A nonlinear hyperbolic Maxwell system using measure-valued functions

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Abstract

We consider a modified antenna's problem with power-type constitutive laws. This consists in a new nonlinear hyperbolic system that extends a Duvaut-Lions model. Using the Galerkin approximation, properties of the natural functional spaces, and exploring the L^p-$L^{p'}$ duality, we prove the existence of solutions, in a generalized sense, passing to the limit in a family of approximated problems and using measure-valued functions. In this process the difficulties in obtaining the necessary *a priori* estimates for the solutions of the finite-dimensional problems are overcome through the use of bases with special properties related to the model.

1 Introduction

The starting point are the Maxwell equations

$$\partial_t \boldsymbol{d} + \boldsymbol{j} - \nabla\times\boldsymbol{h} = 0, \tag{1a}$$
$$\partial_t \boldsymbol{b} + \nabla\times\boldsymbol{e} = 0, \tag{1b}$$
$$\nabla\cdot\boldsymbol{d} = q, \tag{1c}$$
$$\nabla\cdot\boldsymbol{b} = 0, \tag{1d}$$

where $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{d}$, $\boldsymbol{b}$, $\boldsymbol{j}$ and q represent the electric and magnetic fields, the electric and magnetic inductions, the current density and the electrical charge, respectively.

Generalizing a linear model of an antenna's problem due to Duvaut and Lions [7], we replace the classical constitutive Ohm's law and the linear magnetization and polarization laws, by analogue laws of power type ([1, 11, 17]), obtaining a nonlinear hyperbolic system.

We adopt as Ohm's law the relation

$$\boldsymbol{j} = \sigma|\boldsymbol{e}|^{p'-2}\boldsymbol{e}, \tag{2}$$

and we consider that the relations between the fields and inductions, both electric and magnetic, are of the type

$$|\boldsymbol{d}|^{p-2}\boldsymbol{d} = \varepsilon\boldsymbol{e} \quad \text{and} \quad |\boldsymbol{b}|^{p'-2}\boldsymbol{b} = \mu\boldsymbol{h}, \tag{3}$$

where p and p' are conjugate exponents, both greater than 1. In order to obtain a consistent model from the mathematical point of view we were forced to choose conjugate exponents in the power-type laws, which emphasize the dual character between the electric and magnetic fields and inductions. Different types of nonlinear constitutive laws were suggested in the literature (see, for instance, [4, 16]), some of them leading to variational and quasi-variational inequalities ([2, 8, 11]).

The permittivity ε and the magnetic permeability μ are time independent functions, bounded from above and below by positive constants. The conductivity σ is also time independent and it is bounded and nonnegative. We observe that the constitutive laws (2)-(3), when $p = p' = 2$, correspond to the linear case treated in [7].

The constitutive laws (3) are rewritten as follows,

$$\boldsymbol{e} = \hat{\varepsilon}\,|\boldsymbol{d}|^{p-2}\boldsymbol{d} \quad \text{and} \quad \boldsymbol{h} = \hat{\mu}\,|\boldsymbol{b}|^{p'-2}\boldsymbol{b}, \tag{4}$$

where $\hat{\varepsilon} = \frac{1}{\varepsilon}$ and $\hat{\mu} = \frac{1}{\mu}$.

In what follows we assume that Ω is an open $\mathscr{C}^{0,1}$ subset of $\mathbb{R}^3$ with bounded boundary Γ and we denote by $\boldsymbol{n}$ the outward unitary normal vector to Γ. Going back to the vectorial equations (1), considering the case not necessarily homogeneous, since no additional difficulty is introduced, using the constitutive laws (2) and (4), the problem is modeled by the equations

$$\begin{aligned} \partial_t\boldsymbol{d} + \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d} - \nabla\times(\hat{\mu}\,|\boldsymbol{b}|^{p'-2}\boldsymbol{b}) &= \boldsymbol{g}_1 \qquad \text{in } Q,\\ \partial_t\boldsymbol{b} + \nabla\times(\hat{\varepsilon}\,|\boldsymbol{d}|^{p-2}\boldsymbol{d}) &= \boldsymbol{g}_2 \qquad \text{in } Q, \end{aligned}$$

where $Q = \Omega\times(0,T)$ and $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are given data. As usual, we look at equation (1c) as the definition of the electrical charge q and we observe that equation (1d) results from equation (1b) if we consider that, by assumption, $\nabla\cdot\boldsymbol{b} = 0$ in some instant t_0.

Considering the initial conditions $\boldsymbol{d}(0) = \boldsymbol{d}_0$ and $\boldsymbol{b}(0) = \boldsymbol{b}_0$ such that $\nabla\cdot\boldsymbol{b}_0 = 0$ in Ω, equation (1d) is verified, if we impose the condition $\nabla\cdot\boldsymbol{g}_2 = 0$ on Q.

Calling $\Sigma = \Gamma\times(0,T)$ and supposing perfectly conductive walls, the boundary conditions are

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{d}\times\boldsymbol{n} = 0 \qquad \text{on } \Sigma. \tag{5}$$

Assuming some regularity on $\hat{\varepsilon}$ and supposing that $\boldsymbol{b}_0\cdot\boldsymbol{n} = 0$ on Γ, we may substitute (5) by

$$\boldsymbol{g}_2\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{d}\times\boldsymbol{n} = 0 \qquad \text{on } \Sigma.$$

Gathering all the equations and conditions above, given conjugate exponents p and p', the problem consists in the determination of vectorial fields $\boldsymbol{d}$ and $\boldsymbol{b}$, solutions of the system

$$\begin{aligned} \partial_t\boldsymbol{d} + \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d} - \nabla\times(\hat{\mu}\,|\boldsymbol{b}|^{p'-2}\boldsymbol{b}) &= \boldsymbol{g}_1 \quad \text{in} \quad Q, && \text{(6a)}\\ \partial_t\boldsymbol{b} + \nabla\times(\hat{\varepsilon}\,|\boldsymbol{d}|^{p-2}\boldsymbol{d}) &= \boldsymbol{g}_2 \quad \text{in} \quad Q, && \text{(6b)}\\ \boldsymbol{d}\times\boldsymbol{n} &= 0 \quad \text{on} \quad \Sigma, && \text{(6c)}\\ \boldsymbol{d}(0) &= \boldsymbol{d}_0 \quad \text{in} \quad \Omega, && \text{(6d)}\\ \boldsymbol{b}(0) &= \boldsymbol{b}_0 \quad \text{in} \quad \Omega, && \text{(6e)} \end{aligned}$$

where $\hat{\varepsilon}$, $\hat{\mu}$ and σ are scalar functions, the initial conditions $\boldsymbol{d}_0$ and $\boldsymbol{b}_0$ are vectorial functions defined in Ω, $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are vectorial functions defined in Q, all given and satisfying the assumptions

$$\begin{aligned} \nabla\cdot\boldsymbol{g}_2 &= 0 \quad \text{in} \quad Q, \qquad \boldsymbol{g}_2\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Sigma, && \text{(6f)}\\ \nabla\cdot\boldsymbol{b}_0 &= 0 \quad \text{in} \quad \Omega, \qquad \boldsymbol{b}_0\cdot\boldsymbol{n} = 0 \quad \text{on} \quad \Gamma. && \text{(6g)} \end{aligned}$$

Here, if $p < 2$, $|\boldsymbol{d}|^{p-2}\boldsymbol{d}$ is assumed to be zero when $\boldsymbol{d}$ is zero.

The problem (6) in the weak sense leads us to the equations

$$\begin{aligned} -\int_Q \boldsymbol{d}\cdot\partial_t\boldsymbol{\varphi} + \int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}\cdot\boldsymbol{\varphi} - \int_Q \hat{\mu}|\boldsymbol{b}|^{p'-2}\boldsymbol{b}\cdot\nabla\times\boldsymbol{\varphi} &= \int_Q \boldsymbol{g}_1\cdot\boldsymbol{\varphi} + \int_\Omega \boldsymbol{d}_0\cdot\boldsymbol{\varphi}(0)\\ -\int_Q \boldsymbol{b}\cdot\partial_t\boldsymbol{\psi} + \int_Q \hat{\varepsilon}|\boldsymbol{d}|^{p-2}\boldsymbol{d}\cdot\nabla\times\boldsymbol{\psi} &= \int_Q \boldsymbol{g}_2\cdot\boldsymbol{\psi} + \int_\Omega \boldsymbol{b}_0\cdot\boldsymbol{\psi}(0) \end{aligned}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are test functions in convenient functional spaces.

A natural approach to solve this problem consists in approximating it by a family of finite-dimensional problems. The scarcity of *a priori* estimates implies that the use of measure-valued functions seems to be an adequate tool to interpret the limit of the composition of a nonlinear function with controlled growth with a sequence of functions weakly convergent. A first use of these functions can be found in [12, 15, 6] and a detailed study of measure-valued functions in [9]. As a consequence, we define a weaker version of the problem, which consists to find $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{h}$, solutions of the problem

$$-\int_Q \boldsymbol{d}\cdot\partial_t\boldsymbol{\varphi}+\int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}\cdot\boldsymbol{\varphi}-\int_Q \boldsymbol{h}\cdot\nabla\times\boldsymbol{\varphi}=\int_Q \boldsymbol{g}_1\cdot\boldsymbol{\varphi}+\int_\Omega \boldsymbol{d}_0\cdot\boldsymbol{\varphi}(0),$$
$$-\int_Q \boldsymbol{b}\cdot\partial_t\boldsymbol{\psi}+\int_Q \boldsymbol{e}\cdot\nabla\times\boldsymbol{\psi}=\int_Q \boldsymbol{g}_2\cdot\boldsymbol{\psi}+\int_\Omega \boldsymbol{b}_0\cdot\boldsymbol{\psi}(0),$$

for all $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ belonging to test functions spaces, where $\boldsymbol{e}$ and $\boldsymbol{h}$ are the weak limits, in the sense of measure-valued functions, of $\hat{\varepsilon}_m\,|\boldsymbol{d}_m|^{p-2}\boldsymbol{d}_m$ and $\hat{\mu}_m\,|\boldsymbol{b}_m|^{p'-2}\boldsymbol{b}_m$, respectively, with $\boldsymbol{d}_m$ and $\boldsymbol{b}_m$ solutions of approximated problems in dimension m, defined later, after an appropriate choice of bases of the test functions spaces.

In what follows, if E denotes a functional space, then $\boldsymbol{E}$ represents the space E^3.

In Section 2 we present the existence result for the weak formulation of problem (6), introducing the adequate functional framework. A density result and the characterizations of the traces of functions in the space $\boldsymbol{W}^p(\nabla\times,\Omega)$, the space of functions $\boldsymbol{v}$ in $\boldsymbol{L}^p(\Omega)$ such that $\nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$, are a key tool for the proof of the existence result presented in the last section. The proofs of the density and trace results are done in Section 3, following some ideas of [5].

To prove existence of solution for the weak problem we apply, in Section 5, the Galerkin method, defining suitable approximated problems in finite dimensional subspaces. However the fact that the unknowns $\boldsymbol{d}$ and $\boldsymbol{b}$ belong to dual spaces and their expected lack of regularity force us to choose *regular* topological bases for the spaces of test functions. As the problem is nonlinear we introduce a projection operator. Since we are working in the $\boldsymbol{L}^p$-$\boldsymbol{L}^{p'}$ duality, we cannot guarantee the uniform boundedness of these projections. In fact, the natural boundedness of the orthogonal projections in $\boldsymbol{L}^2$ of the approximated solutions, with respect to the inner product, is no longer true when we substitute the duality $\boldsymbol{L}^2$-$\boldsymbol{L}^2$ by $\boldsymbol{L}^p$-$\boldsymbol{L}^{p'}$ and the inner product by the duality operator (see [14]) and so the choice of the bases, done in Section 4, must be carefully done. The difficulties in the identification of the limits of the nonlinear terms are solved passing to the limit in the sense of measure-valued functions.

