ACE Journal — Applied Computing Engineering Journal

ACE *Journal* — *Applied Computing Engineering Journal 3(1):13-17, 2004*

ISSN 1645-3999 (to be available on-line at http://www.ace.co.pt)

Analytical Solutions for Rigid Block Structures Under Small Displacements Regime

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Received: May 14^{th}, 2003; Accepted: October 12^{th}, 2003; In revised form: October 23^{rd}, 2003

Abstract. *The methodology presented in this work combines the theoretical framework developed for the dynamics of coupled rigid elements with the characteristics of rocking motion, in order to reproduce the behaviour of blocky masonry structures under seismic actions. As an application, a closed kinematic chain structure is analysed for the case of harmonic forcing.*

Keywords: Rigid body; coupled rotations; Dirac-delta; impact; masonry; rocking motion.

1 INTRODUCTION

For masonry structures under seismic actions, the theory of coupled rocking rotations (CRR) for multiple bodies can be applied if some basic hypotheses are satisfied [1], namely:

1 The bodies are not deformable;

2 There is no relative sliding between two adjacent contact surfaces;

3 The main sources for dissipation of energy occur at the impact processes between the rigid parts.

The first two requirements are equivalent to those that Limit Analysis establishes for the case of masonry construction [2, 3], while the third adds the damping characteristics of rocking motion.

Although these hypotheses are not realistic for near-source ground motions (where sliding mechanisms are present and friction plays an essential role for energy loss), historical buildings under seismic hazard can be assessed at the light of CRR in low-seismicity regions and, in general, at those situations in which the displacements are considered small.

Regarding to the third requirement above, it may seem that rocking motion seldom occurs for a small displacement regime. All the work done until now about the Rocking Motion (RM) is referred to single block rotations (or two blocks at the most) as if the whole structure was rocking. Nevertheless, RM does not need this limit situation to appear. In rigour, if there is no sliding, RM is present at every crack and impact mechanisms appear as the only way for the blocky masonry system to dissipate energy.

In [4], the theory of n-body coupled rigid body rotations was presented. There, the basic dynamical functions and displacements were elaborated. Concepts such as dynamic chains, generalized matrices or symmetry reduction were introduced and succeeded in both simplifying and highlighting the main features of the process. A computer subroutine based in Maple environment was built specifically for this problem. Finally, a FORTRAN90 code was written in order to integrate the resulting differential equations.

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Regarding RM, a compact mathematical form for the impact interaction present in rocking motion was found [5]. This result was incorporated to the theory and CRR was built from the results obtained in [4, 6]. Perhaps, the power of CRR consists in its independence from the continuum mechanics based algorithms. As a consequence, it can be considered as a feasible alternative to the existing procedures based on continuum considerations (as FEM or BEM). Eigenfrequencies (based on the energy approach performed by Housner [7]), stability, resonance regions and other relevant mechanical consequences can be obtained by using CRR.

Within the scope of CRR, a great number of practical applications can be included. Examples that CRR can deal with are four-hinged masonry arches, a closed kinematical chain structure representing Greek temples, monolithic columns, water or nuclear tanks and offshore equipment. From small number of blocks involved, treated analytically, to an arbitrary number of them, in the domain of statistical mechanics, the theory can hold an adequate basis. In each case, additional hypotheses and results can strengthen the hypotheses.

2 ADOPTED METHODOLOGY

After the geometry of the problem is given (say through a CAD scheme), the plastic hinges are determined via a Limit Analysis program directly [8] (an alternative is to determine the zones of lowest tensile strength by a DEM, FEM or BEM routines). This information is given to the Maple program (Mathematica or any other symbolic environment would be equivalent). Then, the partition of the continuum into rigid domains according to the data is performed. At this stage, a first number of degrees of freedom (DOF(1)) is established, being built a potential dynamic chain.

Constrains are introduced by imposing holonomic geometric conditions to the hinges. The initial DOF(1) are reduced to DOF(2). In addition, arguments based on symmetry reduce even more the DOF(2). The result of this process is DOF(3). Finally, a FORTRAN routine integrates the differential equations by using an ODE integrator. Fourth-order Runge-Kutta, Gear or Variable step size integrators are more or less convenient depending on the final type of differential equations obtained. A basic scheme of the proposed algorithm is shown in figure 1.

The Maple program with the final number DOF(3) helps on performing algebraic manipulations in order to reduce the problem to its differential equations of motion. The main parameters of the motion are found in this process, as schematically shown in figure 2.

