Congruences on Nilpotent-generated Partial Transformation Semigroups

(To Douglas Muna with respect and gratitude)

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Abstract. In 1987, Sullivan characterized the elements of the semigroup NP(X) generated by the nilpotents in P(X), the semigroup (under composition) consisting of all partial transformations of a set X; and in 1999, Marques-Smith and Sullivan determined all the ideals of NP(X) for arbitrary X. In this paper, we use that work to describe all the congruences on NP(X).

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1 Introduction

Throughout this paper, X is a non-empty set. In addition, P(X) denotes the semigroup under composition of all partial transformations of X (that is, all transformations whose domain is a proper subset of X). Note that P(X) contains a zero (namely, the empty mapping $\emptyset$). We say that $\alpha \in P(X)$ is nilpotent with index $r$ if $\alpha^r = \emptyset$ and $\alpha^r \neq \emptyset$, and we let NP(X) denote the semigroup generated by all nilpotents in P(X). In like manner, if I(X) denotes the symmetric inverse semigroup on X, we write NI(X) for the semigroup generated by all nilpotents in I(X).

In [4] the authors described the ideals of NP(X) and NI(X) as a prelude to determining all congruences on these semigroups. The congruences on NI(X) were described in [5], and here we do the same for NP(X). The case when X is finite is considered in Section 3, and we cover the cases when X has infinite regular or singular cardinality in Sections 4 and 5.

2 Preliminary Results

All notation and terminology will be from [1] and [4] unless specified otherwise. In particular, if $\alpha \in P(X)$, we let $r(\alpha)$ denote the rank of $\alpha$ (that is, $|X(\alpha)|$) and put $D(\alpha) = X\setminus X(\alpha)$, $d(\alpha) = |D(\alpha)|$, $G(\alpha) = X\setminus \text{dom } \alpha$, $g(\alpha) = |G(\alpha)|$.

The cardinal numbers $d(\alpha)$ and $g(\alpha)$ are called the defect and the gap of $\alpha$ and were used by Sullivan to characterize the elements of NP(X) for arbitrary X.

To state his result for the infinite case, first we remark that a cardinal $k$ is regular if $\sum_{i=1}^{\kappa} |a_i| = \kappa$ implies that $\kappa = |a_i|$ for each $i$. For each cardinal $\kappa$, there exists $\mu \in 2^{\lambda}$ with $|\mu^{-1}| > \mu$. The following two results summarize Corollary 3 and Theorem 4 in [6] and Lemmas 2.5 and 3.2 in [5].

Theorem 2.1. Let $|X| = \kappa$ be regular and $\alpha \in P(X)$. Then $\alpha \in NP(X)$ if and only if $d(\alpha) \neq 0$, $d(\alpha) = k$, and $g(\alpha) = k$ or $g(\alpha) = k + 1$ for some $\lambda$. Moreover, when this occurs, NP(X) is a regular semigroup and each $\alpha \in NP(X)$ is a product of three or fewer nilpotents with index at most 3.

Theorem 2.2. Let $|X| = \kappa$ be singular and $\alpha \in P(X)$. Then $\alpha \in NP(X)$ if and only if $d(\alpha) \neq 0$, $d(\alpha) = k$, and either $g(\alpha) \geq r(\alpha)$ or $\alpha$ is spread over its rank. Moreover, when this occurs, NP(X) is a regular semigroup and each $\alpha \in NP(X)$ is a product of four or fewer nilpotents with index at most 4.

For the finite case (see Theorems 1 and 2 in [6]), we need some additional notation. If X is an arbitrary set with cardinality $\kappa$ and $1 \leq r \leq \kappa$, we write

$\beta_r = \{ \alpha \in P(X) : r(\alpha) < r \}$

$\beta_r = \{ \alpha \in P(X) : r(\alpha) = r \}$

and recall that the $\beta_r$ constitute all the proper ideals of P(X) and each $\beta_r$ is a D-class of P(X). Moreover, if $k = < \kappa$, then each $\alpha \in \beta_r \cap D_{\kappa-1}$ has a unique completion $S \in G(X)$, the symmetric group on X, defined by

$s\alpha = \begin{cases} \alpha & \text{if } \alpha \in \text{dom } \alpha, \\ \emptyset & \text{if } \alpha = \emptyset, \\ \emptyset & \text{if } \emptyset \in \text{dom } \alpha \end{cases}$

where $X\setminus \text{dom } \alpha = [\emptyset]$ and $X\setminus \text{ran } \alpha = \{ \emptyset \}$ (see [2, p. 388]). We write

$E_{\kappa-1} = \{ \alpha \in I(X) \cap D_{\kappa-1} : S = \text{an even permutation} \}$

Theorem 2.3. Suppose $\kappa \geq 3$ and $\alpha \in P(X)$.
Ideals of $NP(X)$ are precisely the sets $NP_{\gamma} = \{ a \in NP(X) : r(a) < \gamma \}$, where $1 \leq \gamma < \kappa$.

Thus, if $\rho$ is a non-identity and non-universal congruence on $NP(X)$, then $NP_{\rho}$ is non-empty for some $\gamma$ such that $1 \leq \gamma < \kappa$: we call $\gamma$ the rank of $\rho$ and denote it by $\gamma(\rho)$. We also recall the characterization of Green's $D$ relation on $NP(X)$ given in [4, Theorem 11]. We let $NP_{\rho}^D$ denote the $D$-class of $NP(X)$ which contains all elements with rank $\gamma$.

**Theorem 2.6.** If $X$ is any set (finite or infinite) and $\alpha, \beta \in NP(X)$, then $\beta = \alpha \rho_\gamma$ for some $\gamma \in NP(X)$ if and only if $r(\beta) \leq r(\alpha)$.

