

String-corrected dilatonic black holes in d dimensions

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Abstract

We solve the dilaton field equation in the background of a spherically symmetric black hole in bosonic or heterotic string theory with curvature-squared corrections in arbitrary d spacetime dimensions. We then apply this result to obtain a spherically symmetric black hole solution with dilatonic charge and curvature-squared corrections in bosonic or heterotic string theory compactified on a torus. For this black hole we obtain its free energy, entropy, temperature, specific heat and mass.

1 Introduction

Black holes have been over the last years in many ways the best object of predictions from string theory. Some of such predictions are based on considering stringy effects, and how they affect the classical black hole solutions and their properties.

A frequently considered stringy effect is the result of corrections in the inverse string tension (α') in the form of higher-derivative terms in the effective action. Curvature-squared corrections to spherically symmetric d -dimensional black holes in string theory were first discussed in [1]. This article only addresses the effect of the α' corrections; no other string effects are considered. More recently, article [2] considers other typical stringy effects, namely string momentum and winding after compactification of a fundamental string on an internal circle and T -dualization.

In this article we wish to study the effects on spherically symmetric black holes of string compactification on a torus from 10 (or 26) to arbitrary d dimensions. In such compactification, one must pass to the string to the Einstein frame, by a conformal transformation on the original 10 (or 26) dimensions involving the dilaton field. Therefore we need to be in the presence of a dilaton field. We must then determine the solution to the dilaton in the background of a spherically symmetric black hole. We show that the dilaton vanishes classically and, therefore, one must really consider higher-curvature terms. This result had already been anticipated in [1], where the authors take black holes directly in d dimensions and just suggest, but do not fully consider, the effects of string compactification and of the presence of the dilaton.

The article is organized as follows: in section 2, we will solve the dilaton field equation, in the background of a spherically symmetric black hole in d dimensions, in the presence of curvature-squared corrections. Next, in section 3 we find out how the presence of such dilaton actually changes the black hole in bosonic or heterotic string theory, by considering such strings compactified on a torus. Finally we derive some thermodynamical properties of such black hole solution (free energy, entropy, temperature, specific heat and mass), which we compare to the equivalent results of the similar (nondilatonic) solution of [1].

2 The dilaton in the background of a d -dimensional black hole with \mathcal{R}^2 corrections

The most general static, spherically symmetric metric in d spacetime dimensions can be written in spherical coordinates as

$$ds^2 = -f(r) dt^2 + g^{-1}(r) dr^2 + r^2 d\Omega_{d-2}^2. \quad (1)$$

f, g are arbitrary functions of the radius r ; $d\Omega_{d-2}^2 = \sum_{i=2}^{d-1} \frac{\prod_{j=2}^i \sin^2 \theta_j}{\sin^2 \theta_i} d\theta_i^2$ is the element of solid angle in the $(d-2)$ -sphere. For pure Einstein-Hilbert gravity in vacuum, the solution to the Einstein equations gives [3]

$$f(r) = g(r) = 1 - \left(\frac{r_H}{r}\right)^{d-3}, \quad (2)$$

r_H being the horizon radius. This is the d -dimensional extension of Schwarzschild's solution.

We are interested in extending this solution in the presence of a dilaton, but considering string-theoretical α' corrections. We are focusing in particular in $\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ corrections, to first order in α' , which are present in bosonic and heterotic string effective actions (but not on type II superstring) [4]. The effective action we are thus considering, in the Einstein frame, is

$$\frac{1}{16\pi G} \int \sqrt{-g} \left(\mathcal{R} - \frac{4}{d-2} (\partial^\mu \phi) \partial_\mu \phi + e^{\frac{4}{d-2}\phi} \frac{\lambda}{2} \mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} \right) d^d x. \quad (3)$$