2 Weak formulation

Given $p\geq 1$, consider the space

$$\boldsymbol{W}^p(\nabla\times,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\big\}$$

endowed with the natural norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$.

The next propositions are useful to obtain the weak formulation of the problem (6) and their proofs will be presented in Section 3.

Proposition 1 *For $1\leq p<\infty$, $\boldsymbol{\mathscr{D}}(\bar{\Omega})$ is dense in $\boldsymbol{W}^p(\nabla\times,\Omega)$.*

It seems natural, after the former density result, to define $\boldsymbol{W}^p_0(\nabla\times,\Omega)$ as the closure of $\boldsymbol{\mathscr{D}}(\Omega)$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$. In the next proposition we characterize the trace of functions in $\boldsymbol{W}^p(\nabla\times,\Omega)$ and we also identify $\boldsymbol{W}^p_0(\nabla\times,\Omega)$ as the subset of functions in $\boldsymbol{W}^p(\nabla\times,\Omega)$ with null trace.

Proposition 2 *Given $1\leq p<\infty$ we have:*

1. *The trace function $\gamma_\tau:\Big(\boldsymbol{\mathscr{D}}(\bar{\Omega}),\|\cdot\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}\Big)\longrightarrow\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$, defined by $\gamma_\tau\boldsymbol{v}=\boldsymbol{v}_{|\Gamma}\times\boldsymbol{n}$, is continuous and linear and it can be extended, by density, to a continuous linear application defined in $\boldsymbol{W}^p(\nabla\times,\Omega)$, still denoted by γ_τ.*

2. *The kernel of this function γ_τ is the space $\boldsymbol{W}^p_0(\nabla\times,\Omega)$.*

Remark 1 *From now on, we use for the trace of any $\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega)$ the notation $\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}$. As a consequence of the previous proposition we obtain for $\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'}(\Omega)$ the following formula of integration by parts*

$$\int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi}-\int_\Omega\nabla\times\boldsymbol{v}\,\cdot\boldsymbol{\varphi}=\langle\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma},\boldsymbol{\varphi}\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality paring between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$.

Remark 2 *For $p\geq 1$ let*

$$\boldsymbol{W}^p(\nabla\cdot,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\nabla\cdot\boldsymbol{v}\in L^p(\Omega)\big\}$$

endowed with the natural norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\cdot,\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^p(\Omega)}$.

The space $\mathscr{D}(\bar\Omega)$ is also dense in $\boldsymbol{W}^p(\nabla\cdot,\Omega)$ and a function $\boldsymbol{v}$ in this space has a trace belonging to $W^{-\frac{1}{p},p}(\Gamma)$ that we denote by $\boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma}$.

In order to present the weak formulation of the problem (6) we introduce the spaces

$$\mathcal{V}^{p'}_0(Q)=\big\{\boldsymbol{v}\in L^{p'}\big(0,T;\boldsymbol{W}^{p'}_0(\nabla\times,\Omega)\big):\partial_t\boldsymbol{v}\in\boldsymbol{L}^{p'}(Q)\ \text{ and }\ \boldsymbol{v}(\cdot,T)=0\big\}$$

and

$$\mathcal{V}^{p}(Q)=\big\{\boldsymbol{v}\in L^{p}\big(0,T;\boldsymbol{W}^{p}(\nabla\times,\Omega)\big):\partial_t\boldsymbol{v}\in\boldsymbol{L}^{p}(Q)\ \text{ and }\ \boldsymbol{v}(\cdot,T)=0\big\}.$$

We observe that if a function $\boldsymbol{v}\in\boldsymbol{L}^q(Q)$ is such that $\partial_t\boldsymbol{v}\in\boldsymbol{L}^q(Q)$, then $\boldsymbol{v}\in\mathscr{C}\big([0,T];\boldsymbol{L}^q(\Omega)\big)$, which gives a sense to $\boldsymbol{v}(\cdot,T)$.

The above spaces of test functions were chosen to allow us to pass, integrating by parts, the operators ∂_t and $\nabla\times$ to the test functions. As the problem (6) does not impose any condition on the trace of $\boldsymbol{b}$, equation (6a) is tested against functions of $\mathcal{V}^{p'}_0(Q)$. The space $\mathcal{V}^p(Q)$ was adopted for equation (6b) since, by (6c), $\boldsymbol{d}$ has null trace.

We will define a family of finite-dimensional problems. We start by fixing topological bases, $\big(\boldsymbol{\varphi}_n\big)_n$ and $\big(\boldsymbol{\psi}_n\big)_n$ of $\boldsymbol{W}^{p'}_0(\nabla\times,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$, respectively. We represent by P_m the projection operators, both from $\boldsymbol{W}^{p'}_0(\nabla\times,\Omega)$ onto the finite-dimensional subspace $\langle\boldsymbol{\varphi}_1,\ldots,\boldsymbol{\varphi}_m\rangle$ and from $\boldsymbol{W}^p(\nabla\times,\Omega)$ onto the subspace $\langle\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_m\rangle$. In what follows $\hat\varepsilon_m$ and $\hat\mu_m$ denote regularizations of $\hat\varepsilon$ and $\hat\mu$, $\boldsymbol{g}_{1,m}$ and $\boldsymbol{g}_{2,m}$ regularizations of $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, $\boldsymbol{b}_{0,m}$ and $\boldsymbol{d}_{0,m}$ regularizations of $\boldsymbol{b}_0$ and $\boldsymbol{d}_0$.

The approximated problem is to find $\boldsymbol{b}_m$ and $\boldsymbol{d}_m$ satisfying

$$\int_\Omega\partial_t\boldsymbol{d}_m(t)\cdot\boldsymbol{\varphi}_j+\int_\Omega\sigma\hat\varepsilon_m^{p'-1}\boldsymbol{d}_m(t)\cdot\boldsymbol{\varphi}_j-\int_\Omega P_m\big(\hat\mu_m\,|\boldsymbol{b}_m(t)|^{p'-2}\boldsymbol{b}_m(t)\big)\cdot\nabla\times\boldsymbol{\varphi}_j=\int_\Omega\boldsymbol{g}_{1,m}(t)\cdot\boldsymbol{\varphi}_j,\tag{7a}$$

$$\int_\Omega\partial_t\boldsymbol{b}_m(t)\cdot\boldsymbol{\psi}_j+\int_\Omega P_m\big(\hat\varepsilon_m|\boldsymbol{d}_m(t)|^{p-2}\boldsymbol{d}_m(t)\big)\cdot\nabla\times\boldsymbol{\psi}_j=\int_\Omega\boldsymbol{g}_{2,m}(t)\cdot\boldsymbol{\psi}_j,\tag{7b}$$

$$\boldsymbol{b}_m(0)=\boldsymbol{b}_0,\quad \boldsymbol{d}_m(0)=\boldsymbol{d}_0,\tag{7c}$$

with $j=1,\ldots,m$ and $t\in(0,T)$.

We are now able to present the existence result.

Theorem 1 *Under the assumptions*

σ *is a positive function in* $L^\infty(\Omega)$,
$\hat\mu\in L^\infty(\Omega)$ *and is bounded from below by* $\hat\mu_*>0$,
$\hat\varepsilon\in L^\infty(\Omega)\cap\boldsymbol{W}^{1,p}(\Omega)$ *and is bounded from below by* $\hat\varepsilon_*>0$,
$\boldsymbol{g}_1\in L^\infty\big(0,T;\boldsymbol{L}^p(\Omega)\big)$,
$\boldsymbol{g}_2\in L^\infty\big(0,T;\boldsymbol{L}^{p'}(\Omega)\big),\quad\nabla\cdot\boldsymbol{g}_2=0,\quad\boldsymbol{g}_2\cdot\boldsymbol{n}_{|\Sigma}=0$,
$\boldsymbol{d}_0\in\boldsymbol{L}^p(\Omega),\qquad\boldsymbol{b}_0\in\boldsymbol{L}^{p'}(\Omega),\quad\nabla\cdot\boldsymbol{b}_0=0,\quad\boldsymbol{b}_0\cdot\boldsymbol{n}_{|\Gamma}=0$,

the problem

$$-\int_Q \boldsymbol{d}\cdot\partial_t\boldsymbol{\varphi}+\int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}\cdot\boldsymbol{\varphi}-\int_Q \boldsymbol{h}\cdot\nabla\times\boldsymbol{\varphi}=\int_Q \boldsymbol{g}_1\cdot\boldsymbol{\varphi}+\int_\Omega \boldsymbol{d}_0\cdot\boldsymbol{\varphi}(0)\quad\forall\boldsymbol{\varphi}\in\mathcal{V}_0^{p'}(Q),\tag{8a}$$

$$-\int_Q \boldsymbol{b}\cdot\partial_t\boldsymbol{\psi}+\int_Q \boldsymbol{e}\cdot\nabla\times\boldsymbol{\psi}=\int_Q \boldsymbol{g}_2\cdot\boldsymbol{\psi}+\int_\Omega \boldsymbol{b}_0\cdot\boldsymbol{\psi}(0)\quad\forall\boldsymbol{\psi}\in\mathcal{V}^{p}(Q),\tag{8b}$$

has a solution $(\boldsymbol{b},\boldsymbol{h})\in\boldsymbol{L}^{p'}(Q)^2$ and $(\boldsymbol{d},\boldsymbol{e})\in\boldsymbol{L}^{p}(Q)^2$ such that $\boldsymbol{b}$ and $\boldsymbol{d}$ are, respectively, the weak limit of subsequences of $(\boldsymbol{b}_m)_m$ and $(\boldsymbol{d}_m)_m$, solutions of the problem (7), and $\boldsymbol{h}$ and $\boldsymbol{e}$ are identified, respectively, as the weak limit, in the sense of measure-valued functions, of subsequences of $(|\boldsymbol{b}_m|^{p'-2}\boldsymbol{b}_m)_m$ and of $(|\boldsymbol{d}_m|^{p-2}\boldsymbol{d}_m)_m$.

Remark 3 *The meaning of the weak limit, in the sense of measure-valued functions, referred in the above theorem, will be precised later, in Section 6.*

3 Density and trace results

Now we present the proofs of the propositions stated in the previous section. For more detailed versions of the proofs presented in this section, see [10] or [13].

Proof of Proposition 1 We start by first assuming that Ω is a star-like domain with respect to a fixed point x_0. For simplicity we set $x_0=0$.

Given a function $\boldsymbol{u}\in\boldsymbol{W}^p(\nabla\times,\Omega)$ we consider, for $m\in\mathbb{N}$, the domain $\Omega_m=\left(1+\frac{1}{m}\right)\Omega$ and the function $\boldsymbol{u}_m\in\boldsymbol{W}^p(\nabla\times,\Omega_m)$ defined by $\boldsymbol{u}_m(x)=\boldsymbol{u}\left(\frac{m}{m+1}x\right)$, $\forall x\in\Omega_m$.

