Sign pattern matrices that admit P_0–matrices [*†]

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Abstract

For sign patterns corresponding to directed or undirected cycles, we identify conditions under which the patterns admit or require P_0–matrices.

1 Introduction

In qualitative and combinatorial matrix theory, the use of combinatorial information such as the signs of the elements of a matrix is very often useful in the study of some properties of matrices. A matrix all of whose entries are chosen from the set $\{+, -, 0\}$ is called a *sign pattern matrix*. Given an $n \times m$ real matrix $A = (a_{ij})$, we denote by $\text{sign}(A)$ the sign pattern matrix obtained from A by replacing each one of its positive entries by $+$ and each

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one of its negative entries by $-$. For an $n \times m$ sign pattern matrix P, we define the *sign pattern class* $\mathcal{C}(P)$ by

$$\mathcal{C}(P) = \left\{ A \in \mathbb{R}^{m\times n} : \ \operatorname{sign}(A) = P \right\}.$$

A sign pattern matrix P is said to *require* a certain property $\mathcal{P}$ referring to real matrices if all real matrices in $\mathcal{C}(P)$ have the property $\mathcal{P}$, and is said to *allow* that property $\mathcal{P}$ if some real matrix in $\mathcal{C}(P)$ has the property $\mathcal{P}$. In the literature, one can find, in the last few years, an increasing interest in problems that arise from the basic question of whether a certain sign pattern matrix requires (or allows) a certain property (see, for instance, [2], [6], [7] and [8]).

In this paper, we shall consider the class of P_0–matrices. A *P_0–matrix* is a real square matrix all of whose principal minors are nonnegative. Our aim is to determine which sign pattern matrices are admissible for this class of real matrices. In other words, we shall focus on the question 'which sign pattern matrices allow the property of belonging to the class of P_0–matrices?'.

For an $n \times n$ matrix A, the submatrix of A lying in rows α and columns β, $\alpha, \beta \subseteq \{1, ..., n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. Hence, a real $n \times n$ matrix A is a P_0–matrix if $\det A[\alpha] \geq 0$, for all $\alpha \subseteq \{1, ..., n\}$.

In [8], the authors characterized the sign pattern matrices that admit N–matrices, P–matrices and M–matrices. Some partial results related to the description of the sign pattern matrices that allow the property of belonging to the class of inverse M–matrices were also presented. Recall that an $n \times n$ real matrix A is called an *N–matrix* if all of its principal minors are negative while A is said to be a *P–matrix* if all of its principal minors are positive. If Z_n is the set of all square real matrices of order n whose off-diagonal entries are non-positive, a matrix $A \in Z_n$ is an M–matrix if and only if A is a P–matrix. A nonsingular matrix A is said to be an *inverse M-matrix* if A^{-1} is an M–matrix. See [1] and [5] for more information on these classes of matrices. Since P_0–matrices are defined by means of the signs of principal minors, the natural question that arises now is of whether we are able to give a conclusive answer to the similar problem referring to this particular class of matrices.

2 Notation and preliminaries

A natural way to describe an $n \times n$ sign pattern matrix $P = (p_{ij})$ is via a loop-free graph $G(P) = (V(G), E(G))$, where the set of vertices $V(G)$ is $\{1, \ldots, n\}$ and (i, j) is an edge or arc in $E(G)$ if and only if $p_{ij} \neq 0$ and $i \neq j$. A graph-theoretical approach will be quite useful in the study of the problem addressed in the next section.

A sign pattern matrix $P = (p_{ij})$ is said to be *combinatorially symmetric* if $p_{ij} \neq 0$ if and only if $p_{ji} \neq 0$, for all choices of i, j, $i \neq j$, and *not combinatorially symmetric* otherwise.

In this study we will use directed graphs, but in the case of combinatorially symmetric sign pattern matrices we will treat the graphs as undirected when convenient.