**Proof.** It is easy to see that $NP_{\rho}^D \subseteq \rho$, so let $(a, b) \in \rho$ and assume $r(b) < r(a) = r$. If $\alpha$ is infinite, then $X$ is finite and we note that the $\gamma$ defined in case (a) for the proof of [5, Lemma 2] has gap and defect equal to $\kappa$. Hence, by Theorems 2.1 and 2.2 above, this $\gamma$ belongs to $NP(X)$ and, as before, we conclude that $r < \gamma$. If $\alpha$ is finite, then $X$ may be finite or infinite. However, for both possibilities, the $\gamma$ and $\gamma$ defined in case (b) for the proof of [5, Lemma 2] belong to $NP(X)$. Hence, that argument holds for this case, and we again conclude that $r < \gamma$. □

The $L$- and $R$-relations on $P(X)$ are well known: namely, $a \rho \beta$ if and only if $r(a) \leq r(\beta)$ and $a \rho_D \beta$ if and only if $a \rho \beta$ and $a \rho_D \beta$.

**Theorem 2.7.** If $\rho$ is a non-identity congruence on $NP(X)$ and $\gamma = \gamma(\rho)$, then $NP_{\rho} \subseteq \rho \subseteq NP_{\rho}^D$.

**Proof.** It is easy to see that $NP_{\rho} \subseteq \rho$, so let $(a, b) \in \rho$ and assume $r(b) < r(a) = r$. If $\alpha$ is infinite, then $X$ is finite and we note that the $\gamma$ defined in case (a) for the proof of [5, Lemma 2] has gap and defect equal to $\kappa$. Hence, by Theorems 2.1 and 2.2 above, this $\gamma$ belongs to $NP(X)$ and, as before, we conclude that $r < \gamma$. If $\alpha$ is finite, then $X$ may be finite or infinite. However, for both possibilities, the $\gamma$ and $\gamma$ defined in case (b) for the proof of [5, Lemma 2] belong to $NP(X)$. Hence, that argument holds for this case, and we again conclude that $r < \gamma$. □

The proper ideals of $NP(X)$ were described in Theorems 6 and 15 of [4] as follows. In [5, Section 2], the authors remarked that if $X$ is infinite and $r \leq \kappa$, then the proper ideals of $NP(X)$ are simply those of $I(X)$. However, this is not true for $NP(X)$ because each $P_\gamma$ contains total transformations (that is, $\gamma \in P_\gamma$) with domain $\alpha = \omega$, so $\gamma(\alpha) = 0$ and, by Theorems 2.1 and 2.2, these elements do not belong to $NP(X)$.

**Theorem 2.5.** For any set $X$ with (finite or infinite) cardinal $\kappa \geq 2$, the proper ideals of $NP(X)$ are precisely the sets $NP_{\gamma} = \{ a \in NP(X) : r(a) < \gamma \}$, where $1 \leq \gamma < \kappa$.
of whether \( X \) is finite or infinite), hence we conclude, as before, that \( \alpha \neq B \).

To show \( \alpha \neq B \), first we suppose \( \alpha \neq B \). Choose \( r \in \text{dom}(\alpha) \setminus \text{dom}(\beta) \), let \( A_1 \in \text{N}(\alpha) \) be a cross-section of \( A_0 \alpha^{-1} \) which contains \( r \). Then \( id_{A_1} \in N(P_X) \) (since \( |A_1|=r \alpha^{-1} \)). We consider the \( \beta \)-connectedness of \( \delta \in N_X(\beta) \) in the proof of [5, Lemma 3] is also valid here. Moreover, \( r(id_{A_1}) = r \), but \( r(id_{A_1}) \leq r - 1 \) (since \( r \neq \text{dom}(\beta) \)). Since \( \text{dom}(\alpha) \subseteq \text{dom}(\beta) \), Lemma 2.7 implies \( r < \gamma(\beta) \), a contradiction. Therefore, \( \alpha \subseteq \text{dom}(\beta) \) and similarly \( \beta \subseteq \text{dom}(\alpha) \), so \( \alpha \neq B \).

Next suppose \( \alpha \neq B \). Then there exists \( x \in \alpha \alpha^{-1} \) \( \neq \beta \) \( \beta^{-1} \) which contains \( x \) and \( \beta \). Then \( id_{A_1} \in N(P_X) \) (since \( \beta \neq 0 \)). The same \( \alpha \neq B \) holds. Therefore, \( \alpha \neq B \) and similarly \( \beta \neq A \), and for the reverse inclusion, we have shown \( \alpha \neq A \).

The next result is similar to [5, Lemma 4], but we include a proof for this new context.

**Lemma 2.10.** Let \( \alpha \) be a non-identity congruence on \( N(P_X) \) and suppose \( \eta(\beta) = \beta \) is finite. If \( (\alpha, \beta) \in \mu \), then \( \alpha \beta \subseteq \text{dom}(\beta) \) and \( \eta(\alpha) \leq \eta(\beta) \), then \( \eta(\alpha) = \eta(\beta) \).

**Proof.** By Lemma 2.9, \( (\alpha, \beta) \in \mu \), so \( \beta \subseteq \text{dom}(\alpha) \) and \( \eta(\beta) \leq \eta(\alpha) \). Therefore, \( \alpha \subseteq \text{dom}(\beta) \), and \( \eta(\alpha) = \eta(\beta) \).

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3 \ Finite Primary Rank

In [4, p.316], the authors observed that if \( X \) is finite and \( r \neq X \), then \( N_L(N_L) \) is a finite group. For what follows, we require a similar result for \( N(P_X) \) but slightly more general (compare [5, Lemma 3]).