Representing by $\tilde{\boldsymbol{u}}_m$ the extension of $\boldsymbol{u}_m$, by zero, to $\mathbb{R}^3$, let $\tilde{\boldsymbol{u}}_{mn}=\rho_n*\tilde{\boldsymbol{u}}_m$, being ρ_n a mollifier, $n\in\mathbb{N}$ and $*$ the convolution product. Obviously, $\tilde{\boldsymbol{u}}_{mn}\in\boldsymbol{\mathscr{D}}(\mathbb{R}^3)$, $\forall m,n\in\mathbb{N}$, so $\boldsymbol{u}_{mn}:=\tilde{\boldsymbol{u}}_{mn|_\Omega}\in\boldsymbol{\mathscr{D}}\left(\bar{\Omega}\right)$.

We are going to prove that a subsequence of $\boldsymbol{u}_{mn}$ converges to $\boldsymbol{u}$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$.

Denoting by Γ_m the boundary of Ω_m, since Γ and Γ_m are disjoint compact sets then, for all $x\in\Omega$, $B_{d_m}(x)\subseteq\Omega_m$, where $d_m>0$ is the distance between Γ and Γ_m. Hence, for $n>\frac{1}{d_m}$, we have $\boldsymbol{u}_{mn}=(\rho_n*\boldsymbol{u}_m)_{|\Omega}$ and therefore $\nabla\times\boldsymbol{u}_{mn}=(\rho_n*\nabla\times\boldsymbol{u}_m)_{|\Omega}$. The convergence of $\boldsymbol{u}_{mn}$ to $\boldsymbol{u}_m$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$ follows now at once.

The verification that $\boldsymbol{u}_m$ converges to $\boldsymbol{u}$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$ is straightforward.

For general domains Ω, we consider a covering of $\bar{\Omega}$ by a finite family of open sets $\mathscr{O}_i$, $i=0,\ldots,n$, such that $\mathscr{O}_i\cap\Omega$, for $i=1,\ldots,n$, are $\mathscr{C}^{0,1}$ star-like domains with bounded boundary, $\mathscr{O}_0\subseteq\bar{\mathscr{O}}_0\subseteq\Omega$. Let α_i, $i=0,\ldots,n$, be a partition of the unity subordinated to the above covering, more precisely,

$$\alpha_i\in\mathscr{C}^\infty(\mathbb{R}^3),\quad \operatorname{supp}\ \alpha_i\subseteq\mathscr{O}_i,\quad 0\le\alpha_i\le 1,\quad \sum_{i=0}^n\alpha_i(x)=1,\quad\forall x\in\Omega.$$

Denoting, for $i=0,\ldots,n$, $\Omega_i=\mathscr{O}_i\cap\Omega$ and $\boldsymbol{u}^i=\boldsymbol{u}_{|\Omega_i}$, we have $\boldsymbol{u}^i\in\boldsymbol{W}^p(\nabla\times,\Omega_i)$. For $i=1,\ldots,n$, since Ω_i is a star-like domain, there exists a sequence of functions of $\boldsymbol{\mathscr{D}}(\bar{\Omega}_i)$, $\left(\boldsymbol{u}^i_m\right)_m$, such that $\boldsymbol{u}^i_m\underset{m}{\longrightarrow}\boldsymbol{u}^i$ in $\boldsymbol{W}^p(\nabla\times,\Omega_i)$.

We note that the boundaries Γ_0 of Ω_0 and Γ are compact and disjoint, therefore the distance d_0 between them is positive.

When Ω_0 is bounded, it is clear that for $m>\frac{1}{d_0}$, $\boldsymbol{u}^0_m:=(\rho_m*\boldsymbol{u})_{|\Omega_0}\in\boldsymbol{\mathscr{D}}(\bar{\Omega}_0)$ and $\boldsymbol{u}^0_m\underset{m}{\longrightarrow}\boldsymbol{u}_0$ in $\boldsymbol{W}^p(\nabla\times,\Omega_0)$.

If Ω_0 is not bounded, we use the above regularizing process jointly with a truncation process, in order to guarantee the compactness of the support of the functions $\boldsymbol{u}^i_m$ in the sequence.

Extending $\boldsymbol{u}^i_m$ to $\bar{\Omega}$ in such a way that this extension (still denoted by $\boldsymbol{u}^i_m$) belongs to $\boldsymbol{\mathscr{D}}(\bar{\Omega})$, it is easy to verify that $\boldsymbol{u}_m:=\displaystyle\sum_{i=0}^n\alpha_i\boldsymbol{u}^i_m\in\boldsymbol{\mathscr{D}}(\bar{\Omega})$ converges to $\boldsymbol{u}$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$. □

Proof of Proposition 2

1. The Green's formula

$$\int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi}\ -\int_\Omega\nabla\times\boldsymbol{v}\ \cdot\boldsymbol{\varphi}=\int_\Gamma\boldsymbol{v}\times\boldsymbol{n}\ \cdot\boldsymbol{\varphi},\tag{9}$$

verified for all $\boldsymbol{v}$, $\boldsymbol{\varphi} \in \mathscr{D}\left(\bar{\Omega}\right)$, may be extended, by density, for all $\boldsymbol{v} \in \mathscr{D}(\bar{\Omega})$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$. Then, given $\boldsymbol{v} \in \mathscr{D}\left(\bar{\Omega}\right)$ and $\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega)$,

$$\left|\int_\Gamma \boldsymbol{v} \times \boldsymbol{n} \ \cdot \boldsymbol{\varphi}\right| \leq \big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \big)\big(\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\nabla\times\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} \big),$$

so

$$\left|\int_\Gamma \boldsymbol{v} \times \boldsymbol{n} \ \cdot \boldsymbol{\varphi}\right| \leq \|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \quad \forall\, \boldsymbol{v} \in \mathscr{D}\left(\bar{\Omega}\right) \ \forall\, \boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'}(\Omega).$$

It is well known that, given $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p'}(\Omega)$, its trace, $\boldsymbol{\mu} := \boldsymbol{\psi}_{|_\Gamma}$, belongs to $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$. Besides, it is well known that the trace operator $\boldsymbol{W}^{1,p'}(\Omega) \longrightarrow \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$ has a linear continuous right inverse. Given $\boldsymbol{\mu} \in \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$, denoting by $\boldsymbol{\varphi}_\mu$ a function in $\boldsymbol{W}^{1,p'}(\Omega)$ such that

$$\boldsymbol{\mu} = \boldsymbol{\varphi}_{\mu_{|_\Gamma}} \quad \text{and} \quad \left\|\boldsymbol{\varphi}_\mu\right\|_{\boldsymbol{W}^{1,p'}(\Omega)} \leq k\, \|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)},$$

with $k > 0$ independent of $\boldsymbol{\mu}$, it is straightforward to conclude that

$$\|\gamma_\tau \boldsymbol{v}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} = \sup_{\|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} \leq 1} \left|\int_\Gamma \gamma_\tau \boldsymbol{v} \ \cdot \boldsymbol{\mu}\right| \leq k\, \|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}\,.$$

Proved the continuity of γ_τ, arguments of density guarantee its continuous extension to $\boldsymbol{W}^p(\nabla\times,\Omega)$.

2. Observe that the Green's formula represented in (9) may be generalized, by density, to any function $\boldsymbol{v} \in \boldsymbol{W}^p(\nabla\times,\Omega)$, obtaining, for all $\boldsymbol{\varphi}$ in $\boldsymbol{W}^{1,p'}(\Omega)$

$$\int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi} - \int_\Omega \nabla\times\boldsymbol{v} \ \cdot \boldsymbol{\varphi} = \langle \gamma_\tau \boldsymbol{v}, \boldsymbol{\varphi}\rangle,$$

denoting $\langle \ \cdot\ ,\ \cdot\ \rangle$ the duality paring between $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)$.

Given $\boldsymbol{v} \in \boldsymbol{W}^p(\nabla\times,\Omega)$ such that $\boldsymbol{v} \in \ker\gamma_\tau$, we have

$$\int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi} - \int_\Omega \nabla\times\boldsymbol{v} \ \cdot \boldsymbol{\varphi} = 0 \quad \forall \boldsymbol{\varphi} \in \mathscr{D}(\bar{\Omega}). \tag{10}$$

To prove that a function $\boldsymbol{v}$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$ verifying the equality (10) belongs to $\boldsymbol{W}^p_0(\nabla\times,\Omega)$ we need to verify that $\boldsymbol{v}$ is the limit, in $\boldsymbol{W}^p(\nabla\times,\Omega)$, of a sequence of functions in $\mathscr{D}(\Omega)$.

The technique used here is similar to the one used in the proof of Proposition 1, so we will describe it briefly.

We begin observing that the extension by zero to $\mathbb{R}^3$ of $\boldsymbol{v}$ satisfying (10) is a function of $\boldsymbol{W}^p(\nabla\times,\mathbb{R}^3)$, whose support is contained in $\bar{\Omega}$. In fact, calling $\boldsymbol{w} = \nabla\times\boldsymbol{v}$, representing by $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{w}}$ the extensions to $\mathbb{R}^3$, by zero outside Ω, of $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively, from (10) we have

$$\int_{\mathbb{R}^3} \tilde{\boldsymbol{v}}\cdot\nabla\times\boldsymbol{\varphi} - \int_{\mathbb{R}^3} \tilde{\boldsymbol{w}} \ \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi} - \int_\Omega \nabla\times\boldsymbol{v}\cdot\boldsymbol{\varphi} = 0 \quad \forall \boldsymbol{\varphi} \in \mathscr{D}(\mathbb{R}^3),$$

which means that $\tilde{\boldsymbol{w}} = \nabla\times\tilde{\boldsymbol{v}}$ in $\mathscr{D}'(\mathbb{R}^3)$, so $\tilde{\boldsymbol{v}} \in \boldsymbol{W}^p(\nabla\times,\mathbb{R}^3)$.

Considering first the case where Ω is a star-like domain with respect to $x_0 \in \Omega$ (we assume $x_0 = 0$ for simplicity), for each $m \in \mathbb{N}$, the function $\tilde{\boldsymbol{v}}_m(x) := \tilde{\boldsymbol{v}}\left(\frac{m+1}{m}x\right)$ belongs also to $\boldsymbol{W}^p(\nabla\times,\mathbb{R}^3)$ and its support is contained in Ω.

Defining, for $n \in \mathbb{N}$, $\tilde{\boldsymbol{v}}_{mn} := \rho_n * \tilde{\boldsymbol{v}}_m$, obviously $\tilde{\boldsymbol{v}}_{mn} \in \mathscr{D}(\mathbb{R}^3)$ and, for n big enough, $\operatorname{supp}\tilde{\boldsymbol{v}}_{mn} \subset \Omega$.

We have $\nabla\times\tilde{\boldsymbol{v}}_{mn} = \rho_n * \nabla\times\tilde{\boldsymbol{v}}_m$, and so $\tilde{\boldsymbol{v}}_{mn} \underset{n}{\longrightarrow} \tilde{\boldsymbol{v}}_m$ in $\boldsymbol{W}^p(\nabla\times,\mathbb{R}^3)$.

On the other hand,

$$\|\nabla\times\tilde{\boldsymbol{v}}_m - \nabla\times\tilde{\boldsymbol{v}}\|^p_{\boldsymbol{L}^p(\Omega)} \leq 2^{p-1}\left(\tfrac{1}{m^p}\left(\tfrac{m}{m+1}\right)^3 \|\nabla\times\tilde{\boldsymbol{v}}\|^p_{\boldsymbol{L}^p(\Omega)} + \left\|(\nabla\times\tilde{\boldsymbol{v}})(\tfrac{m+1}{m}\,x) - \nabla\times\tilde{\boldsymbol{v}}\right\|^p_{\boldsymbol{L}^p(\Omega)}\right)$$

and we conclude that
$$\|\tilde{\boldsymbol{v}}_m - \boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)} \underset{m}{\longrightarrow} 0.$$

We can find then a sequence $(n_m)_m$ such that $\tilde{\boldsymbol{v}}_{mn_m} \underset{m}{\longrightarrow} \boldsymbol{v}$ in $\boldsymbol{W}^p(\nabla\times,\Omega)$ and $\tilde{\boldsymbol{v}}_{mn_m} \in \mathscr{D}(\Omega)$, $\forall m \in \mathbb{N}$.

