices, therefore, to prove that ζ is surjective; equivalently, that if $\vartheta \in [\omega, \pi_{\lambda,\mu}]$ then $(\lambda, \vartheta, \mu)$ is a weighted triple. Now, by Theorem 3, $\hat{\vartheta}$ is represented by the weighted triple

$$(\iota_{\alpha E}, \vartheta, \iota_{E\alpha}),$$

whereas $\Psi\Psi^+(\bar{\lambda})$ is represented by the weighted triple

$$(\lambda, \pi_{\lambda,\mu}, \bar{\lambda}|_{E\alpha}),$$

and $\Psi\Psi^+(\bar{\mu})$ is represented by the weighted triple

$$(\bar{\mu}|_{\alpha E}, \pi_{\lambda,\mu}, \mu).$$

It follows by Theorem 2 that the *-congruence $\hat{\vartheta} \cap \Psi\Psi^+(\bar{\lambda}) \cap \Psi\Psi^+(\bar{\mu})$ is represented by the intersection of these three triples which, since $\lambda \subseteq \bar{\mu}|_{\alpha E}$ and $\mu \subseteq \bar{\lambda}|_{E\alpha}$, is $(\lambda, \vartheta, \mu)$.

COROLLARY. *For every* $(\lambda, \vartheta, \mu) \in$ WT(S) *we have*

$$\Psi(\lambda, \vartheta, \mu) = \hat{\vartheta} \cap \Psi\Psi^+(\bar{\lambda}) \cap \Psi\Psi^+(\bar{\mu}).$$

Given $\varphi \in \operatorname{Con} \alpha E$ consider now the relation λ_φ defined on αE by

$$(a, b) \in \lambda_\varphi \Leftrightarrow (a\alpha, b\alpha) \in \varphi.$$

and therefore $\lambda_{\varphi_1} \vee \lambda_{\varphi_2} = \lambda_{\varphi_1 \vee \varphi_2}$. Similarly we have $\mu_{\varphi_1} \cap \mu_{\varphi_2} = \mu_{\varphi_1 \cap \varphi_2}$ and $\mu_{\varphi_1} \vee \mu_{\varphi_2} = \mu_{\varphi_1 \vee \varphi_2}$. It follows by Theorem 10 that if $\varphi_1, \varphi_2 \in \mathrm{Con}\ \alpha E\alpha$ are special then so also are $\varphi_1 \cap \varphi_2$ and $\varphi_1 \vee \varphi_2$. Hence the set of special congruences on $\alpha E\alpha$ forms a lattice which we shall denote by $\mathrm{SpCon}\ \alpha E\alpha$.

Note now that if $\lambda \in \mathrm{SpCon}\ \alpha E$ then $\lambda|_{\alpha E\alpha} \in \mathrm{SpCon}\ \alpha E\alpha$. In fact,

$$\begin{aligned}(a,b) \in \lambda &\Rightarrow (a\alpha, b\alpha) \in \lambda \\ &\Rightarrow (\forall x \in S) \quad (x^*a\alpha x, x^*b\alpha x) \in \lambda \\ &\Rightarrow (\forall x \in S) \quad (x^*ax\alpha, x^*bx\alpha) \in \lambda|_{\alpha E\alpha},\end{aligned}$$

and

$$\begin{aligned}(a,b) \in \lambda \Rightarrow (\forall x \in S) \quad &(x^{**}ax^*, x^{**}bx^*) \in \lambda|_{\alpha E\alpha} = \mu|_{\alpha E\alpha} \\ \Rightarrow (\forall x \in S) \quad &(xax^*, xbx^*) = (xx^*x^{**}ax^*, xx^*x^{**}bx^*) \in \mu \\ \Rightarrow (\forall x \in S) \quad &(\alpha xax^*, \alpha xbx^*) \in \mu|_{\alpha E\alpha} = \lambda|_{\alpha E\alpha}.\end{aligned}$$

This observation, together with Theorems 9 and 10, gives immediately the following result.