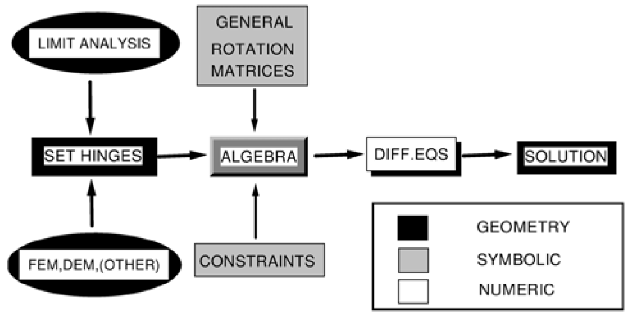

Figure 1: Proposed methodology.

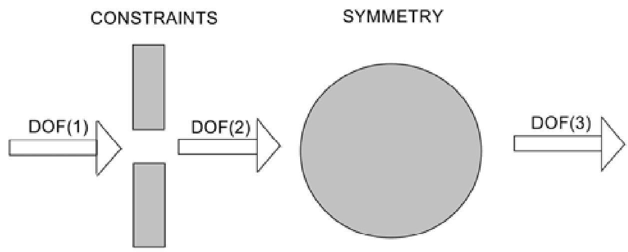

Figure 2: Reduction of degrees of freedom.

3 THEORY OF MULTIPLE-BODY COUPLED ROCKING ROTATIONS

3.1 Dynamical Functions

In [4], a novel formulation has been presented regarding compact expressions for the three dimensional displacements, kinetic and potential functions for an arbitrary number of blocks coupled as functions of their relative angles. There, also the concept of dynamic chain has been introduced, as the compatible subset of the set of the potential axes of rotation that is chosen for a given movement. The expressions found are again provided here for completeness.

The expressions for the generalized three-dimensional displacements are given by

$$\vec{r}^n(\vec{\phi}) = \sum_{j=0}^{n-1} (a^j)\Gamma^j(\vec{e}) + (x^n)\Gamma^n(\vec{e}) \tag{1}$$

As discussed in [4], this expression allows computing the position of any point of the n-body as function of the movement performed by the dynamic chain. Here, a represents a row-matrix of vectors which connect two consecutive hinges, x is the row-matrix for the position of an arbitrary point of the last body of the chain measured from a system of reference fixed at one of its hinges, and Γ is the matrix of the rotation operators involved expressed in the canonical basis $\{e\}$, see also figure 3.

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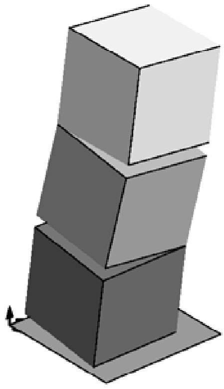

Figure 3: Three-coupled body rotations.

The expression for the kinetic energy is given by

$$T = \frac{1}{2}\sum_{n=1}^{N}\left[M^n\sum_{j,k=0}^{n-1}a_o^j\gamma^{jh}a_o^{hT}+\int_{V_n}x_o^n\gamma^{nn}x_o^{nT}\right. \\ \left. +2M^n\sum_{j=0}^{n-1}a_o^j\gamma^{jn}x_o^{nT}\right] \tag{2}$$

where; M^n and V_n are the mass and volume of body n respectively, and the γ matrices are inner products of the general Γ. For a detailed explanation, see [4].
The expression for the potential energy is given by

$$U = g\sum_{n=1}^{N}M^n\left[\sum_{j=1}^{n-1}a_{oi}^j\Gamma_{i2}^j + x_{coi}^N\Gamma_{i2}^N\right] \tag{3}$$

For some special cases, the initial three-dimensional configuration can be reduced to a planar problem. This happens when the force acts through a plane, or when the geometry of the structure exhibits a high symmetry. As a consequence, the rotation matrices can be substantially reduced and the expressions take an easier handling. In addition to the symmetry arguments other forms of simplification can be considered if some basic assumptions about the possible displacements are made. In particular, if the last hinge to occur in the system of masonry blocks is assumed to remain fixed during the motion, there are two additional equations to be incorporated to the system. For selected structures, this reduction process leads to Single-Degree-of-Freedom systems (SDOF). Under the hypotheses above, the Lagrangian for the n-bodies takes the form as function of the generalized coordinates of angle and velocity, given by

$$L^n = \frac{1}{2}(1 \pm \nu^n\theta + \xi^n\theta^2) \mp \eta^n\theta + \frac{1}{2}\chi^n\theta^2 \tag{4}$$

Here, ν^n, ξ^n, η^n and χ^n are geometric parameters associated to the body. This expression can be applied for the three examples analyzed in this work and, in particular, for the closed kinematical chain structure.