In the general case, the sequence of functions in $\mathscr{D}(\Omega)$ that approximates $\boldsymbol{v}$ is obtained considering a covering of $\bar{\Omega}$ by a finite family of open sets whose closures are contained in Ω or whose intersection with Ω is a star-like domain with bounded and lipschitzian boundary. Working with a partition of the unity subordinated to this covering, we argue as in Proposition 1 to get the desired conclusion here.

The other inclusion is an immediate consequence of the definitions of $\boldsymbol{W}_0^p(\nabla\times,\Omega)$ and γ_τ. □

4 Special bases

The necessary *a priori* estimates to pass to the limit in the approximated problem (7) will be achieved by constructing special topological bases of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$.

Proposition 3 *There exist $\boldsymbol{\varphi}_n \in \mathscr{D}(\Omega)$ and $\boldsymbol{\psi}_n \in \mathscr{D}(\bar{\Omega})$, $n \in \mathbb{N}$, such that $(\boldsymbol{\varphi}_n)_n$ and $(\boldsymbol{\psi}_n)_n$ are topological bases of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$, respectively.*

Proof Since $\mathscr{D}(\Omega)$ is dense in the separable space $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ and $\mathscr{D}(\bar{\Omega})$ is dense in the separable space $\boldsymbol{W}^p(\nabla\times,\Omega)$, the result follows. □

It is well known that a finite-dimensional subspace of a Banach space E has a topological supplement. Besides, if G and L are topological supplements of E, given $z \in E$, there exist unique $x \in G$ and $y \in L$ such that $z = x + y$. The projection operators $z \longmapsto x$ and $z \longmapsto y$ are linear and continuous. For details about the above results and the following definition see [3].

Definition 1 *Let E be an infinite-dimensional separable Banach space and $(e_n)_n$ a topological basis of E. If L_m is a topological supplement of $G_m = \langle e_1, \dots, e_m\rangle$, we represent by P_m the projection operator from E onto G_m, in the above sense.*

Remark 4 *The projection operators P_m are linear and continuous but, although in the hilbertian case they are uniformly bounded (i.e., bounded independently of m), this is no longer true if E is only a separable Banach space. In particular, given a bounded sequence $(\boldsymbol{z}_n)_n$ in $\boldsymbol{L}^p(\Omega)$, we cannot assure that $\{P_m\boldsymbol{z}_m : m \in \mathbb{N}\}$ is bounded in $\boldsymbol{L}^p(\Omega)$ independently of m, if $p \neq 2$ (see [14]).*

4.1 Closed bases for the $\nabla\times$ operator

The difficulty raised in the last remark forces us to refine our choice of bases.

Proposition 4 *There exist topological bases $(\boldsymbol{\varphi}_n)_n$ and $(\boldsymbol{\psi}_n)_n$ of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$, respectively, where $\boldsymbol{\varphi}_n \in \mathscr{D}(\Omega)$ and $\boldsymbol{\psi}_n \in \mathscr{D}(\bar{\Omega})$ and a nondecreasing sequence $(k_m)_m$ such that, for all $m \in \mathbb{N}$,*

$$\{\nabla\times\boldsymbol{\varphi}_1, \dots, \nabla\times\boldsymbol{\varphi}_m\} \subseteq \langle\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_{k_m}\rangle, \tag{11a}$$

and

$$\{\nabla\times\boldsymbol{\psi}_1, \dots, \nabla\times\boldsymbol{\psi}_m\} \subseteq \langle\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_{k_m}\rangle. \tag{11b}$$

Proof We consider a basis $(\hat{\boldsymbol{\varphi}}_n)_n$ of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ consisting of functions in $\mathscr{D}(\Omega)$.

Given $\boldsymbol{u} \in \mathscr{D}(\Omega)$, we represent by $(\nabla\times)^n\boldsymbol{u}$, $n \in \mathbb{N}_0$, the function obtained by applying n times the operator $\nabla\times$ to the function $\boldsymbol{u}$.

As the set $\big\{(\nabla\times)^n\hat{\boldsymbol{\varphi}}_m : (n,m) \in \mathbb{N}_0 \times \mathbb{N}\big\}$ is countable we consider an ordering of its elements. The set of all finite linear combinations of elements of this sequence is dense in $\mathscr{D}(\Omega)$. We construct a topological basis, inductively, dropping the element in the position k if it is a linear combination of the previous ones. We denote the topological basis obtained by $(\boldsymbol{\varphi}_n)_n$.

Using similar arguments we construct the topological basis $(\boldsymbol{\psi}_n)_n$.

For each $m \in \mathbb{N}$, the process of construction of the topological bases guarantees the existence of $k_m \in \mathbb{N}$ such that the inclusions (11) are satisfied. $\square$

4.2 Bi-orthogonal sequences in the duality $\boldsymbol{L}^p(\Omega)$-$\boldsymbol{L}^{p'}(\Omega)$

Given topological bases $(\boldsymbol{\varphi}_n)_n$ of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ such that $\boldsymbol{\varphi}_n \in \boldsymbol{\mathscr{D}}(\Omega)$ and $(\boldsymbol{\psi}_n)_n$ of $\boldsymbol{W}^p(\nabla\times,\Omega)$ such that $\boldsymbol{\psi}_n \in \boldsymbol{\mathscr{D}}(\bar{\Omega})$, we are going to construct, for each sufficiently small $\theta > 0$, a new sequence of linearly independent functions $(\boldsymbol{\psi}_n^\theta)_n$ verifying, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\boldsymbol{\psi}_n^\theta \in \boldsymbol{\mathscr{D}}(\bar{\Omega}) \quad \text{and} \quad & \left\|\boldsymbol{\psi}_n - \boldsymbol{\psi}_n^\theta\right\|_{\boldsymbol{W}^p(\nabla\times,\Omega)} \le \theta,\\
& \det\left[\int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\psi}_j^\theta\right]_{i,j=1,\dots,n} \neq 0.
\end{aligned} \tag{12}
$$

We proceed, in a second step, to the bi-orthogonalization of the sequence $\left((\boldsymbol{\varphi}_n, \boldsymbol{\psi}_m^\theta)\right)_{n,m}$, in the duality $\boldsymbol{L}^p(\Omega)$-$\boldsymbol{L}^{p'}(\Omega)$, obtaining a new sequence $\left((\tilde{\boldsymbol{\varphi}}_n, \tilde{\boldsymbol{\psi}}_m^\theta)\right)_{n,m}$, still in $\boldsymbol{\mathscr{D}}(\Omega) \times \boldsymbol{\mathscr{D}}(\bar{\Omega})$, satisfying, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\tilde{\boldsymbol{\varphi}}_n \text{ is a linear combination of } \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n,\\
&\tilde{\boldsymbol{\psi}}_n^\theta \text{ is a linear combination of } \boldsymbol{\psi}_1^\theta, ..., \boldsymbol{\psi}_n^\theta,\\
&\int_\Omega \tilde{\boldsymbol{\varphi}}_i \cdot \tilde{\boldsymbol{\psi}}_j^\theta = \delta_{ij}, \quad i,j = 1, \dots, n,
\end{aligned} \tag{13}
$$

where δ_{ij} is the Kronecker symbol.

Let us implement the first step which gives us the sequence $(\boldsymbol{\psi}_n^\theta)_n$.

For $n = 1$, if $\displaystyle\int_\Omega \boldsymbol{\varphi}_1 \cdot \boldsymbol{\psi}_1 \neq 0$, we take $\boldsymbol{\psi}_1^\theta = \boldsymbol{\psi}_1$. In the other case we choose $x_0 \in \Omega$ such that $\boldsymbol{\varphi}_1(x_0) \neq 0$. Let φ_{1_i} be a component of $\boldsymbol{\varphi}_1$ nonzero in x_0. Consider $\theta > 0$ such that $B_\theta(x_0) \subseteq \Omega$ and $\varphi_{1_i|_{B_\theta(x_0)}}$ does not change the signal. Fix $\beta_{1_i} \in \mathscr{C}_0^\infty(B_\theta(x_0))$ such that $\beta_{1_i}(x) > 0$, for all $x \in B_\theta(x_0)$. Extend β_{1_i} by zero outside of $B_\theta(x_0)$ and define $\boldsymbol{\psi}_1^\theta = \boldsymbol{\psi}_1 + \theta\boldsymbol{\beta}_1/\left\|\boldsymbol{\beta}_1\right\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}$, where $\boldsymbol{\beta}_1 = (\delta_{ij}\beta_{1_i})_{j=1,2,3}$. Then $a_{1,1}^\theta := \displaystyle\int_\Omega \boldsymbol{\varphi}_1 \cdot \boldsymbol{\psi}_1^\theta \neq 0$ and $\left\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_1^\theta\right\|_{\boldsymbol{W}^p(\nabla\times,\Omega)} = \theta$.

Suppose we have constructed n functions $\boldsymbol{\psi}_1^\theta, ..., \boldsymbol{\psi}_n^\theta$ belonging to $\boldsymbol{\mathscr{D}}(\bar{\Omega})$, such that, for $i,j = 1, ..., n$,

$$
\left\|\boldsymbol{\psi}_i - \boldsymbol{\psi}_i^\theta\right\|_{\boldsymbol{W}^p(\nabla\times,\Omega)} \le \theta \quad \text{and} \quad \det\left[\int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\psi}_j^\theta\right]_{i,j} := a_{n,n}^\theta \neq 0.
$$

Let us construct $\boldsymbol{\psi}_{n+1}^\theta$. If

$$
\det\begin{bmatrix}
\int_\Omega \boldsymbol{\varphi}_1 \cdot \boldsymbol{\psi}_1^\theta & \dots & \int_\Omega \boldsymbol{\varphi}_1 \cdot \boldsymbol{\psi}_n^\theta & \int_\Omega \boldsymbol{\varphi}_1 \cdot \boldsymbol{\psi}_{n+1}\\
\vdots & \ddots & \vdots & \vdots\\
\int_\Omega \boldsymbol{\varphi}_n \cdot \boldsymbol{\psi}_1^\theta & \dots & \int_\Omega \boldsymbol{\varphi}_n \cdot \boldsymbol{\psi}_n^\theta & \int_\Omega \boldsymbol{\varphi}_n \cdot \boldsymbol{\psi}_{n+1}\\
\int_\Omega \boldsymbol{\varphi}_{n+1} \cdot \boldsymbol{\psi}_1^\theta & \dots & \int_\Omega \boldsymbol{\varphi}_{n+1} \cdot \boldsymbol{\psi}_n^\theta & \int_\Omega \boldsymbol{\varphi}_{n+1} \cdot \boldsymbol{\psi}_{n+1}
\end{bmatrix} \neq 0, \tag{14}
$$

we choose $\boldsymbol{\psi}_{n+1}^\theta = \boldsymbol{\psi}_{n+1}$.