**Lemma 3.1.** If \( X \) is any set and \( 0 \leq r \leq X \), then \( N_P(N_P \cup \{r\}) \) is a finite group, and it contains a primitive idempotent if and only if \( r \neq 0 \).

**Proof.** Choose cross-section \( \{a_1, \ldots, a_\gamma\} \) of \( \alpha \alpha^{-1} \) and \( \beta \beta^{-1} \), respectively, and write

\[
\alpha = \left( \begin{array}{c}
A_1 & \ldots & A_\gamma \\
B_1 & \ldots & B_\gamma
\end{array} \right), \quad \beta = \left( \begin{array}{c}
B_1 & \ldots & B_\gamma \\
A_1 & \ldots & A_\gamma
\end{array} \right), \quad \gamma = \left( \begin{array}{c}
A_1 & \ldots & A_\gamma \\
B_1 & \ldots & B_\gamma
\end{array} \right), \quad \lambda = \left( \begin{array}{c}
B_1 & \ldots & B_\gamma \\
A_1 & \ldots & A_\gamma
\end{array} \right).
\]

---

If \( X \subseteq H \cap B \), then \( |P| = k < \infty \) implies \( \alpha \beta = \beta \). Also, since \( \gamma > 1 \), \( \beta \leq \beta^2 \), and \( \gamma(\beta) = \beta(\beta) \), the gap of \( \gamma \) satisfies the conditions of Theorem 2.1. Theorem 2.2 (depending on the nature of \( k \) and \( \gamma \)) in [N(P_X)], \( \lambda \subseteq \gamma \), \( \lambda(\beta) = \gamma(\beta) \), and \( \gamma(\beta) \) is an \( \alpha \)-isomorphism.

If \( X \subseteq n \leq H_0 \), then \( \gamma(n \gamma) = \beta(\gamma) \), so \( \gamma(n \gamma) \neq 0 \), so Theorem 2.3 implies \( \gamma(n \gamma) \neq 0 \), when \( n \) is even, and \( n \) is odd and \( e < n - 1 \). On the other hand, if \( n \) is odd and \( e < n - 1 \), then \( \alpha \beta \) is not \( \alpha \)-isomorphic to \( E_{n-1} \) (since their gaps are non-zero). Moreover, in this case, \( N_P(N_P \cup \{r\}) = E_{n-1} \cup \{0\} \), and this is 0-modal by [5, Lemma 5].

Suppose \( \gamma \) is finite and \( \alpha \neq \gamma(\beta) \) for non-zero idempotents \( \alpha, \beta \in P_X \), with rank \( r \). Then \( \gamma(\alpha) = \gamma(\beta) \), and \( \gamma(\alpha) \) is a non-zero idempotent, so \( \gamma(\alpha) = \gamma(\beta) \). Therefore, for each \( e \in \text{dom}(\alpha) \), \( \alpha(e) = \beta(e) = \gamma(e) = \gamma(e) \), hence \( \gamma(\alpha) = \gamma(\beta) \). Also, if \( \gamma \neq \gamma(\beta) \), then \( \gamma(e) = \gamma(e) \) for some \( e \in \text{dom}(\alpha) \), so \( \gamma(\alpha) = \gamma(\beta) \). Then, \( \gamma(\alpha) = \gamma(\beta) \), and it follows that \( \alpha = \beta \).

In other words, every non-zero idempotent in \( N_P(N_P \cup \{r\}) \) is primitive. Conversely, suppose \( \alpha \neq \beta \) is a non-zero idempotent in \( N_P(N_P \cup \{r\}) \), and \( \gamma(\alpha) = \gamma(\beta) \). Then we can write

\[
\beta = \left( \begin{array}{c}
B_1 & \ldots & B_\gamma \\
A_1 & \ldots & A_\gamma
\end{array} \right), \quad \alpha = \left( \begin{array}{c}
B_1 & \ldots & B_\gamma \\
A_1 & \ldots & A_\gamma
\end{array} \right),
\]

where \( [P] = r \in \{0, 1\} \) for some fixed \( 0 \neq 1 \) and \( b_i \subseteq B_i \) for each \( i \). Since \( \beta \neq N(P_X) \), its gap satisfies the conditions of Theorem 2.1. Since \( \gamma(\alpha) = \gamma(\beta) \) and \( r(\alpha) = r(\beta) \), the same is true for \( \alpha \), and \( \gamma(\beta) = \gamma(\beta) \). In other words, if \( r \leq H_0 \), then non-zero idempotent in \( N_P(N_P \cup \{r\}) \) is primitive.

Next we prove a result which is similar to [5, Lemma 4] and, in doing so, we do not assume any prior knowledge of the congruences on a completely 0-simple semigroup.

**Lemma 3.2.** Suppose \( \alpha \subseteq X \) is any set and \( r \) is any positive integer with \( 0 \leq r \leq X \). If \( \gamma \) is a non-universal congruence on \( N_P(N_P \cup \{r\}) \), then the relation \( \sigma^* \) defined on \( N(P_X) \) by \( \sigma^* = \{id_{N(P_X)} \cup \{n \in \text{D}(D_X \cup \{r\})\} \cup \{\gamma(\beta) \neq \gamma(\beta) \} \} \) is a congruence on \( N(P_X) \).