THEOREM 11. *The mapping* $\Delta_{\alpha E} : \mathrm{SpCon}\ \alpha E \to \mathrm{SpCon}\ \alpha E\alpha$ *given by* $\Delta_{\alpha E}(\vartheta) = \vartheta|_{\alpha E\alpha}$ *is surjective and residuated, with residual* $\Delta^+_{\alpha E}$ *given by* $\Delta^+_{\alpha E}(\varphi) = \lambda_\varphi$.

Note also that if $\vartheta \in \mathrm{Con}\ S$ then $\vartheta|_{\alpha E\alpha}$ is special. In fact, by Theorem 5, $\vartheta|_{\alpha E}$ is special and therefore, by the above observation, so is $\vartheta|_{\alpha E\alpha}$. We can therefore use Theorems 6 and 11, and their duals to obtain the following result.

THEOREM 12. *The mapping* $\Gamma : \mathrm{Con}^*\ S \to \mathrm{SpCon}\ \alpha E\alpha$ *given by* $\Gamma(\vartheta) = \vartheta|_{\alpha E\alpha}$ *is surjective and residuated, with residual* Γ^+ *given by*

$$\Gamma^+(\varphi) = \Psi(\lambda_\varphi, \pi_{\lambda_\varphi, \mu_\varphi}, \mu_\varphi).$$

Proof. Consider the diagram

$$\begin{array}{ccc} \mathrm{Con}^*\ S & \xrightarrow{\Phi_{\alpha E}} & \mathrm{SpCon}\ \alpha E \\ {\scriptstyle \Phi_{E\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{\alpha E}} \\ \mathrm{SpCon}\ E\alpha & \xrightarrow[\Delta_{E\alpha}]{} & \mathrm{SpCon}\ \alpha E\alpha \end{array}$$

which is commutative since, by definition, $\Delta_{\alpha E}\Phi_{\alpha E} = \Gamma = \Delta_{E\alpha}\Phi_{E\alpha}$. That Γ is surjective follows from Theorems 6 and 11. By [**2**, Theorem 2.8] we have on the one hand, for every $\varphi \in \mathrm{SpCon}\ \alpha E\alpha$,

$$\Gamma^+(\varphi) = \Phi^+_{\alpha E}\Delta^+_{\alpha E}(\varphi) = \Phi^+_{\alpha E}(\lambda_\varphi) = \Psi\Psi^+(\overline{\lambda_\varphi}) = \Psi(\overline{\varphi_\varphi}|_{\alpha E}, \overline{\lambda_\varphi}|_{H_\alpha}, \overline{\lambda_\varphi}|_{E\alpha}), \tag{1}$$

and, on the other,

$$\Gamma^+(\varphi) = \Phi^+_{E\alpha}\Delta^+_{E\alpha}(\varphi) = \Phi^+_{E\alpha}(\mu_\varphi) = \Psi\Psi^+(\overline{\mu_\varphi}) = \Psi(\overline{\mu_\varphi}|_{\alpha E}, \overline{\mu_\varphi}|_{H\alpha}, \overline{\mu_\varphi}|_{E\alpha}). \tag{2}$$

Since $\overline{\lambda_\varphi}|_{\alpha E} = \lambda_\varphi$ and $\overline{\mu_\varphi}|_{E\alpha} = \mu_\varphi$, it follows from (1), (2) that $\overline{\lambda_\varphi}|_{E\alpha} = \overline{\mu_\varphi}$ and $\overline{\mu_\varphi}|_{\alpha E} = \lambda_\varphi$ so that $\Gamma^+(\varphi) = \Psi(\lambda_\varphi, \pi_{\lambda_\varphi, \mu_\varphi}, \mu_\varphi)$ as asserted.

If $\vartheta, \vartheta' \in \operatorname{Con} S$ then it is clear that $(\vartheta \cap \vartheta')|_{\alpha E} = \vartheta|_{\alpha E} \cap \vartheta'|_{\alpha E}$. If now $(a, b) \in (\vartheta \vee \vartheta')|_{\alpha E}$ then there exist $z_1, \ldots, z_n$ such that

$$a \equiv z_1 \equiv z_2 \equiv \ldots \equiv z_n \equiv b$$

where each $\equiv$ signifies either ϑ or ϑ'. It follows that

$$a = \alpha a \equiv \alpha z_1 \equiv \alpha z_2 \equiv \ldots \equiv \alpha z_n \equiv \alpha b = b$$

and therefore $(a, b) \in \vartheta|_{\alpha E} \vee \vartheta'|_{\alpha E}$. Hence we have that $(\vartheta \vee \vartheta')|_{\alpha E} \subseteq \vartheta|_{\alpha E} \vee \vartheta'|_{\alpha E}$, whence we have equality since the reverse inclusion is trivial. It now follows from the Corollary to Theorem 13 that $\operatorname{Con}_1^* S$ is a sublattice of $\operatorname{Con}^* S$.