4 FORMULATION FOR ROCKING MOTION ANALYSIS

When a rigid body changes instantaneously its center of rotation from one point to another, the resulting motion is usually described in terms of differential equations for certain domains of the dynamical variables involved. For simple planar motions, the sign of the angular variables considered can define these domains and the differential equations for the motion become piecewise. The impact marks the transition from rocking around one edge to rocking around the other, and, therefore, the transition from one governing equation to another. In addition, the sudden changes of velocity at the critical point (as shown in figure 4) can be associated with an impulsive force.

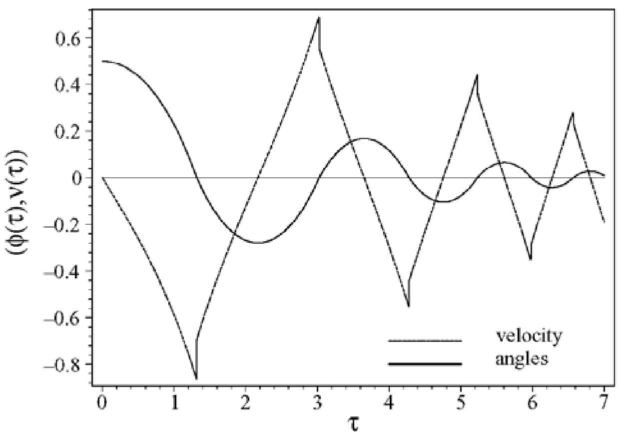

Figure 4: Rocking motion behaviour.

The traditional formulation of impact mechanisms introduces a restitution coefficient and resets the initial conditions every time of impact, according to a reduction in velocity proportional to this coefficient. The main drawback of the traditional process is the difficulty in its generalization to a higher number of blocks. In fact there is no effective form for damping in the differential equation.

4.1 Implementation of Dirac-Delta Forces

In [6] a Dirac-delta type interaction was introduced qualitatively. In [5], it was improved and successfully compared with the traditional approach and experimental data (see figure 5). Its implementation explained the behaviour observed in a single rocking block and allowed

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to propose a more compact formulation. The expression for the interaction found was

$$F_\delta = C \ln(r)\dot{\theta} \left|\dot{\theta}\right| \delta(\theta) \qquad (5)$$

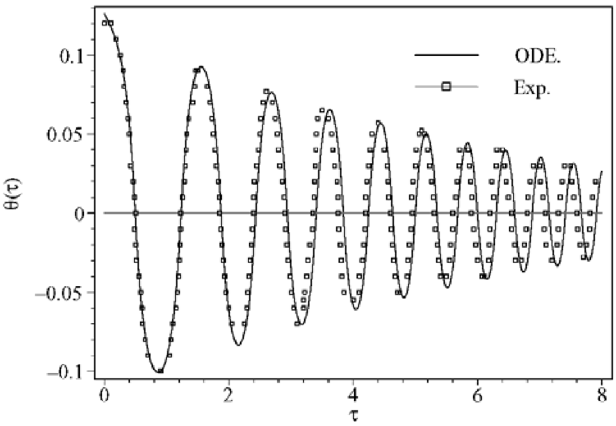

Figure 5: Comparison of Dirac-delta force with experimental data.

4.2 Configuration of Energetic Levels

A useful conclusion was derived in [5] regarding the energetic distribution of the free rocking motion. The diagrams illustrated in figure 6 offer a useful way for visualizing the physics of the problem and can be used for obtaining quantities with practical relevance, namely the rate of energy loss per impact.

5 APPLICATIONS TO SYSTEMS REDUCIBLE TO ONE DEGREE OF FREEDOM

Within the possibilities of CRR, some simple cases are analysed in this work. All of them belong to the class of systems that can be reduced to SDOF. In [4], it was shown that the mechanical behaviour of all of the present cases could be understood by means of the single block rocking dynamics. In particular a generalization of the Housner-Hogan equations was found [7, 9] and used for the case of the closed kinematic chain structures of figures 7 to 9.

6 APPLICATION

For the case of kinematic chains with one degree-of-freedom, the theory of coupled rocking rotations was applied to a simple model representing one-storey masonry buildings during earthquakes. The frequency as function of the forcing amplitude and a representative height

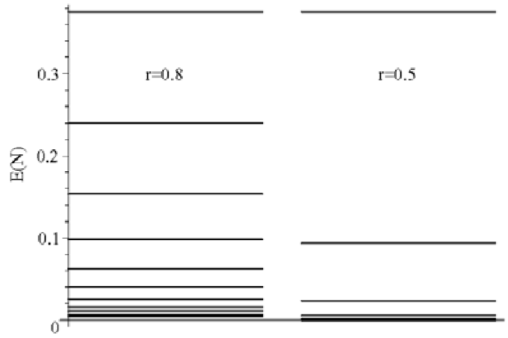

Figure 6: Energetic levels of the Rocking Motion.