**Proof.** Clearly, \( \sigma^* \) is an equivalence, so we aim to show that it is left and right compatible with composition on \( N(P_X) \). To do this, we consider only the case when \( (\alpha, \beta) \in \sigma^* \) and \( r(\alpha) = r(\beta) \) (the other possibilities are easy to check). First suppose \( (\alpha, \beta, n) \in \sigma^* \) and \( \gamma(\alpha) = \gamma(\beta) \). Then \( n(\alpha) = n(\beta) \). Since \( \gamma \) is a congruence on \( N_P(N_P \cup \{r\}) \), it follows that \( (\alpha, \beta) \in \sigma^* \) and hence \( r(\alpha) = r(\beta) \) is universal on \( N_P(N_P \cup \{r\}) \). A contradiction. Thus, \( r(\alpha) = r(\beta) \) and this implies \( r(\alpha) = r(\beta) ) = \gamma(\beta) ) = \gamma(\beta) \).

On the other hand, if \( (\alpha, \beta, n) \in \sigma^* \) and \( r(\alpha) = r(\beta) \), then \( n(\alpha) = n(\beta) = r(\alpha) = r(\beta) \).
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of (disjoint) sets in the partition of \( d \) determined by the equivalence \( \mu \circ \sigma^{-1} \) on \( d \). Hence, if \( \mu = \mu' \), then \( g(\mu') \Rightarrow g(\mu) \), and \( \mu' = \mu \) if \( |X| = n \) is finite and odd, and \( r = n - 1 \). That is, the usual argument shows that \( \rho' \in \Delta \). Clearly, \( \rho' = \rho \circ \rho' \). Therefore, \( \langle \rho, \psi \rangle 
\in \sigma \cap (\Delta \cap \Delta) \subseteq \sigma^* \). Hence, \( \sigma^* \) is right compatible.

Now let \( \lambda \in N_{\mu}(X) \) and suppose \( r(\lambda) = r(\lambda') = r \) for the same \( \alpha, \beta \) as at the start. Let \( \{\alpha, \beta\} \cap \Omega(X) = \emptyset \). Then a similar argument leads us to conclude that \( t = r \) and hence \( d = \emptyset \). Also, say. Moreover, since \( r(\lambda) = r = r(\lambda') \), there exists a subset \( J \) of \( J \) which is a right semigroup of \( \mathbb{Z}/(a,c) \). Let \( \lambda_0 = \lambda(a,c) \). Then \( \{x \in X : x \in a(c) \} \subseteq \{x \in \lambda_0 \} \) and \( r(\lambda_0) \neq t(\lambda) \). Thus, when \( X \) is infinite, if \( g(\lambda) \geq r(\lambda) \) or \( r(a,c) \geq 0 \) for some \( x \in \lambda \), then \( \lambda_0 \) satisfies the same conditions and so \( \lambda_0 \in \Delta \). Suppose \( \lambda \) is spread over its ranks, but \( \lambda_0 \) is not, that is, there exists a cardinal \( p < r(\lambda_0) \) such that \( \lambda_0 \cdot \lambda_0 \leq p \) for all \( x \in a(c) \). This means that \( d = \bigcup \lambda_0 \cdot \lambda_0 \). Hence, \( g(\lambda_0) = k \). Therefore, in this case, \( \lambda_0 \) also belongs to \( \Delta \).

In fact, the same is true when \( |X| = n \), that is, when \( n \) is odd and \( r = n - 1 \) (since then \( \lambda \in N_{\mu}(X) \), \( g(\lambda) \neq 0 \) and \( r(\lambda) = n - 1 \) together imply \( \lambda \in \Delta \). Hence, \( \lambda_0 = \lambda \). Since \( \lambda_0 \cdot \lambda_0 \leq p \) for all \( x \in a(c) \), we conclude that \( \lambda_0 \cdot \lambda_0 \in \sigma^* \).

**Remark 3.3.** Recall that every non-universal congruence \( \rho \) on a \( 0 \)-simple semigroup is 0-regular; that is, \( \rho = \emptyset \); and clearly, by Lemma 3.1, \( N_{\mu}(\Delta) \langle 0 \rangle \), is 0-simple for each (finite or infinite) \( r \geq 4 \). Consequently, in the above result, \( \sigma(\lambda) \neq \emptyset \) implies \( \sigma(\lambda) = \emptyset \). For, if \( \sigma(\lambda) \), then, by their definition, \( \sigma(\lambda) \cap (\Delta \cap \Delta) = \emptyset \). Thus, since \( \rho = \emptyset \), 0-regularity, this implies \( \sigma(\lambda) \subseteq \emptyset \).

**Using the results in Section 2, we now determine all congruences \( \rho \) on \( N_{\mu}(X) \) for which \( g(\rho) \) is finite. Again, our argument closely follows that for \( 5 \), Theorem 5, but we include all the details for this more general context.**

**Theorem 3.4.** Let \( \rho \) be a non-identity and non-universal congruence on \( N_{\mu}(X) \) and suppose \( r = g(\rho) \) is finite. Then \( \rho = \sigma^* \), where \( \sigma \) is a non-universal congruence on \( N_{\mu}(\Delta) \).

**Proof.** Suppose \( \rho(\mu) \neq 0 \). By the definition of \( g(\rho) \), if one of \( \alpha \) or \( \beta \) has rank less than \( r \), then the other also has rank less than \( r \), and thus \( \rho(\mu) \neq 0 \). By Lemma 2.7, if the rank of \( x \) or \( z \) is at least \( r \), then \( r(x) = r(z) = s \). We assert that if \( x \) is a finite, then \( \alpha = \beta \).