If (14) is not verified, we choose a function $\boldsymbol{\beta}_{n+1} \in \langle \boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_n \rangle^\perp$ in $\boldsymbol{L}^p(\Omega)$ verifying $\displaystyle\int_\Omega \boldsymbol{\varphi}_{n+1} \cdot \boldsymbol{\beta}_{n+1} \neq 0.$ Defining $\boldsymbol{\psi}^\theta_{n+1} = \boldsymbol{\psi}_{n+1} + \theta \boldsymbol{\beta}_{n+1}$ we have

$$\det \left[\int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\psi}^\theta_j \right]_{1,j=1,\ldots,n+1} = \theta\, a^\theta_{n,n} \int_\Omega \boldsymbol{\varphi}_{n+1} \cdot \boldsymbol{\beta}_{n+1} := a^\theta_{n+1,n+1} \neq 0.$$

Considering now a regularizing sequence $(\rho_k)_k$, let

$$\boldsymbol{\psi}^{\theta,k}_{n+1} = \boldsymbol{\psi}_{n+1} + \theta\ \rho_k * (\chi_{B_k(0)} \boldsymbol{\beta}_{n+1}),$$

and calling $a^{\theta,k}_{n+1,n+1}$ to the determinant of the corresponding matrix, we have

$$\boldsymbol{\psi}^{\theta,k}_{n+1} \underset{k}{\longrightarrow} \boldsymbol{\psi}^\theta_{n+1} \ \text{ in } \boldsymbol{L}^p(\Omega) \quad \Longrightarrow \quad a^{\theta,k}_{n+1,n+1} \underset{k}{\longrightarrow} a^\theta_{n+1,n+1} \neq 0,$$

so there exists $k_0 \in \mathbb{N}$ such that $a^{\theta,k_0}_{n+1,n+1} \neq 0$.

Taking

$$\boldsymbol{\psi}^\theta_{n+1} = \boldsymbol{\psi}_{n+1} + \theta \, \frac{\rho_{k_0} * (\chi_{B_{k_0}(0)} \boldsymbol{\beta}_{n+1})}{\left\| \rho_{k_0} * (\chi_{B_{k_0}(0)} \boldsymbol{\beta}_{n+1}) \right\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}}$$

we have proved the existence of a sequence $(\boldsymbol{\psi}^\theta_n)_n$ verifying (12).

We remark that, although $(\boldsymbol{\psi}^\theta_n)_n$ is a linearly independent system, it may not be a basis of $\boldsymbol{W}^p(\nabla\times,\Omega)$.

We continue by applying inductively the Gauss elimination method to the sequence $\big((\boldsymbol{\varphi}_n, \boldsymbol{\psi}^\theta_m)\big)_{n,m}$, to construct a sequence $\big((\tilde{\boldsymbol{\varphi}}_n, \tilde{\boldsymbol{\psi}}^\theta_m)\big)_{n,m}$ satisfying (13), by transforming the block of dimension $n \in \mathbb{N}$, $\left[\displaystyle\int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\psi}^\theta_j \right]_{i,j=1,\ldots,n}$, of the matrix $\left[\displaystyle\int_\Omega \boldsymbol{\varphi}_i \cdot \boldsymbol{\psi}^\theta_j \right]_{i,j\in\mathbb{N}}$, in $\left[\ \delta_{ij}\ \right]_{i,j=1,\ldots,n}$.

Remark 5 *Let $\boldsymbol{F}_m = \langle \tilde{\boldsymbol{\varphi}}_1, \ldots, \tilde{\boldsymbol{\varphi}}_m \rangle$ and $\boldsymbol{G}_m = \langle \tilde{\boldsymbol{\psi}}^\theta_1, \ldots, \tilde{\boldsymbol{\psi}}^\theta_m \rangle$. These are closed vectorial subspaces of $\boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{L}^p(\Omega)$, respectively. Consider, for $i = 1, \ldots, m$, the continuous linear forms $\Phi_i(\boldsymbol{u}) = \displaystyle\int_\Omega \boldsymbol{u} \cdot \tilde{\boldsymbol{\varphi}}_i$, for $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and $\Psi_i(\boldsymbol{u}) = \displaystyle\int_\Omega \boldsymbol{u} \cdot \tilde{\boldsymbol{\psi}}^\theta_i$, for $\boldsymbol{u} \in \boldsymbol{L}^{p'}(\Omega)$.*

Defining

$$\boldsymbol{H}_m = \bigcap_{i=1}^m \ker \Psi_i \qquad \textit{and} \qquad \boldsymbol{L}_m = \bigcap_{i=1}^m \ker \Phi_i,$$

we observe that $\boldsymbol{F}_m$ and $\boldsymbol{H}_m$ are topological supplements in $\boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{G}_m$ and $\boldsymbol{L}_m$ are topological supplements in $\boldsymbol{L}^p(\Omega)$.

In fact

$$\boldsymbol{L}^{p'}(\Omega) = \boldsymbol{F}_m \oplus \boldsymbol{H}_m = \boldsymbol{F}_m \oplus \boldsymbol{G}_m^\perp,$$
$$\boldsymbol{L}^p(\Omega) = \boldsymbol{G}_m \oplus \boldsymbol{L}_m = \boldsymbol{G}_m \oplus \boldsymbol{F}_m^\perp.$$

If $\boldsymbol{u} \in \boldsymbol{F}_m$ (respectively $\boldsymbol{u} \in \boldsymbol{G}_m$) and $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ (respectively $\boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$), then

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{u} \cdot P_m \boldsymbol{v} + \int_\Omega \boldsymbol{u} \cdot (\boldsymbol{v} - P_m \boldsymbol{v}) = \int_\Omega \boldsymbol{u} \cdot P_m \boldsymbol{v}, \tag{15}$$

since $\boldsymbol{v} - P_m \boldsymbol{v} \in \boldsymbol{L}_m = \boldsymbol{F}_m^\perp$ (respectively $\boldsymbol{v} - P_m \boldsymbol{v} \in \boldsymbol{H}_m = \boldsymbol{G}_m^\perp$).

From the construction of the sequences $(\tilde{\boldsymbol{\varphi}}_n)_n$ and $(\tilde{\boldsymbol{\psi}}^\theta_n)_n$, we have, for all $m \in \mathbb{N}$, $\boldsymbol{F}_m = \langle \boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m \rangle$ and $\boldsymbol{G}_m = \langle \boldsymbol{\psi}^\theta_1, \ldots, \boldsymbol{\psi}^\theta_m \rangle$.

5 Approximate problem in finite dimension

We denote $J_p(\boldsymbol{u}) = |\boldsymbol{u}|^{p-2}\boldsymbol{u}$ and we introduce a regularization of this operator defining

$$\begin{array}{rccc} J_p^\eta : & \boldsymbol{L}^p(\Omega) & \longrightarrow & \boldsymbol{L}^{p'}(\Omega) \\ & \boldsymbol{u} & \longmapsto & \left(|\boldsymbol{u}|^2+\eta\right)^{\frac{p-2}{2}}\boldsymbol{u} \end{array}$$

where η is a positive constant.

We choose $\big((\boldsymbol{\varphi}_n,\boldsymbol{\psi}_m)\big)_{n,m}$ a topological basis of $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)\times\boldsymbol{W}^p(\nabla\times,\Omega)$ satisfying the conditions of Proposition 4 and we consider a sequence $\big((\boldsymbol{\varphi}_n,\boldsymbol{\psi}_m^\theta)\big)_{n,m}$ satisfying the conditions (13). As before, we represent by P_m the projection operators, both from $\boldsymbol{W}_0^{p'}(\nabla\times,\Omega)$ onto the finite-dimensional subspace $\langle\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_m\rangle$ and from $\boldsymbol{W}^p(\nabla\times,\Omega)$ onto the subspace $\langle\boldsymbol{\psi}_1^\theta,\dots,\boldsymbol{\psi}_m^\theta\rangle$.

Represent by $\hat\varepsilon_m$ and $\hat\mu_m$ functions of $\mathscr{D}(\bar\Omega)$ obtained by regularizing $\hat\varepsilon$ and $\hat\mu$, having the same upper and lower bounds, respectively.

For this particular choice of bases, we redefine the approximated problem (7). We want to find

$$\boldsymbol{d}_m(t) = \boldsymbol{d}_m^{\eta,\theta}(t) = \sum_{j=1}^m \xi_{j,m}(t)\boldsymbol{\psi}_j^\theta, \tag{16a}$$

$$\boldsymbol{b}_m(t) = \boldsymbol{b}_m^{\eta,\theta}(t) = \sum_{j=1}^m \zeta_{j,m}(t)\boldsymbol{\varphi}_j, \tag{16b}$$

verifying the following $2m$ ordinary differential equations

$$\int_\Omega \partial_t\boldsymbol{d}_m(t)\cdot\boldsymbol{\varphi}_j + \int_\Omega \sigma\hat\varepsilon_m^{p'-1}\boldsymbol{d}_m(t)\cdot\boldsymbol{\varphi}_j - \int_\Omega \nabla\times P_m\big(\hat\mu_m\, J_{p'}^\eta\boldsymbol{b}_m(t)\big)\cdot\boldsymbol{\varphi}_j = \int_\Omega \boldsymbol{g}_{1,m}(t)\cdot\boldsymbol{\varphi}_j, \tag{17a}$$

$$\int_\Omega \partial_t\boldsymbol{b}_m(t)\cdot\boldsymbol{\psi}_j^\theta + \int_\Omega \nabla\times P_m\left(\hat\varepsilon_m J_p^\eta\boldsymbol{d}_m(t)\right)\cdot\boldsymbol{\psi}_j^\theta = \int_\Omega \boldsymbol{g}_{2,m}(t)\cdot\boldsymbol{\psi}_j^\theta, \tag{17b}$$

with $j=1,\dots,m$. For a.e. $t\in(0,T)$, $\boldsymbol{g}_{1,m}(t)\in\langle\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_m\rangle$ is such that $\boldsymbol{g}_{1,m}\underset{m}{\longrightarrow}\boldsymbol{g}_1$ in $L^\infty\big(0,T;\boldsymbol{L}^p(\Omega)\big)$ and $\boldsymbol{g}_{2,m}(t)\in\langle\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_m\rangle$ verifies $\boldsymbol{g}_{2,m}\underset{m}{\longrightarrow}\boldsymbol{g}_2$ in $L^\infty\big(0,T;\boldsymbol{L}^{p'}(\Omega)\big)$. We consider as initial conditions

$$\boldsymbol{d}_m(0) = \boldsymbol{d}_{0,m} \qquad \text{and} \qquad \boldsymbol{b}_m(0) = \boldsymbol{b}_{0,m} \tag{17c}$$

where $\boldsymbol{d}_{0,m}$ belongs to $\langle\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_m\rangle$ and $\boldsymbol{b}_{0,m}$ belongs to $\langle\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_m\rangle$ being such that $\boldsymbol{d}_{0,m}\underset{m}{\longrightarrow}\boldsymbol{d}_0$ in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{b}_{0,m}\underset{m}{\longrightarrow}\boldsymbol{b}_0$ in $\boldsymbol{L}^{p'}(\Omega)$.

We remark that $\boldsymbol{d}_m$ and $\boldsymbol{b}_m$ depend, as we point out in (16), on η and θ, but we omit this dependence whenever possible.