A *path* in a graph is a sequence of edges (i_1, i_2), $(i_2, i_3), \ldots$, (i_{k-1}, i_k) in which all vertices are distinct, except, possibly, the first and the last. The *length* of a path is the number of edges in the path. A *cycle* is a closed path, that is a path in which the first and the last vertices coincide.

The directed graph associated with a not combinatorially symmetric sign pattern matrix is either acyclic (possessing no cycles) or cyclic (with at least one cycle).

A *permutation pattern* is simply a sign pattern matrix with exactly one entry in each row and column equal to + and the remaining entries equal to 0. A product of the form $S^T PS$, where S is a square permutation pattern and P is a sign pattern matrix of the same order as S, is called a *permutation similarity.* The directed graph of $S^T PS$ is obtained by applying the permutation corresponding to S to the vertex labels in the directed graph of P.

If G is an acyclic directed graph, then under an appropriate permutation of the vertex labels, G has an upper triangular adjacency matrix. This well-known result in combinatorial graph theory leads us to the following result concerning sign pattern matrices: If P is a not combinatorially symmetric sign pattern matrix whose associated graph is acyclic, there exists a permutation pattern Q such that the sign pattern matrix $\tilde{P} = Q^T PQ$ is upper triangular. (see [4]).

If G is a directed n–cycle, it is also well-known that there exists a permutation similarity that transforms the adjacency matrix into the following standard form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

When it comes to sign pattern matrices, we may, then, conclude that if P is a sign pattern matrix whose associated graph is a directed n–cycle, then there is a permutation similarity that transforms P into the following form:

$$\begin{bmatrix} 0 & p_{12} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & p_{23} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & p_{34} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & p_{n-1n} \\ p_{n1} & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $p_{n1} \neq 0$ and $p_{ii+1} \neq 0$ for $i = 1, \dots, n-1$. Let P_β denote the $n \times n$ sign pattern matrix obtained by taking $p_{n1} = \beta$ and $p_{ii+1} = +$ for $i = 1, \dots, n-1$ in the previous sign pattern matrix.

3 Results

In this section we will focus on the question of whether there exists a P_0–matrix $A \in \mathcal{C}(P)$ for a given $n \times n$ sign pattern matrix P. Throughout we will mainly consider sign pattern matrices with zero diagonal elements.

We begin by analyzing the existence of a P_0–matrix in $\mathcal{C}(P)$ where P is a sign pattern matrix of the form

$$\begin{bmatrix} 0 & p_{12} \\ p_{21} & 0 \end{bmatrix}$$

with $p_{12}, p_{21} \in \{0, -, +\}$. Given a matrix $A = (a_{ij}) \in \mathcal{C}(P)$, $\det A = -a_{12}a_{21}$. This means that there exists a P_0–matrix in $\mathcal{C}(P)$ if and only if $p_{12}p_{21} \leq 0$ (where the operation of multiplication and the relations '$\leq$', '$<$'

and '=' in $\{0, -, +\}$ are defined in a natural way). Moreover, if $p_{12}p_{21} = 0$, all matrices in $\mathcal{C}(P)$ are P_0–matrices. Note that if $p_{12}p_{21} \neq 0$ then P can be described by a directed cycle of length 2. Motivated by these facts, we introduce the following definition.

Definition 3.1. *We say that a sign pattern matrix $P = (p_{ij})$ has the 2–cycle property if $p_{ij}p_{ji} < 0$ whenever $(i, j), (j, i) \in E(G)$, where G is the graph describing P.*

It is obvious that having the 2-cycle property is a necessary and sufficient condition for the existence of a P_0–matrix in $\mathcal{C}(P)$ in case $n = 2$ with no zero off-diagonal entries and diagonal entries equal to zero. This can be extended for sign pattern matrices whose associated graphs are undirected cycles, as the following theorem states.