To see this, assume \( x \in \Delta \) and \( z \neq \emptyset \) for some \( x \in d \) (without loss of generality). Write \( xs = xz \) and choose a partial cross-section \( Y \) of \( a(c) \) such that \( x \in Y \), \( Y \) is an \( \emptyset \). Choose \( x \in Y \) (this is possible since \( x \in \Delta \) and \( x \in \emptyset \) and \( x \notin \emptyset \)). Let \( Y \in \emptyset \) and observe that \( x \in Z(c) \) if and only if \( \emptyset(c) \) has rank at most \( r - 1 \). Therefore, both \( x \) and \( y \) belong to \( \emptyset(Y) \) (since their ranks are finite). Therefore, \( (\alpha', \beta') \in \rho \). Since this contradicts the choice of \( r = g(\rho) \), the assertion follows. Consequently, if \( z \in \emptyset \), then \( (\alpha', \beta') \in \emptyset(X) \). On the other hand, if \( r \leq s \in \emptyset \), and \( \alpha \neq \beta \), then Lemma 2.9 implies \( r = s \). That is, \( (\alpha', \beta') \in (\Delta \cap \Delta) \mid \emptyset \). We assert that \( \rho \in \Delta \cap \Delta \). For, clearly it is an equivalence on \( N_{\mu}(\emptyset) \). Also, if \( (\alpha', \beta') \in (\Delta \cap \Delta) \mid \emptyset \) and \( \rho \in \Delta \), then \( (\alpha', \beta') \in \emptyset \), where the ranks of \( \alpha, \beta \) are at most \( r \). Moreover, by the choice of \( \rho = g(\rho) \), either \( \rho(\alpha) = \rho(\beta) \) or \( \rho(\alpha) = \rho(\beta) \) and \( \rho(\beta) = \rho(\beta) \). However, the choice of \( \rho = g(\rho) \), either \( \rho(\alpha) = \rho(\beta) \) or \( \rho(\beta) = \rho(\beta) \) and \( \rho(\beta) = \rho(\beta) \). Hence, \( \rho \in \emptyset \) and \( \emptyset \) in the former case, \( \rho(\alpha) \rho(\beta) \in (\Delta \cap \Delta) \mid \emptyset \). Hence, \( \emptyset \) is no equi-

\[ r = g(\rho) \]
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$|X| = n$ is odd and $r = n - 1$, then $NP_{n,1}/NP_{n} = E_{n-1} \cup \{0\}$ and this was discussed fully in the proof of [5, Lemma 7]. Moreover, if $a \in A$, for each $i = 1, \ldots, n$, then

$$id \cdot y_{i} \mapsto (y_{i}) \cdot a \cdot (x_{i}) \mapsto (y_{i}) \cdot a \cdot z \cdot (x_{i}) \mapsto (y_{i}) \cdot b \cdot (x_{i}) = (y_{i}) = \gamma \cdot (x_{i}).$$

Since this pair belongs to $\beta$, it follows that $\gamma \in N$ and thus $(a, \beta) \in \tau$.

The next result extends [5, Corollary 1] to arbitrary sets.

Corollary 3.6. For any set $X$, the set of all congruences on $NP_{n}(X)$ with finite primary rank forms a chain with respect to $\subseteq$.

Proof. Let $\rho_{1}$ and $\rho_{2}$ be distinct congruences on $NP_{n}(X)$, neither of which equals the identity or the universal congruence on $N_{n}(X)$, and write $[\rho_{1}] = \rho_{1}$, where $r_{1}$ is a positive integer for $i = 1, 2$. Then $r_{1} = r_{2}$, and $\rho_{1}^{*} = \rho_{2}^{*}$ for some (unique) congruence $\sigma$ on $NP_{n}/NP_{n}$, where $NP_{n}/NP_{n} \subseteq NP_{n,1}/NP_{n}$, if $r_{1} < r_{2}$, then $NP_{n,1} \subseteq NP_{n}$, and $[\sigma] \cap (NP_{n} \times NP_{n}) \subseteq NP_{n,1} \times NP_{n}$, from which we deduce that $\rho_{1} \subseteq \rho_{2}$. Suppose $r_{1} = r_{2} = r$, say. By Lemma 3.5, $\sigma$ is determined by some $N_{1} \subseteq G(Y)$, and $\sigma$ is determined by some $N_{2} \subseteq G(Y)$, where $|Y| = r$ (note that the same $Y$ can be used). Since the normal subgroup $G(Y)$ is a chain, it follows from Lemmas 3.5 that $\sigma_{1} \subseteq \sigma_{2}$ or $\sigma_{2} \subseteq \sigma_{1}$, and hence $\rho_{1} \subseteq \rho_{2}$ or $\rho_{2} \subseteq \rho_{1}$.

4 Infinite Primary Rank for $NP_{n}(X)$ when $|X|$ is Regular

For the regular case, X is an infinite set with cardinal $\aleph_0$. Suppose $\rho$ is a congruence on $NP_{n}(X)$ and let $\rho = \rho^{1} \cap N_{n}(X)$. Clearly, $\rho$ is a congruence on $N_{n}(X)$, and if $\rho^{2}$ is infinite, then $|\rho| = \aleph_0$ is also. In this event, we can define a $\beta$ in terms of a finite number of Rees congruences and Malcev congruence as follows.

Theorem 4.1. Suppose $|X| = k \geq n \geq 8$. If $\rho = \rho^{1} \cap N_{n}(X)$ with $|\rho| = \aleph_0$, then

$$\rho = \rho_{1}^{0} \cap \rho_{1}^{1} \cap \rho_{2}^{0} \cap (NP_{n,1} \times NP_{n}) \cap (NP_{n,1} \times NP_{n}),$$

where $\rho_{1}$ is $\rho$-invariant, and the cardinal $\eta_{1}$, $\eta_{2}$, form a sequence $n \leq \eta_{1} < \ldots < \eta_{k} \leq k$, in which $\eta_{k+1}$ is infinite, either $\eta_{1} = 1$, or $\eta_{k}$ is infinite, and $\eta_{k} = k$.