Proposition 5 *The system of ordinary differential equations (17) has a unique solution $(\boldsymbol{b}_m,\boldsymbol{d}_m)$ belonging to $W^{1,\infty}\big(0,T;\boldsymbol{\mathscr{D}}(\Omega)\big)\times W^{1,\infty}\big(0,T;\boldsymbol{\mathscr{D}}(\bar\Omega)\big)$.*

Proof Denoting $\boldsymbol{\xi}_m(t) = \big(\xi_{1,m}(t),\dots,\xi_{m,m}(t)\big)$ and $\boldsymbol{\zeta}_m(t) = \big(\zeta_{1,m}(t),\dots,\zeta_{m,m}(t)\big)$, the system (17) can be rewritten as follows:

$$\begin{cases} \big(\boldsymbol{\xi}_m'(t),\boldsymbol{\zeta}_m'(t)\big) = \Big(\boldsymbol{Z}\big(\boldsymbol{\xi}_m(t),\boldsymbol{\zeta}_m(t)\big),\boldsymbol{W}\big(\boldsymbol{\xi}_m(t),\boldsymbol{\zeta}_m(t)\big)\Big) + \big(\boldsymbol{G}^1(t),\boldsymbol{G}^2(t)\big), \\[2ex] \big(\boldsymbol{\xi}_m(0),\boldsymbol{\zeta}_m(0)\big) = \big(\boldsymbol{\xi}^{0,m},\boldsymbol{\zeta}^{0,m}\big), \end{cases}$$

where, for $j=1,\dots,m$,

$$
\begin{aligned}
Z_j\big(\boldsymbol{\xi}_m(t),\boldsymbol{\zeta}_m(t)\big) &= -\int_\Omega \sigma\hat{\varepsilon}_m^{p'-1}\boldsymbol{d}_m(t)\cdot\boldsymbol{\varphi}_j + \int_\Omega \nabla\times P_m\big(\hat{\mu}_m\, J_{p'}^{\eta}\boldsymbol{b}_m(t)\big)\cdot\boldsymbol{\varphi}_j,\\
W_j\big(\boldsymbol{\xi}_m(t),\boldsymbol{\zeta}_m(t)\big) &= -\int_\Omega \nabla\times P_m\left(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\right)\cdot\boldsymbol{\psi}_j^{\theta},\\
G_j^1(t) &= \int_\Omega \boldsymbol{g}_{1,m}(t)\cdot\boldsymbol{\varphi}_j,\\
G_j^2(t) &= \int_\Omega \boldsymbol{g}_{2,m}(t)\cdot\boldsymbol{\psi}_j^{\theta},\\
\xi_j^{0,m} &= \int_\Omega \boldsymbol{d}_{0,m}\cdot\boldsymbol{\varphi}_j,\\
\zeta_j^{0,m} &= \int_\Omega \boldsymbol{b}_{0,m}\cdot\boldsymbol{\psi}_j^{\theta}.
\end{aligned}
$$

The existence and uniqueness of solution of the system (17) is a consequence of the Cauchy-Lipschitz-Picard Theorem. □

5.1 A priori estimates

Proposition 6 *Let $(\boldsymbol{d}_m,\boldsymbol{b}_m)$ be the unique solution of the problem (17). Then there exists a positive constant C independent of $m\in\mathbb{N}$, $\eta>0$ and $\theta>0$ such that*

$$
\|\boldsymbol{d}_m^{\eta,\theta}\|_{L^\infty(0,T;\boldsymbol{L}^p(\Omega))} \le C \qquad \|\boldsymbol{b}_m^{\eta,\theta}\|_{L^\infty(0,T;\boldsymbol{L}^{p'}(\Omega))} \le C, \tag{18}
$$

$$
\|J_p^{\eta}\boldsymbol{d}_m^{\eta,\theta}\|_{L^\infty(0,T;\boldsymbol{L}^{p'}(\Omega))} \le C \qquad \|J_{p'}^{\eta}\boldsymbol{b}_m^{\eta,\theta}\|_{L^\infty(0,T;\boldsymbol{L}^{p}(\Omega))} \le C. \tag{19}
$$

Proof As $P_m\big(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\big)$ belongs to $\langle\boldsymbol{\varphi}_1,\dots,\boldsymbol{\varphi}_m\rangle$ and $P_m\big(\hat{\mu}_m J_{p'}^{\eta}\boldsymbol{b}_m(t)\big)$ belongs to $\langle\boldsymbol{\psi}_1^{\theta},\dots,\boldsymbol{\psi}_m^{\theta}\rangle$, the unique solution of the system (17) satisfies

$$
\begin{aligned}
\int_\Omega \partial_t\boldsymbol{d}_m(t)\cdot P_m\big(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\big) &+ \int_\Omega \sigma\hat{\varepsilon}_m^{p'-1}\boldsymbol{d}_m(t)\cdot P_m\big(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\big)\\
&+ \int_\Omega \partial_t\boldsymbol{b}_m(t)\cdot P_m\big(\hat{\mu}_m J_{p'}^{\eta}\boldsymbol{b}_m(t)\big) = \int_\Omega \boldsymbol{g}_{1,m}(t)\cdot P_m\big(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\big)\\
&+ \int_\Omega \boldsymbol{g}_{2,m}(t)\cdot P_m\big(\hat{\mu}_m J_{p'}^{\eta}\boldsymbol{b}_m(t)\big).
\end{aligned}
$$

Applying to the last equality the orthogonality relation (15), we have

$$
\begin{aligned}
\int_\Omega \hat{\varepsilon}_m\,\partial_t\boldsymbol{d}_m(t)\cdot J_p^{\eta}\boldsymbol{d}_m(t) &+ \int_\Omega \sigma\hat{\varepsilon}_m^{p'-1}\boldsymbol{d}_m(t)\cdot J_p^{\eta}\boldsymbol{d}_m(t)\\
&+ \int_\Omega \hat{\mu}_m\,\partial_t\boldsymbol{b}_m(t)\cdot J_{p'}^{\eta}\boldsymbol{b}_m(t) = \int_\Omega \boldsymbol{g}_{1,m}(t)\cdot\big(\hat{\varepsilon}_m J_p^{\eta}\boldsymbol{d}_m(t)\big)\\
&+ \int_\Omega \boldsymbol{g}_{2,m}(t)\cdot\big(\hat{\mu}_m J_{p'}^{\eta}\boldsymbol{b}_m(t)\big).
\end{aligned}
$$

Observing that for $\boldsymbol{u}\in W^{1,q}\big(0,T;\boldsymbol{L}^q(\Omega)\big)$, $1<q<\infty$, we have

$$
\partial_t\boldsymbol{u}\cdot J_q^{\eta}\boldsymbol{u} = \tfrac{1}{q}\,\partial_t\big(\,|\boldsymbol{u}|^2+\eta\big)^{\frac{q}{2}} \tag{20}
$$

and

$$
\boldsymbol{d}_m(t)\cdot J_p^{\eta}\boldsymbol{d}_m(t) = \big(\,|\boldsymbol{d}_m(t)|^2+\eta\big)^{\frac{p-2}{2}}\,|\boldsymbol{d}_m(t)|^2 \ge 0, \tag{21}
$$

we obtain

$$\frac{1}{p}\int_\Omega \hat{\varepsilon}_m\,\partial_t\big(\,|\boldsymbol{d}_m(t)|^2+\eta\big)^{\frac{p}{2}}+\frac{1}{p'}\int_\Omega \hat{\mu}_m\,\partial_t\big(\,|\boldsymbol{b}_m(t)|^2+\eta\big)^{\frac{p'}{2}} \\ \le \int_\Omega \boldsymbol{g}_{1,m}(t)\cdot\big(\hat{\varepsilon}_m J_p^\eta \boldsymbol{d}_m(t)\big)+\int_\Omega \boldsymbol{g}_{2,m}(t)\cdot\big(\hat{\mu}_m J_{p'}^\eta \boldsymbol{b}_m(t)\big).$$

The sets $\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}=\bigcup_{j=1}^m \operatorname{supp}\,\tilde{\boldsymbol{\varphi}}_j$ and $\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}=\bigcup_{j=1}^m \operatorname{supp}\,\tilde{\boldsymbol{\psi}}_j^\theta$ are bounded. Applying Hölder and Young inequalities to the second member of the previous relation we obtain

$$\int_\Omega \boldsymbol{g}_{1,m}(t)\cdot\big(\hat{\varepsilon}_m J_p^\eta \boldsymbol{d}_m(t)\big)+\int_\Omega \boldsymbol{g}_{2,m}(t)\cdot\big(\hat{\mu}_m J_{p'}^\eta \boldsymbol{b}_m(t)\big) \\ \le \frac{1}{p}\int_\Omega \big|\boldsymbol{g}_{1,m}(t)\big|^p+\frac{1}{p'}\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m^{p'}\big(\big|\boldsymbol{d}_m(t)\big|^2+\eta\big)^{\frac{p}{2}} \\ +\frac{1}{p'}\int_\Omega \big|\boldsymbol{g}_{2,m}(t)\big|^{p'}+\frac{1}{p}\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m^p\big(\big|\boldsymbol{b}_m(t)\big|^2+\eta\big)^{\frac{p'}{2}}.$$

Using the equalities (20) and (21), we conclude that

$$\frac{1}{p}\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m\,\partial_t\big(\big|\boldsymbol{d}_m(t)\big|^2+\eta\big)^{\frac{p}{2}}+\frac{1}{p'}\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m\,\partial_t\big(\big|\boldsymbol{b}_m(t)\big|^2+\eta\big)^{\frac{p'}{2}} \\ \le \frac{1}{p}\int_\Omega \big|\boldsymbol{g}_{1,m}(t)\big|^p+\frac{1}{p'}\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m^{p'}\big(\big|\boldsymbol{d}_m(t)\big|^2+\eta\big)^{\frac{p}{2}} \\ +\frac{1}{p'}\int_\Omega \big|\boldsymbol{g}_{2,m}(t)\big|^{p'}+\frac{1}{p}\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m^p\big(\big|\boldsymbol{b}_m(t)\big|^2+\eta\big)^{\frac{p'}{2}}. \tag{22}$$

Integrating over $[0,t]$ the inequality (22), we get

$$\frac{1}{p}\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m\big(\big|\boldsymbol{d}_m(t)\big|^2+\eta\big)^{\frac{p}{2}}+\frac{1}{p'}\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m\big(\big|\boldsymbol{b}_m(t)\big|^2+\eta\big)^{\frac{p'}{2}} \\ \le \frac{1}{p}\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m\big(\big|\boldsymbol{d}_{0,m}\big|^2+\eta\big)^{\frac{p}{2}}+\frac{1}{p'}\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m\big(\big|\boldsymbol{b}_{0,m}\big|^2+\eta\big)^{\frac{p'}{2}} \\ +\frac{1}{p}\int_0^t\int_\Omega \big|\boldsymbol{g}_{1,m}(\tau)\big|^p+\frac{1}{p'}\int_0^t\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \hat{\varepsilon}_m^{p'}\big(\big|\boldsymbol{d}_m(\tau)\big|^2+\eta\big)^{\frac{p}{2}} \\ +\frac{1}{p'}\int_0^t\int_\Omega \big|\boldsymbol{g}_{2,m}(\tau)\big|^{p'}+\frac{1}{p}\int_0^t\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \hat{\mu}_m^p\big(\big|\boldsymbol{b}_m(\tau)\big|^2+\eta\big)^{\frac{p'}{2}}$$

obtaining

$$\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \big(\big|\boldsymbol{d}_m(t)\big|^2+\eta\big)^{\frac{p}{2}}+\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \big(\big|\boldsymbol{b}_m(t)\big|^2+\eta\big)^{\frac{p'}{2}} \\ \le C\Big[\int_0^t\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \big(\big|\boldsymbol{d}_m(\tau)\big|^2+\eta\big)^{\frac{p}{2}}+\int_0^t\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \big(\big|\boldsymbol{b}_m(\tau)\big|^2+\eta\big)^{\frac{p'}{2}} \\ +\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \big(\big|\boldsymbol{d}_{0,m}\big|^2+\eta\big)^{\frac{p}{2}}+\int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \big(\big|\boldsymbol{b}_{0,m}\big|^2+\eta\big)^{\frac{p'}{2}} \\ +\int_0^T\int_\Omega \big|\boldsymbol{g}_{1,m}(t)\big|^p+\int_0^T\int_\Omega \big|\boldsymbol{g}_{2,m}(t)\big|^{p'}\Big],$$