Theorem 3.2. *Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix, with $p_{ii} = 0$ for all i, whose associated graph $G(P)$ is an undirected cycle. There exists a P_0–matrix in $\mathcal{C}(P)$ if and only if P has the 2–cycle property.*

Proof. The necessary condition is obvious. Conversely, assume that P has the 2–cycle property. We may assume, by permutation similarity, that P is of the following form

$$P = \begin{bmatrix} 0 & p_{12} & 0 & \dots & 0 & p_{1n} \\ p_{21} & 0 & p_{23} & \dots & 0 & 0 \\ 0 & p_{32} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & p_{n-1n} \\ p_{n1} & 0 & 0 & \dots & p_{nn-1} & 0 \end{bmatrix}.$$

Let D be the diagonal sign pattern matrix defined by

$$D = diag(1, p_{12}, p_{12}p_{23}, p_{12}p_{23}p_{34}, \dots, p_{12}p_{23} \cdots p_{n-1n}).$$

Given that $p_{ii+1}p_{i+1i} < 0$ for $i = 1, \ldots, n-1$, it is easy to see that

$$\begin{aligned} DPD^{-1} &= \begin{bmatrix} 0 & + & 0 & \ldots & 0 & -p_{n1}\prod_{i=1}^{n-1} p_{ii+1} \\ - & 0 & + & \ldots & 0 & 0 \\ 0 & - & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & + \\ p_{n1}\prod_{i=1}^{n-1} p_{ii+1} & 0 & 0 & \ldots & - & 0 \end{bmatrix} \\ &= P_\beta - P_\beta^T, \end{aligned}$$

with $\beta = p_{n1}\prod_{i=1}^{n-1} p_{ii+1}$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & -x \\ -1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ x & 0 & 0\ldots & -1 & 0 & \end{bmatrix},$$

with $x \in \mathbb{R}$ such that $\operatorname{sign}(x) = \beta$.

If n is odd, $\det A = 0$.

If n is even, of the form $n = 2k$, it is not difficult to prove that

$$\begin{aligned} \det A &= (-1)^k \det A\,[\{1,3,5,\ldots,2k-1\}\,|\,\{2,4,6,\ldots,2k\}] \times \\ &\quad \times \det A\,[\{2,4,6,\ldots,2k\}\,|\,\{1,3,5,\ldots,2k-1\}] \\ &= (-1)^k(1-x)((-1)^k + (-1)^{k+1}x) = (1-x)^2 \geq 0. \end{aligned}$$

To conclude that A is a P_0–matrix in $\mathcal{C}(DPD^{-}1)$ we still have to show that $\det A[\alpha] \geq 0$ for all $\alpha \subset \{1, \ldots, n\}$.

Let $\alpha \subset \{1, \ldots, n\}$. It is obvious that for $|\alpha| \leq 2$, $\det A[\alpha] \geq 0$. Assuming $|\alpha| > 2$, we firstly address the case in which $1 \notin \alpha$ or $n \notin \alpha$. In this case

$A[\alpha]$ is a submatrix of the $(n-1) \times (n-1)$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.$$

If the indexes in α are consecutive and $|\alpha|$ is odd, it is not hard to prove that $\det A[\alpha] = 0$. If the indexes in α are consecutive and $|\alpha|$ is even, α is of the form $\alpha = \{i_1, i_1+1, i_1+2, \dots, i_1+2q-1\}$. In that case, $\det A[\alpha] = (-1)^{2q} = 1 > 0$. If the indexes in α are not consecutive, then $A[\alpha]$ is a direct sum of matrices $A[\alpha_1], A[\alpha_2], \dots, A[\alpha_t]$ where $\alpha_1, \alpha_2, \dots, \alpha_t \subset \{1, \dots, n\}$, $\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_t = \alpha$, $\alpha_i \cap \alpha_j = \emptyset$ and the indexes of each α_i are consecutive. Therefore, $\det A[\alpha_i] \geq 0$ and $\det A[\alpha] \geq 0$.

Next we analyze the case in which $1, n \in \alpha$. Let j_{0_α} be the smallest index j in $\{1, \dots, n\}$ such that $j \notin \alpha$.

If $j_{0_\alpha} = 2$, then $\det A[\alpha] = x^2 \det A[\alpha - \{1, n\}]$. We have already seen that $\det A[\alpha - \{1, n\}] \geq 0$. These imply $\det A[\alpha] \geq 0$.