Conversely, if $\rho$ is a relation on $N_{n}(X)$ defined as in (1) for a sequence of cardinalities with the above properties, then $\rho$ is a congruence on $N_{n}(X)$.

In the above, for each proper ideal $L = RX \cap NP_{n} = NL$ of $N_{n}(X)$, $L$ denotes the corresponding Rees congruence on $N_{n}(X)$ (compare [1, Vol. 1, p. 171] and [1, Vol. 2, p. 227]). Also, as in [5], $DL_{n}$ denotes the $D$-class of $N_{n}(X)$ which contains all elements with rank $r$. In addition, for any $a, b \in P_{n}(X)$ and $n \geq 0$, we let

$$D(a, b) = \{ x \in X : x \mapsto 0 \}, \quad D(a, b) = \max \{ D(a, b, 0), D(a, b, 0) \}, \quad D(a, b) = \{ (a, b) \in P_{n}(X) \times P_{n}(X) : D(a, b) < n \}. \quad \Delta_{n} = \{(a, b) \in P_{n}(X) \times P_{n}(X) : D(a, b) < n \}.$$
Conclusions hold. It follows that \(dr(\lambda \alpha, \lambda \beta) = d\) and

\[
\lambda \alpha = \begin{pmatrix} u' & r_x \end{pmatrix} \sim \lambda \beta = \begin{pmatrix} u' & r_y \end{pmatrix},
\]

where \(\lambda \alpha\) may not exist for some \(i\), (that is, when \(y_i \neq \text{dom } \beta\)) and the \(\lambda \beta\) may not be distinct (for example, if \(\beta\) is not injective on \(Y\)). If \(|\{y_i\}| = d\), write \(\lambda \beta = (y_i)\) where the \(y_i\) are distinct and fix \(y_i \in Y\) such that \(y_i \beta = y_i\). If \(X'\) is the identity on \(\{y_i\} \cup \{r_x\}\); then as before, \(X' \in \Pi(X)\) and we obtain

\[
X \alpha = \begin{pmatrix} v_x & r_y \end{pmatrix} \sim X \beta = \begin{pmatrix} v_x & r_y \end{pmatrix},
\]

and these are elements of \(\Pi(X)\) whose difference rank equals \([d \leq d]\). On the other hand, if \(|\{y_i\}| < d\), then \((\alpha \lambda)(y_i) = \alpha(y_i)\), say, has cardinal \(k\). In this event, if \(\mu = \text{the identity on } \{y_i\} \cup \{u\}\), then \(\mu \in \Pi(X)\) (since \(d \leq d(\alpha) = k\)), and from (3) we obtain

\[
\lambda \alpha = \begin{pmatrix} u' & r_x \end{pmatrix} \sim \lambda \beta = \begin{pmatrix} u' & r_y \end{pmatrix}.
\]

Hence, again we find a pair in \(\mathcal{P}\) whose difference rank equals \([d \leq d]\). In other words, if \(\rho\) contains a pair of elements which differ at \(d\) places, then \(\mathcal{P}\) does also. Note that with the above notation, \(r(\beta) = r(\alpha) = r\), say, and

\[
Y \alpha \subseteq D_{\alpha} = D_{\alpha} \cap \text{ran } \alpha \quad \text{and} \quad D_{\alpha} \cap \text{ran } \alpha \subseteq D_{\beta}.
\]

Hence, \([\text{ran } \alpha \cap [\text{ran } \beta]] = (\text{ran } \alpha) \cap (\text{ran } \beta) \subseteq D_{\alpha} \cap \text{ran } \alpha = r(\alpha) \subseteq r(\beta)\).

Clearly, we will reach the same conclusion if \(X = \rho\) is used in the above argument.

Therefore, by Remark 4.4, if \(\lambda \alpha \in \Delta \cap \Gamma\) for some \(\lambda \in \Pi(X)\). In other words, we have shown that if there exists \(\alpha \in \Delta \cap \Gamma\) for which \(r(\beta) \leq r(\alpha) = r\) and \(dr(\alpha, \beta) = d \leq r\), then there exists \(\lambda \beta \in \Delta \cap \Gamma\) for which \(r(\beta) \leq r(\alpha) = r\) and \(dr(\alpha, \beta) = d\). Clearly, the converse also holds since \(r(\beta) = r\), and \(I_\alpha \in \Delta \cap \Gamma\) is the only \(\lambda \beta \in \Delta \cap \Gamma\) as \(\Delta \cap \Gamma\) is the only \(\lambda \beta \in \Delta \cap \Gamma\). Hence,\(\Delta \cap \Gamma\) is the only \(\lambda \beta \in \Delta \cap \Gamma\).

Conversely, if \(\alpha \in \Delta \cap \Gamma\), then \(\alpha \in \Delta \cap \Gamma\) since \(\alpha \in \Delta \cap \Gamma\). If \(r(\beta) \leq r(\alpha) = r\) and \(dr(\alpha, \beta) = d\), then this is the same as \(\alpha \in \Delta \cap \Gamma\). Hence, \(\alpha \in \Delta \cap \Gamma\). From this, it follows that \(\Delta \cap \Gamma\) is the only \(\lambda \beta \in \Delta \cap \Gamma\).