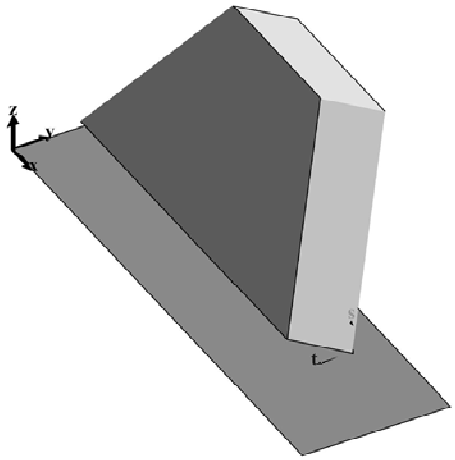

Figure 7: One single block rotations.

is found as, see [4]

$$F^H = \frac{\sqrt{\chi}}{4\cosh^{-1}(\frac{1}{1-\varepsilon\theta_m})} \qquad (6)$$

In the above expression, θ_m is the maximum tilt angle (that is, the amplitude of movement) while χ and ε are two parameters that depend solely on the geometry of the block. Manipulation of equation 6 allows plotting the Housner-frequency versus amplitude and height, as shown in figure 10.

Although this is just a first result, the methodology presented here opens new paths for theoretical research and computation. For instance, systematic Poincar Surface of Section Study revealed patterns with stability islands and chaos for different values of initial conditions and characteristic parameters.

7 CONCLUSIONS

It is likely that on of the best possibilities to handle the difficulties related to the safety and conservation of his-

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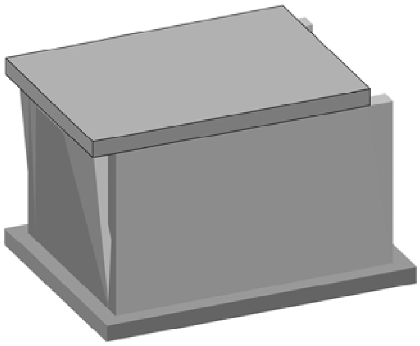

Figure 8: Closed kinematic chain structure.

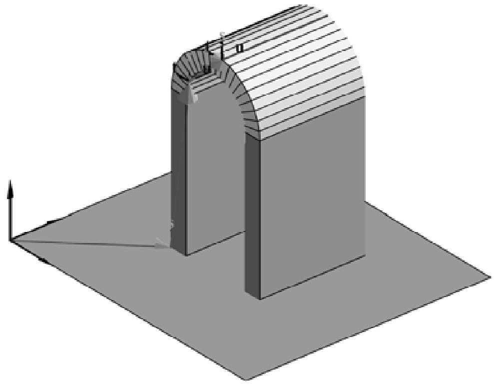

Figure 9: Four-hinged arch.

torical constructions is the usage of structural component models, namely lumped approaches, which are at best a crude approximation of the structure. The simplicity of the geometric model allows increased complexity on the loading side and in the non-linear dynamic response. Such an approach can be used to determine overall dynamic structural response to actual earthquake ground motion input but rely heavily on the correct definition of component hysteresis, which has to include material non-linearity and also effects resulting from the true geometry of the structure.

The present paper addresses the aforementioned approach, aiming at representing the damping and dynamic behaviour of masonry structures by the impact of a single degree-of-freedom block in rocking motion. The problem exhibits a hard singularity at a particular point where the differentiation of the dynamic functions is not well defined. Although this motion may seems very simple, the process hides both a great richness in dynamic behaviour and a wide range of practical relevance. Here, a first step has been made, regarding the proposal of a compact formulation, the formulation of a generalised

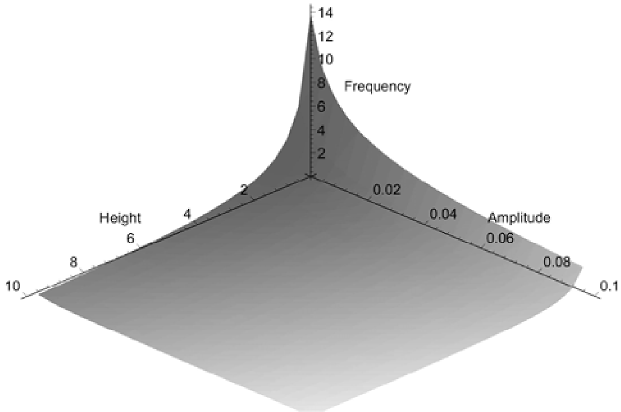

Figure 10: Generalized Housner Frequency.

displacements formwork and the representation of actual structures by a single degree-of-freedom system.

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