where C is a positive constant depending on p, p', $\max\left\{\|\hat{\varepsilon}\|_{L^\infty(\Omega)}, \|\hat{\mu}\|_{L^\infty(\Omega)}, 1\right\}$ and $\min\left\{\hat{\varepsilon}_*, \hat{\mu}_*\right\}$ and so independent of m, η and θ. As

$$\int_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}} \left(\left|\boldsymbol{d}_{0,m}\right|^2+\eta\right)^{\frac{p}{2}} + \int_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}} \left(\left|\boldsymbol{b}_{0,m}\right|^2+\eta\right)^{\frac{p'}{2}} \leq 2^{\frac{p'}{2}}\left(\int_\Omega \left|\boldsymbol{d}_{0,m}\right|^p + \int_\Omega \left|\boldsymbol{b}_{0,m}\right|^{p'} + \eta^{\frac{p}{2}}\left|\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}\right| + \eta^{\frac{p'}{2}}\left|\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}\right|\right),$$

considering, from now on, values of η such that

$$\eta \leq \min\left\{\left|\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}\right|^{-\frac{2}{p}}, \left|\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}\right|^{-\frac{2}{p'}}\right\} \tag{23}$$

and observing that the sequences $\left(\boldsymbol{d}_{0,m}\right)_m$ and $\left(\boldsymbol{b}_{0,m}\right)_m$ are bounded respectively in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{L}^{p'}(\Omega)$ and the sequences $\left(\boldsymbol{g}_{1,m}\right)_m$ and $\left(\boldsymbol{g}_{2,m}\right)_m$ are also bounded respectively in $\boldsymbol{L}^p(Q)$ and $\boldsymbol{L}^{p'}(Q)$, the inequality (22) becomes

$$\begin{aligned}&\int_\Omega \left(\left|\boldsymbol{d}_m(t)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}}\right)^{\frac{p}{2}} + \int_\Omega \left(\left|\boldsymbol{b}_m(t)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}}\right)^{\frac{p'}{2}}\\ &\qquad\leq C_1\left(\int_0^t\int_\Omega \left(\left|\boldsymbol{d}_m(\tau)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}}\right)^{\frac{p}{2}} + \int_0^t\int_\Omega \left(\left|\boldsymbol{b}_m(\tau)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}}\right)^{\frac{p'}{2}}\right) + C_2\end{aligned}$$

where C_1 and C_2 are constants independent of m, η and θ.

Applying the Gronwall inequality, we obtain

$$\int_\Omega \left(\left|\boldsymbol{d}_m(t)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}}\right)^{\frac{p}{2}} + \int_\Omega \left(\left|\boldsymbol{b}_m(t)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\varphi}}_j}}\right)^{\frac{p'}{2}} \leq C$$

for a.e. $t \in (0,T)$, where C is a constant independent of m, t, η and θ, and so

$$\int_\Omega |\boldsymbol{d}_m(t)|^p + \int_\Omega |\boldsymbol{b}_m(t)|^{p'} \leq C$$

concluding the proof of (18).

As, for η satisfying (23) we have for a.e. t

$$\|J_p^\eta \boldsymbol{d}_m(t)\|^{p'}_{\boldsymbol{L}^{p'}(\Omega)} \leq \int_\Omega \left(\left|\boldsymbol{d}_m(t)\right|^2+\eta\,\chi_{\Omega_{m,\tilde{\boldsymbol{\psi}}_j^\theta}}\right)^{\frac{p}{2}} \leq \|\boldsymbol{d}_m(t)\|^p_{\boldsymbol{L}^p(\Omega)}+1$$

and an analogous inequality is satisfied by $J_{p'}^\eta \boldsymbol{b}_m(t)$, the proof of (19) follows. □

6 Existence of solution

Let us introduce a result which is a direct consequence of an important theorem concerning Young measures. Details about Young measures and measure-valued functions can be found in [9].

Here we make the usual identification of the Banach space of the Radom measures $M(\mathbb{R}^3)$ with the dual space of $(\mathscr{C}_0(\mathbb{R}^3), \|\cdot\|_\infty)$.

Proposition 7 *Let $(\boldsymbol{z}_n)_n$ be a sequence of functions uniformly bounded in $\boldsymbol{L}^p(\Omega)$, $p \in]1,+\infty[$. Then there exist a subsequence, still denoted by $(\boldsymbol{z}_n)_n$, and a measure-valued function $\varrho : \Omega \longrightarrow M(\mathbb{R}^3)$, such that, for every function $\tau : \mathbb{R}^3 \longrightarrow \mathbb{R}$ verifying the growth condition*

$$\exists\, c > 0 \quad \forall\, \xi \in \mathbb{R}^3 \qquad |\tau(\xi)| \leq c|\xi|^{p-1}, \tag{24}$$

we have

$$\tau(\boldsymbol{z}_n) \longrightarrow \overline{\tau}, \quad \textit{weakly in } L^{p'}(\Omega), \qquad \overline{\tau}(y) \overset{\text{a.e.}}{=} \langle\, \varrho_y, \varphi\,\rangle.$$

Proof Choosing the Young function $\Psi(u) = u^{p'}$, we have

$$\int_\Omega \Psi\big(|\tau(\boldsymbol{z}_n)|\big)dy = \int_\Omega |\tau(\boldsymbol{z}_n)|^{p'}dy \le c^{p'}\int_\Omega |\boldsymbol{z}_n|^{(p-1)p'}dy = \|\boldsymbol{z}_n\|^p_{\boldsymbol{L}^p(\Omega)} < \infty.$$

Using the Theorem 2.1, page 171, of [9], the conclusion is immediate. □

Proof of Theorem 1 Assuming that η and θ depend on m, satisfying (23) and such that $\eta_m \underset{m}{\longrightarrow} 0$ and $\theta_m \underset{m}{\longrightarrow} 0$, we denote $\boldsymbol{d}_m(t) = \boldsymbol{d}_m^{\eta_m,\theta_m}(t)$ and $\boldsymbol{b}_m(t) = \boldsymbol{b}_m^{\eta_m,\theta_m}(t)$. We proved that $\boldsymbol{d}_m$ and $J_{p'}^{\eta_m}\boldsymbol{b}_m$ are bounded sequences in $L^\infty\big(0,T;\boldsymbol{L}^p(\Omega)\big)$ and $\boldsymbol{b}_m$ and $J_p^{\eta_m}\boldsymbol{d}_m$ are bounded sequences in $L^\infty\big(0,T;\boldsymbol{L}^{p'}(\Omega)\big)$. So, there exists a subsequence ν of m such that

$$\boldsymbol{d}_\nu \underset{\nu}{\longrightarrow} \boldsymbol{d} \quad \text{weak - * in } L^\infty\big(0,T;\boldsymbol{L}^p(\Omega)\big),$$
$$\boldsymbol{b}_\nu \underset{\nu}{\longrightarrow} \boldsymbol{b} \quad \text{weak - * in } L^\infty\big(0,T;\boldsymbol{L}^{p'}(\Omega)\big).$$

Integrating by parts in (17) we have, for each ν and $j = 1,\ldots,\nu$,

$$\int_\Omega \partial_t\boldsymbol{d}_\nu(t)\cdot\boldsymbol{\varphi}_j + \int_\Omega \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}_\nu(t)\cdot\boldsymbol{\varphi}_j - \int_\Omega P_\nu\big(\hat{\mu}_\nu\, J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu(t)\big)\cdot\nabla\times\boldsymbol{\varphi}_j = \int_\Omega \boldsymbol{g}_{1,\nu}(t)\cdot\boldsymbol{\varphi}_j, \tag{25a}$$
$$\int_\Omega \partial_t\boldsymbol{b}_\nu(t)\cdot\boldsymbol{\psi}_j^{\theta_\nu} + \int_\Omega P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu(t)\right)\cdot\nabla\times\boldsymbol{\psi}_j^{\theta_\nu} = \int_\Omega \boldsymbol{g}_{2,\nu}(t)\cdot\boldsymbol{\psi}_j^{\theta_\nu}. \tag{25b}$$

Fixing $n\in\mathbb{N}$, let ν be such that $k_n \le \nu$ (k_n defined in Proposition 4). For $j = 1,\ldots,n$, we have

$$\nabla\times\boldsymbol{\varphi}_j = \sum_{i=1}^{k_j}\alpha_i\boldsymbol{\varphi}_i \in \boldsymbol{F}_\nu, \qquad \nabla\times\boldsymbol{\psi}_j = \sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i \in \boldsymbol{G}_\nu.$$

From (25a), and using (15), we obtain

$$\int_\Omega \partial_t\boldsymbol{d}_\nu(t)\cdot\boldsymbol{\varphi}_j + \int_\Omega \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}_\nu(t)\cdot\boldsymbol{\varphi}_j - \int_\Omega \hat{\mu}_\nu\, J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu(t)\cdot\sum_{i=1}^{k_j}\alpha_i\boldsymbol{\varphi}_i = \int_\Omega \boldsymbol{g}_{1,\nu}(t)\cdot\boldsymbol{\varphi}_j. \tag{26}$$

Observing that

$$\begin{aligned}\nabla\times\boldsymbol{\psi}_j^{\theta_\nu} &= \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big) + \nabla\times\boldsymbol{\psi}_j\\ &= \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big) + \sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i\\ &= \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big) + \sum_{i=1}^{k_j}\beta_i(\boldsymbol{\psi}_i - \boldsymbol{\psi}_i^{\theta_\nu}) + \underbrace{\sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i^{\theta_\nu}}_{\in\boldsymbol{G}_\nu},\end{aligned}$$

from (25b), using again (15), we obtain

$$\begin{aligned}\int_\Omega \partial_t\boldsymbol{b}_\nu(t)\cdot\boldsymbol{\psi}_j^{\theta_\nu} + \int_\Omega P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu(t)\right)\cdot\nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big)&\\ + \int_\Omega P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu(t)\right)\cdot\sum_{i=1}^{k_j}\beta_i(\boldsymbol{\psi}_i - \boldsymbol{\psi}_i^{\theta_\nu})&\\ + \int_\Omega \hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu(t)\cdot\sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i^{\theta_\nu} &= \int_\Omega \boldsymbol{g}_{2,\nu}(t)\cdot\boldsymbol{\psi}_j^{\theta_\nu}.\end{aligned} \tag{27}$$