It remains to consider the last term in (1) and the corresponding one in (2). If \(n = 1\) in (1), then no pair of distinct elements of \(\Pi(X)\) with rank \(k\) is \(\rho\)-equivalent. Suppose there exists \((\alpha, \beta) \in \rho \cap (D_{\alpha} \times D_{\beta})\) where \((\alpha, \beta) \neq (\beta, \alpha)\). Without loss of generality, we assume \(\alpha \neq \beta\) for some \(\alpha \in \Delta\), and let \(A = (\alpha)\) be a cross-section of \(\alpha \in \Delta\) which contains \(\alpha \in \alpha = \alpha\), say. Then as before, \(\lambda \alpha \in \Pi(X)\) and we have

\[
\text{id}_{\Delta \alpha} = \begin{pmatrix} a & a \end{pmatrix} \sim \text{id}_{\Delta \beta} = \begin{pmatrix} a & a \end{pmatrix},
\]

where the \(a, \beta\) are not necessarily distinct. If \(|\{a, \beta\}| = k\), write \((a, \beta) = (a, \beta)\), where the \(a, \beta\) are distinct (if non-empty) \(0 \in \alpha\) and \(|\alpha| = k\). Let \(\beta = (\alpha)\). Then

\[
\text{id}_{\Delta \alpha} \in \Pi(X)\quad \text{and}
\]

\[
\text{id}_{\Delta \beta} = \begin{pmatrix} a \end{pmatrix} \sim \text{id}_{\Delta \beta} = \begin{pmatrix} a \end{pmatrix}.
\]

Since \(a, \beta \in \mathcal{P}, \text{id}_{\Delta \alpha} \neq \text{id}_{\Delta \beta}\) and these are \(\rho\)-equivalent elements of \(\Pi(X)\) with rank \(k\), contradicting our initial assumption that \(n = 1\).

Hence, if \(n = 1\), then \(|\{a, \beta\}| = k\) and so \((\alpha, \beta) = (\alpha, \beta)\), say, has cardinal \(k\). Then \(\text{id}_{\Delta \alpha} \in \Pi(X)\) (since \(|\{\alpha\}| \leq d(\alpha) = k\)), and from (2) we obtain

\[
\text{id}_{\Delta \alpha} \cap \text{id}_{\Delta \beta} \neq \emptyset.
\]

It follows that \(\text{id}_{\alpha} \neq \emptyset\) and \(\rho\) is universal, contradicting our basic assumption.

Suppose instead that \(n > 0 \in 1\), and hence \(\alpha \neq k\) (by the condition on the cardinal). This means that if \(X = \Delta \cup \Gamma\), \(|\alpha| = k\) and \(|\beta| = n\), then

\[
\text{id}_{\Delta \alpha} \cap \text{id}_{\Delta \beta} \neq \emptyset.
\]

From this, it follows that \(\Delta \alpha \cap (D_{\alpha} \times D_{\beta}) \subseteq \rho\).

Consequently, we have proved half of the following result. For its converse, we note that just as in [5], Lemma 4.3 can be used to show that \(\rho\) is a congruence on \(\Pi(X)\), provided the cardinals have the properties stated: the difference between the last paragraph in the proof of [5, Theorem 3] and the current one is simply a matter of notation (that is, in [5, Theorem 3], the difference between \(\Pi(X)\) and \(\Pi(X)\) become \(\Pi(X)\)).

Theorem 4.5. Suppose \(X = k \geq \alpha = \beta\) and \(\beta = \beta\) is regular. If \(\alpha \in \Pi(X)\) is a non-universal congruence on \(\Pi(X)\) for which \(\text{id}_{\alpha} \neq \emptyset\), then

\[
\rho - \Pi(X) \cap (D_{\alpha} \times D_{\beta}) \subseteq \emptyset.
\]

where \(\alpha \in \Pi(X)\) and the cardinals \(\xi \in \alpha\), form a sequence \(\xi \in \xi < \cdots < \xi < \xi < \cdots \leq k\), in which \(\alpha\) is infinite and \(\alpha = k\).

Conversely, if \(\rho\) is a relation on \(\Pi(X)\), defined as in (7) for a sequence of cardinals with the above properties, then \(\rho\) is a non-universal congruence on \(\Pi(X)\).

5 Infinitary Primary Rank for \(\Pi(X)\) when \(|X|\) is Singular

In this section, \(X\) is an infinite set whose cardinal \(k\) is singular, that is, according to [3, Lemma 10.2.2], \(k = \sum_{\alpha < \beta} \kappa_{\alpha}\) for some distinct infinitudes \(\kappa_{\alpha}\), where \(|\alpha| = k\) and \(\kappa_{\alpha} < k\) for each \(\alpha < \beta\). To describe all the congruences on \(\Pi(X)\) for each \(X\), we closely follow the argument in Section 4. In fact, here the only differences will occur when we need to ensure that a specific transformation belongs to \(\Pi(X)\), that is, it satisfies the conditions of Theorem 2.2.

Like before, given a congruence \(\rho\) on \(\Pi(X)\), we let \(\mathcal{P}\) denote the restriction of \(\rho\) to \(\Pi(X)\), and observe that if \(\text{id}_{\alpha} \neq \emptyset\), then \(\text{id}_{\alpha} \neq \emptyset\). In fact, since \(\rho \neq \emptyset\), we know \(\text{id}_{\alpha} \neq \emptyset\). For the reverse inequality, suppose \((a, \beta) \in \rho\) for some
\[ a \in N P(X) \text{ and let } A \text{ be a cross-section of } a \cdot a^{-1}. \text{ Since } k \text{ is singular and } a \in N P(X), \ \text{Theorem 2.2 implies that } g(\alpha) \neq 0, \text{ and either } g(\alpha) \geq r(\alpha) \text{ or } a \text{ is spread over its rank. If } r(\alpha) < k, \text{ then } |A| < k, \text{ so } |X \cdot A| = k \text{ and hence } id_A \in N(X). \]