If $j_{0_\alpha} = 3$, we can write $\det A[\alpha] = \det A[\alpha - \{1, 2\}]$. We have also seen that $\det A[\alpha - \{1, 2\}] \geq 0$. Therefore $\det A[\alpha] \geq 0$.

For $j_{0_\alpha} = 4$, it is easy to prove that $\det A[\alpha] = \det A[\alpha - \{2, 3\}]$. Observe that $j_{0_{\alpha - \{2,3\}}} = 2$. Hence $\det A[\alpha - \{2, 3\}] \geq 0$ and $\det A[\alpha] \geq 0$.

For any $j_{0_\alpha} \geq 4$, we can write

$$\det A[\alpha] = \det A\left[\alpha - \{j_{0_\alpha} - 2, j_{0_\alpha} - 1\}\right].$$

Note that $j_{0_{\alpha - \{j_{0_\alpha} - 2, j_{0_\alpha} - 1\}}} = j_{0_\alpha} - 2$. By induction, we have

$$\det A\left[\alpha - \{j_{0_\alpha} - 2, j_{0_\alpha} - 1\}\right] \geq 0$$

and consequently $\det A[\alpha] \geq 0$.

Since A is a P_0–matrix in $\mathcal{C}(DPD^{-}1)$, by using diagonal similarity, we conclude that there exists a P_0–matrix in $\mathcal{C}(P)$. $\square$

Among the directed graphs associated with not combinatorially symmetric sign pattern matrices, the acyclic graphs and the directed cycles play an important role.

The next theorem states that all sign pattern matrices whose associated graphs are acyclic require the property of belonging to the class of P_0–matrices.

Theorem 3.3. *If $P = (p_{ij})$ is an $n \times n$ sign pattern matrix, with $p_{ii} \in \{0, +\}$ for all i, whose associated graph $G(P)$ is acyclic, all matrices in $\mathcal{C}(P)$ are P_0–matrices.*

Proof. Since $G(P)$ is acyclic, there exists a permutation pattern Q such that $\tilde{P} = Q^T PQ$ is upper triangular. Let B be the permutation matrix in $\mathcal{C}(Q)$. Given $A \in \mathcal{C}(P)$, it is easy to see that $\tilde{A} = B^T AB$ is an element of $\mathcal{C}(\tilde{P})$. Since $\tilde{A}$ is upper triangular it is obvious that $\tilde{A}$ is a P_0–matrix. Hence, $A = B\tilde{A}B^T$ is also a P_0–matrix. $\square$

We now focus on the case of directed cycles. Given a cycle $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ in a graph $G(P)$, where $P = (p_{ij})$ is a sign pattern matrix, we define the *sign of the cycle* as 1 if $p_{i_1 i_2} p_{i_2 i_3} \ldots p_{i_{k-1}, i_k} p_{i_k, i_1} = +$ and as -1 if $p_{i_1 i_2} p_{i_2 i_3} \ldots p_{i_{k-1}, i_k} p_{i_k, i_1} = -$.

Theorem 3.4. *Let $P = (p_{ij})$ be an $n \times n$ sign pattern matrix, with $p_{ii} = 0$ for all i, whose associated graph $G(P)$ is a directed cycle. Then the following statements are equivalent:*

1. *The sign of the cycle is $(-1)^{n+1}$.*

2. *There exists a P_0–matrix in $\mathcal{C}(P)$.*

3. *All matrices in $\mathcal{C}(P)$ are P_0–matrices.*

Proof. Without loss of generality, we may assume that any matrix $A \in \mathcal{C}(P)$ is of the form

$$A = \begin{bmatrix} 0 & a_{12} & 0 & \ldots & 0 & 0 \\ 0 & 0 & a_{23} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_{n-1n} \\ a_{n1} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

where $\text{sign}(a_{ij}) = p_{ij}$ for all choices of i and j. We have

$$\det A = (-1)^{n+1} a_{12} a_{23} \ldots a_{n-1n} a_{n1}.$$

Let $\alpha \subset \{1, \ldots, n\}$. If $1 \notin \alpha$ or $n \notin \alpha$, $A[\alpha]$ is upper triangular with zero diagonal elements. If $j \in \alpha$ and $j+1 \notin \alpha$ $(j \leq n-2)$, all the components of the jth–line of $A[\alpha]$ are zero. Hence $\det A[\alpha] = 0$. This implies A is a P_0–matrix if and only if the sign of the cycle is $(-1)^{n+1}$. $\square$