Considering functions

$$\mathbf{\Phi}_n(t)=\sum_{j=1}^n \xi_j(t)\boldsymbol{\varphi}_j, \qquad \text{where } \xi_j(t)\in\mathscr{C}^1([0,T]), \quad \xi_j(T)=0$$

and

$$\mathbf{\Psi}_n^{\theta_\nu}(t)=\sum_{j=1}^n \zeta_j(t)\boldsymbol{\psi}_j^{\theta_\nu}, \qquad \text{where } \zeta_j(t)\in\mathscr{C}^1([0,T]), \quad \zeta_j(T)=0,$$

multiplying in (26) and in (27), the j-th equation, respectively by $\xi_j(t)$ and $\zeta_j(t)$, with $j=1,\dots,n$, summing over j and integrating between 0 and T, we have

$$\int_Q \partial_t\boldsymbol{d}_\nu\cdot\mathbf{\Phi}_n+\int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}_\nu\cdot\mathbf{\Phi}_n-\int_Q \hat{\mu}_\nu\, J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu\cdot\nabla\times\mathbf{\Phi}_n=\int_Q \boldsymbol{g}_{1,\nu}(t)\cdot\mathbf{\Phi}_n, \tag{28a}$$

$$\begin{aligned}\int_Q \partial_t\boldsymbol{b}_\nu\cdot\mathbf{\Psi}_n^{\theta_\nu}&+\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\right)\cdot\sum_{j=1}^n\zeta_j\nabla\times\left(\boldsymbol{\psi}_j^{\theta_\nu}-\boldsymbol{\psi}_j\right)\\&+\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\right)\cdot\sum_{j=1}^n\zeta_j\sum_{i=1}^{k_j}\beta_i(\boldsymbol{\psi}_i-\boldsymbol{\psi}_i^{\theta_\nu})\\&+\int_Q \hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\cdot\sum_{j=1}^n\zeta_j\sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i^{\theta_\nu}=\int_Q \boldsymbol{g}_{2,\nu}\cdot\mathbf{\Psi}_n^{\theta_\nu}.\end{aligned} \tag{28b}$$

Integrating by parts, in time, the first term of equations (28), we have

$$-\int_Q \boldsymbol{d}_\nu\cdot\partial_t\mathbf{\Phi}_n+\int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}_\nu\cdot\mathbf{\Phi}_n-\int_Q \hat{\mu}_\nu\, J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu\cdot\nabla\times\mathbf{\Phi}_n=\int_Q \boldsymbol{g}_{1,\nu}(t)\cdot\mathbf{\Phi}_n+\int_\Omega \boldsymbol{d}_{\nu,0}\cdot\mathbf{\Phi}_n(0), \tag{29a}$$

$$\begin{aligned}-\int_Q \boldsymbol{b}_\nu\cdot\partial_t\mathbf{\Psi}_n^{\theta_\nu}&+\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\right)\cdot\sum_{j=1}^m\zeta_j\nabla\times(\boldsymbol{\psi}_j^{\theta_\nu}-\boldsymbol{\psi}_j)\\&+\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\right)\cdot\sum_{j=1}^m\zeta_j\sum_{i=1}^{k_j}\beta_i(\boldsymbol{\psi}_i-\boldsymbol{\psi}_i^{\theta_\nu})+\int_Q \hat{\varepsilon}_\nu J_p^{\eta_\nu}\boldsymbol{d}_\nu\cdot\sum_{j=1}^m\zeta_j\sum_{i=1}^{k_j}\beta_i\boldsymbol{\psi}_i^{\theta_\nu}\\&=\int_Q \boldsymbol{g}_{2,\nu}\cdot\mathbf{\Psi}_n^{\theta_\nu}+\int_\Omega \boldsymbol{b}_{\nu,0}\cdot\mathbf{\Psi}_n^{\theta_\nu}(0).\end{aligned} \tag{29b}$$

Before passing to the limit in ν in equations (29), let us study the behavior of some terms. Writing

$$\int_Q \hat{\mu}_\nu\, J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu\cdot\nabla\times\mathbf{\Phi}_n=\int_Q \hat{\mu}_\nu\,(J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu-J_{p'}\boldsymbol{b}_\nu)\cdot\nabla\times\mathbf{\Phi}_n+\int_Q \hat{\mu}_\nu\, J_{p'}\boldsymbol{b}_\nu\cdot\nabla\times\mathbf{\Phi}_n,$$

observing that, for a fixed ν (recall that the functions $\boldsymbol{b}_\nu$ depend on η_ν and θ_ν), we have

$$J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu(x,t)-J_{p'}\boldsymbol{b}_\nu(x,t)\xrightarrow[\eta_\nu\to 0]{}0, \qquad \text{for a.e. } (x,t)\in Q,$$

applying the Lebesgue Theorem we get

$$\int_Q \hat{\mu}_\nu\,(J_{p'}^{\eta_\nu}\boldsymbol{b}_\nu-J_{p'}\boldsymbol{b}_\nu)\cdot\nabla\times\mathbf{\Phi}_n\xrightarrow[\eta_\nu\to 0]{}0\cdot$$

Observe now that the vectorial operator $J_{p'}$ is such that each component verifies the growth condition (24) for p'. Then, for a subsequence, still denoted by $\boldsymbol{b}_\nu$,

$$J_{p'}(\boldsymbol{b}_\nu)\xrightarrow[\nu]{}\bar{J}_{p',(\boldsymbol{b}_\nu)}=\bar{J}_{p',\boldsymbol{b}}:=\mu\boldsymbol{h}\ \text{in}\ \boldsymbol{L}^{p'}(\Omega),\ \ \bar{J}_{p'}(y)\overset{\text{a.e.}}{=}\langle\,\varrho_y,J_{p'}\,\rangle. \tag{30}$$

The function ϱ in (30) is vectorial, more precisely, $\varrho : \Omega \longrightarrow M(\mathbb{R}^3)^3$.
So

$$\int_Q \hat{\mu}_\nu\, J_{p'}^{\eta_\nu} \boldsymbol{b}_\nu \cdot \nabla\times \boldsymbol{\Phi}_n \xrightarrow[\nu\to\infty]{} \int_Q \hat{\mu}\, \bar{J}_{p',\boldsymbol{b}} \cdot \nabla\times\boldsymbol{\Phi}_n,$$

as long as η_ν goes to zero sufficiently quick, in order to satisfy (23).

Applying analogous arguments, since we also have

$$J_p(\boldsymbol{d}_\nu) \xrightarrow[\nu]{} \bar{J}_{p,(\boldsymbol{d}_\nu)} = \bar{J}_{p,\boldsymbol{d}} := \varepsilon \boldsymbol{e} \text{ in } \boldsymbol{L}^p(\Omega),\ \ \bar{J}_p(y) \overset{\text{a.e.}}{=} \langle\, \vartheta_y, J_p \,\rangle, \tag{31}$$

then

$$\int_Q \hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu \cdot \sum_{j=1}^{n} \zeta_j \sum_{i=1}^{k_j} \beta_i \boldsymbol{\psi}_i^{\theta_\nu} \xrightarrow[\nu\to\infty]{} \int_Q \hat{\varepsilon}\, \bar{J}_{p,\boldsymbol{d}} \cdot \nabla\times\boldsymbol{\Psi}_n.$$

The option we have made, denoting $\bar{J}_{p',\boldsymbol{b}}$ and $\bar{J}_{p,\boldsymbol{d}}$ by $\mu\boldsymbol{h}$ and $\varepsilon\boldsymbol{e}$, respectively, will be clarified in Remark 6.

The properties of the operator P_ν, linear and continuous, and of the functions ζ_j, $\boldsymbol{\psi}_j^{\theta_\nu}$ and $\boldsymbol{\psi}_j$, guarantee the existence of constants C_ν and C_n such that

$$\begin{aligned}
&\int_Q \left| P_\nu\big(\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\big) \cdot \sum_{j=1}^{n} \zeta_j \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big)\right| \\
&\qquad\qquad \le \left\| P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\right)\right\|_{\boldsymbol{L}^{p'}(Q)} \left\| \sum_{j=1}^{n} \zeta_j \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big)\right\|_{\boldsymbol{L}^p(Q)} \\
&\qquad\qquad\qquad\qquad \le C_\nu \left\|\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\right\|_{\boldsymbol{L}^{p'}(Q)} C_n \theta_\nu.
\end{aligned}$$

Since $\left\|\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\right\|_{\boldsymbol{L}^{p'}(Q)}$ is bounded independently of ν, η_ν and θ_ν, we conclude that

$$\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\right) \cdot \sum_{j=1}^{n} \zeta_j \nabla\times\big(\boldsymbol{\psi}_j^{\theta_\nu} - \boldsymbol{\psi}_j\big) \xrightarrow[\nu\to\infty]{} 0,$$

as long as $\theta_\nu \xrightarrow[\nu\to\infty]{} 0$ sufficiently quick.

Analogously we conclude that

$$\int_Q P_\nu\left(\hat{\varepsilon}_\nu J_p^{\eta_\nu} \boldsymbol{d}_\nu\right) \cdot \sum_{j=1}^{m} \zeta_j \sum_{i=1}^{k_j} \beta_i (\boldsymbol{\psi}_i - \boldsymbol{\psi}_i^{\theta_\nu}) \xrightarrow[\nu\to\infty]{} 0.$$

Passing to the limit in ν in equations (29), we obtain that $\boldsymbol{d}$, $\boldsymbol{b}$, $\boldsymbol{h}$ and $\boldsymbol{e}$ verify

$$-\int_Q \boldsymbol{d}\cdot\partial_t\boldsymbol{\Phi}_n + \int_Q \sigma\hat{\varepsilon}^{p'-1}\boldsymbol{d}\cdot\boldsymbol{\Phi}_n - \int_Q \boldsymbol{h}\cdot\nabla\times\boldsymbol{\Phi}_n = \int_Q \boldsymbol{g}_1\cdot\boldsymbol{\Phi}_n + \int_\Omega \boldsymbol{d}_0\cdot\boldsymbol{\Phi}_n(0)$$

and

$$-\int_Q \boldsymbol{b}\cdot\partial_t\boldsymbol{\Psi}_n + \int_Q \boldsymbol{e}\cdot\nabla\times\boldsymbol{\Psi}_n = \int_Q \boldsymbol{g}_2\cdot\boldsymbol{\Psi}_n + \int_\Omega \boldsymbol{b}_0\cdot\boldsymbol{\Psi}_n(0),$$

using the identifications introduced in (30)-(31).

By density, we may substitute the functions $\boldsymbol{\Phi}_n$ by arbitrary functions in $\mathcal{V}_0^{p'}(Q)$ and $\boldsymbol{\Psi}_n$ by arbitrary functions in $\mathcal{V}^p(Q)$, obtaining existence of solution of problem (8). □

Remark 6 *The constitutive laws (3) allow us to write the Maxwell system in the form (6a), (6b). It is then natural to denote, in the approximate problem (17),* $\boldsymbol{h}_m = \hat{\mu} J_{p'}\boldsymbol{b}_m$ *and* $\boldsymbol{e}_m = \hat{\varepsilon} J_p \boldsymbol{d}_m$. *As the sequences* $(\boldsymbol{h}_m)_m$ *and* $(\boldsymbol{e}_m)_m$ *are bounded in* $L^\infty\big(0,T;\boldsymbol{L}^p(\Omega)\big)$ *and* $L^\infty\big(0,T;\boldsymbol{L}^{p'}(\Omega)\big)$ *respectively, then*

$$\begin{array}{ccc} \hat{\mu}_\nu J_{p'}\boldsymbol{b}_\nu & = & \hat{\mu}_\nu \mu \boldsymbol{h}_\nu \\ \downarrow & & \downarrow \\ \hat{\mu}\bar{J}_{p',\boldsymbol{b}} & & \boldsymbol{h} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \hat{\varepsilon}_\nu J_p \boldsymbol{d}_\nu & = & \hat{\varepsilon}_\nu \varepsilon \boldsymbol{e}_\nu \\ \downarrow & & \downarrow \\ \hat{\mu}\bar{J}_{p,\boldsymbol{d}} & & \boldsymbol{e} \end{array}$$

and so the identifications $\bar{J}_{p',\boldsymbol{b}} := \mu\boldsymbol{h}$ *and* $\bar{J}_{p,\boldsymbol{d}} := \varepsilon\boldsymbol{e}$ *are natural.*

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