Suppose \( r(\alpha) = k \). If \( g(\alpha) \geq r(\alpha) \), then \(|X \cdot A| \geq g(\alpha) = k\); and if \( a \) is spread over its rank, then for each \( m \in M \) (see the start of this section), there exists \( \mu_m \in X \) such that \(|\mu_m \cdot a^{-1}| > \mu_m\). Since \( A \) contains exactly one element from each \( \mu_m \cdot a^{-1} \), we see that for each \( m \), \(|\mu_m \cdot a^{-1}| > k_m\). Hence, \( \kappa = \sum k_m \leq \sum_{\mu_m \cdot a^{-1}} |A| \), and it follows that \(|X \cdot A| = k\). Thus, \( id_A \in N(X) \) in all cases and, as in Section 4, we deduce that \( g(\alpha) \leq g(\beta) \) and equality follows. Moreover, since \( \eta(\alpha) \leq \eta(\beta) \), we know \(|X \cdot A| = k\) for each \( A \subseteq X \) with cardinal less than \( \eta_k \), hence \( id_A \in N(X) \) and, as before, we conclude that \( N P(X) \subseteq \rho \).

Next, both Lemmas 4.2 and 4.3 hold for any set \( X \), so they can be applied in the present situation. In particular, Remark 4.4 remains valid.

Now using the same notation as before, let \( A \) be the identity on \( B \cup \{e\} \).

Since \( k \) is singular and \( a \in N P(X) \), Theorem 3.2 implies that \( g(\alpha) \neq 0 \), and either \( g(\alpha) \geq r(\alpha) \) or \( a \) is spread over its rank. If \( r(\alpha) < k \), then \(|B| + |e| < k \), hence \( g(\alpha) = 0 \) and \( A \in N(X) \). Suppose instead that \( r(\alpha) = k \). Then the above argument for the set \( A \) applies equally here for the set \( B \), and we deduce that \( A \in N(X) \) in all cases. As \( \alpha(3) \), this implies that \( (\alpha, \lambda, \lambda) \in \rho \), where \( \alpha(3) = 3 \) and, as before, the same previos holds. Then the same \( A \) belongs to \( N(X) \) (since \( \{x \cup \{e\} \in X \cup \{y \cup \{e\} \} : |X \cup \{e\}| = k \}) \), and so we obtain (4). On the other hand, if \( \rho \) is the identity on the set \( \{x \cup \{e\} \cup \{e\} \} \), then \( \rho \not\in N(X) \) (since by Theorem 2.2, \( d(\rho) \geq d(a) = k \)) and thus we obtain (5).

Thus, we have obtained a result which is exactly the same as Theorem 4.5, except \(|X| = k \) is a singular cardinal.

We now deduce a result similar to [6, Corollary 2]. Our proof follows the one for \( N(X) \) but, since it depends on Theorem 4.5 (and the corresponding result for singular cardinals), we include all the details.

**Corollary 5.1.** Suppose \(|X| = k \geq k_0 \) and write \( \Delta^*_a = \Delta_a \cap (N P(X) \times N P(X)) \).

Then \( \Delta^*_a \) is the only maximal congruence on \( N P(X) \), and hence \( N P(X)/\Delta^*_a \) is a congruence-free nilpotent-generated regular semigroup.

**Proof.** First we note that \( \Delta^*_a \) is a non-universal congruence on \( N P(X) \).

Since \( N P(X) \) is nilpotent-generated and regular (by Theorem 2.1 and 2.2), and \( \Delta^*_a \) is a congruence on \( N P(X) \), it follows that \( N P(X)/\Delta^*_a \) is also nilpotent-generated and regular.

Suppose \( \Delta_a^* \subseteq \rho \) for some non-universal congruence on \( N P(X) \). Now \( \eta(\rho) \) equals

\[
\text{the least cardinal greater than } r(\alpha) \text{ for each } \alpha \in N P(X) \text{ such that } (\alpha, \emptyset) \not\subseteq \rho. 
\]

But if \( A \subseteq X \) is cardinal less than \( k \), then \( d(id_A) = k \) and \( g(id_A) = k \), so \( id_A \in \Delta^*_a \subseteq \rho \). In particular, since \( k_0 \leq k_0 \), we can conclude that \( \eta(\rho) \geq k_0 \). Therefore, the form displayed in (7), regardless of whether \( k \) is regular or singular. Clearly, \( (\alpha, \emptyset) \in \Delta^*_a \subseteq \rho \) for each \( \alpha \in N P \), so \( \eta_k = k_0 \). Moreover, if \( X = A \cup B \cup C \), where \(|A| = \{c\} = k_0 \) and \(|B| < k_0 \), then both \( id_{A \cup B} \) and \( id_{B \cup C} \) have gap and defect equal to \( k_0 \), so they belong to \( DP \), and hence \( id_{A \cup B} \in DP \). If \( id_{B \cup C} \in DP \), then \( \Delta^*_a \subseteq DP \), this implies that each term in (7) is contained in \( \Delta^*_a \), hence \( \rho \subseteq \Delta^*_a \) and equality follows.

Finally, suppose \( \rho \) is a maximal congruence on \( N P(X) \) for which there exists \((\alpha, \emptyset) \not\subseteq \rho \) with \( d(\alpha, \emptyset) = k_0 \). Then \( r(\alpha) = r(\emptyset) = k_0 \). Since such pairs \((\alpha, \emptyset) \not\subseteq \rho \) do not belong to the congruences described in Theorem 3.4, we deduce that \( \eta(\rho) \geq k_0 \). However, then (7) implies that \( k = k_0 \), and so we have a contradiction:

\[
k' \leq k \\

Thus, \( d(\alpha, \emptyset) \not\subseteq \rho \) for all \( \alpha \in \rho \), hence \( \rho \subseteq \Delta^*_a \), and equality follows by the maximality of \( \rho \) and the fact that \( \Delta^*_a \) is non-universal.

**References**