4 Remarks

The general case where P is a sign pattern matrix whose associated graph is cyclic is still open. Taking into account the necessary and sufficient condition presented for directed cycles of length n, we expected that the generalization of such condition given by

(*) the directed cycles of length k in $G(P)$ have sign $(-1)^{k+1}$

would be a necessary condition for the admissibility of a sign pattern matrix P, whose associated graph is cyclic, for the class of P_0–matrices. But it is not. In fact, the sign pattern matrix

$$P = \begin{bmatrix} 0 & - & 0 & - \\ + & 0 & - & + \\ 0 & + & 0 & - \\ + & - & + & 0 \end{bmatrix}$$

is admissible for the class of P_0–matrices, since

$$A = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 2 & -2 & 3 & 0 \end{bmatrix}$$

is a P_0–matrix in $\mathcal{C}(P)$. Still, the sign of the directed cycle $(1,2),(2,4),(4,1)$ is $-1 \neq (-1)^{3+1}$.

It is not surprising, however, that the condition (*) is sufficient for the admissibility of any sign pattern matrix P with zero diagonal entries and, moreover, that all matrices in the class $\mathcal{C}(P)$ are P_0–matrices. In fact, consider $A \in \mathcal{C}(P)$ and recall

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Since all p_{ii} are zero, the summand corresponding to each permutation σ is zero if σ does not satisfy $\sigma(i) \neq i$, for all choices of i. Each permutation σ for which the summand is nonzero corresponds to a product of directed cycles $C_1, C_2, \ldots, C_k$ in $G(P)$ of length $l_1, l_2, \ldots, l_k$, respectively, where $l_1 + l_2 + \ldots + l_k = n$. From the condition on the directed cycles of length k, we know that the sign of each one of the cycles C_i is $(-1)^{l_i+1}$. Therefore, for such a permutation σ, both $\operatorname{sign}(\sigma)$ and $\operatorname{sign}\left(\prod_{i=1}^{n} a_{i\sigma(i)}\right)$ are equal to $(-1)^{l_1+1}(-1)^{l_2+1}\ldots(-1)^{l_k+1}$. This means that the summand corresponding to σ is positive. Hence, $\det A \geq 0$. Given $\alpha \subseteq \{1, \ldots, n\}$, if $|\alpha| = 1$ then $\det A[\alpha] = 0$ and if $|\alpha| \geq 2$ using the same reasoning as for $\det A$ we get $\det A[\alpha] \geq 0$. These allows us to assert that A is a P_0–matrix.

Obviously, condition (*) is too strong. Yet, it is a necessary condition in some very special cases. A sign pattern $P = (p_{ij})$ is said to be *asymmetric* if $p_{ij} \neq 0$ implies $p_{ji} = 0$, for all distinct i, j. It is not hard to check that condition (*), besides being a sufficient condition for the admissibility of an asymmetric sign pattern matrix with all diagonal entries zero for the class of P_0–matrices, it is also necessary for $n = 2, 3, 4$. However it is not necessary for such sign pattern matrices in general. In fact,

$$P = \begin{bmatrix} 0 & + & + & 0 & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & + \\ - & 0 & 0 & 0 & 0 \end{bmatrix}$$

is an asymmetric sign pattern matrix which is admissible for the class of P_0–matrices, since

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a P_0–matrix in $\mathcal{C}(P)$. Still, the sign of the directed cycle $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $(5,1)$ is $-1 \neq (-1)^{5+1}$.

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