Observe that if $\Psi(\lambda, \vartheta, \mu) \in \operatorname{Con}^* S$ and $\varphi = \lambda|_{\alpha E\alpha} = \mu|_{\alpha E\alpha}$ then it follows from Theorem 14 that $\Psi(\lambda_\varphi, \vartheta, \mu_\varphi) \in \operatorname{Con}_1^* S$ and the mapping $\Psi(\lambda, \vartheta, \mu) \mapsto \Psi(\lambda_\varphi, \vartheta, \mu_\varphi)$ is a closure on $\operatorname{Con}^* S$.

Although a description of the lattice $\operatorname{Con}^* S$ appears to be very difficult, we can describe the sublattice $\operatorname{Con}_1^* S$. For this purpose, we denote by $\operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha$ the set

$$\{(\vartheta, \varphi) \in \operatorname{Con} H_\alpha \times \operatorname{SpCon} \alpha E\alpha;\ \vartheta \subseteq \pi_{\lambda_\varphi, \mu_\varphi}\}.$$

THEOREM 15. $\operatorname{Con}_1^* S \simeq \operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha$.

Proof. For every $\varphi \in \operatorname{SpCon} \alpha E\alpha$ we have, by Theorem 12 and the fact that $\lambda_\varphi, \mu_\varphi$ are unitary,

$$\Gamma^+(\varphi) = \Psi(\lambda_\varphi, \pi_{\lambda_\varphi, \mu_\varphi}, \mu_\varphi) \in \operatorname{Con}_1^* S.$$

It follows by Theorem 8 that for $(\vartheta, \varphi) \in \operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha$ we have

$$\Psi(\lambda_\varphi, \vartheta, \mu_\varphi) \in \operatorname{Con}_1^* S.$$

We can therefore define a mapping $\zeta : \operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha \to \operatorname{Con}_1^* S$ by the prescription

$$\zeta(\vartheta, \varphi) = \Psi(\lambda_\varphi, \vartheta, \mu_\varphi).$$

Suppose now that $\Psi(\lambda, \vartheta, \mu) \in \operatorname{Con}_1^* S$. If $\varphi = \lambda|_{\alpha E\alpha} = \mu|_{\alpha E\alpha}$ then since λ and μ are unitary we have $\lambda = \lambda_\varphi$ and $\mu = \mu_\varphi$; for example, by Theorem 13,

$$(a, b) \in \lambda_\varphi \Leftrightarrow (a\alpha, b\alpha) \in \varphi = \lambda|_{\alpha E\alpha} \Leftrightarrow (a, b) \in \lambda.$$

Since $\vartheta \subseteq \pi_{\lambda,\mu} = \pi_{\lambda_\varphi, \mu_\varphi}$ we therefore have $(\vartheta, \varphi) \in \operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha$. Consequently we can define a mapping $\eta : \operatorname{Con}_1^* S \to \operatorname{Con} H_\alpha \,|\!\times\!|\, \operatorname{SpCon} \alpha E\alpha$ by the prescription

$$\eta \Psi(\lambda, \vartheta, \mu) = (\vartheta, \varphi).$$

Now each of ζ, η is isotone; and we have

$$\eta\zeta(\vartheta, \varphi) = \eta\Psi(\lambda_\varphi, \vartheta, \mu_\varphi) = (\vartheta, \varphi);$$

$$\zeta\eta\Psi(\lambda, \vartheta, \mu) = \zeta(\vartheta, \varphi) = \Psi(\lambda_\varphi, \vartheta, \mu_\varphi) = \Psi(\lambda, \vartheta, \mu).$$

Thus η, ζ are mutually inverse isomorphisms.