Here $\lambda = \frac{\alpha'}{2}, \frac{\alpha'}{4}$ and 0, for bosonic, heterotic and type II strings, respectively. We are only considering gravitational terms: we can consistently settle all fermions and gauge fields to zero for the moment. That is not the case of the dilaton, as it can be seen from the field equations (neglecting terms which are quadratic in ϕ):

$$\nabla^2 \phi - \frac{\lambda}{4} e^{\frac{4}{d-2}\phi} (\mathcal{R}_{\rho\sigma\lambda\tau} \mathcal{R}^{\rho\sigma\lambda\tau}) = 0, \quad (4)$$

$$\mathcal{R}_{\mu\nu} + \lambda e^{\frac{4}{d-2}\phi} \left(\mathcal{R}_{\mu\rho\sigma\tau} \mathcal{R}_\nu{}^{\rho\sigma\tau} - \frac{1}{2(d-2)} g_{\mu\nu} \mathcal{R}_{\rho\sigma\lambda\tau} \mathcal{R}^{\rho\sigma\lambda\tau} \right) = 0. \quad (5)$$

From (4), a constant-dilaton solution would imply the vanishing of the source term $\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$: $\phi = 0$ is a consistent solution only if one takes $\lambda = 0$. This is the solution for ϕ we take at this order. For the particular, spherically symmetric case we are considering, we take (1) with $f(r) = g(r)$ given by (2) as the $\lambda = 0$ metric. We are interested in computing the first α' corrections to ϕ and $g_{\mu\nu}$, using (4) and (5) and always working perturbatively in λ , neglecting λ^2 and higher order terms.

In order to avoid ghosts, the gravitational correction should in principle be given by the Gauss-Bonnet combination $\mathcal{R}_{GB}^2 := \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2$ but, since we are only interesting in computing first-order perturbative in α' corrections to a classical solution, one can neglect the Ricci terms in the low-energy effective action, which from (5) would only contribute at a higher order in α' ; also, as a simple computation shows, for the particular solution (2) which we take as a background both terms are equivalent, since for this case one has $4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} \equiv \mathcal{R}^2$ and, therefore,

$$\mathcal{R}_{GB}^2 \equiv \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} = \frac{2(d-2)(d-3)}{r^4} (1-f)^2 + \frac{2(d-2)}{r^3} f'^2 + (f'')^2 = (d-2)^2 (d-3)(d-1) \frac{r_H^{2d-6}}{r^{2d-2}}. \quad (6)$$

We get then $\nabla^\mu \nabla_\mu \phi(r) = (f\phi')' + \frac{d-2}{r} f\phi'$, from which $r^{d-2} \nabla^\mu \nabla_\mu \phi(r) = (r^{d-2} f\phi')'$. We can take the $\lambda = 0$ metric in (4) in order to compute both $\nabla^\mu \nabla_\mu \phi(r)$, since ϕ is of order λ , and $\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$ (given in (6)), since this term is already multiplied by λ . Putting everything together, we write (4) as

$$\left((r^{d-2} - r_H^{d-3} r) \phi' \right)' = \lambda \frac{(d-2)^2 (d-3)(d-1)}{4} \frac{r_H^{2d-6}}{r^d}, \quad (7)$$

which we simply integrate to obtain

$$(r^{d-2} - r_H^{d-3} r) \phi' = -\lambda \frac{(d-2)^2 (d-3)}{4} \frac{r_H^{2d-6}}{r^{d-1}} - (d-3)\Sigma. \quad (8)$$

The integration constant Σ , as will become clear below, is the dilatonic charge. Integrating again, and defining the incomplete Euler beta function as $B(x; a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt$, we find

$$\begin{aligned} \phi(r) &= -\frac{\Sigma}{r_H^{d-3}} \ln \left(1 - \left(\frac{r_H}{r} \right)^{d-3} \right) \\ &- \frac{\lambda}{r_H^2} \frac{(d-2)^2}{8} \left[(d-3) \left(\frac{r_H}{r} \right)^2 + 2 \frac{d-3}{d-1} \left(\frac{r_H}{r} \right)^{d-1} - 2B \left(\left(\frac{r_H}{r} \right)^{d-3}; \frac{2}{d-3}, 0 \right) \right]. \end{aligned} \quad (9)$$

At asymptotic infinity this solution is approximately given by

$$\phi(r) \approx \frac{\Sigma}{r^{d-3}} + \frac{\Sigma r_H^{d-3}}{2r^{2d-6}} + \frac{\lambda}{8} (d-2)(d-3) \frac{r_H^{2d-6}}{r^{2d-4}}, \quad (10)$$

which is the asymptotic limit found in [5]. This solution depends on another parameter, the dilatonic charge Σ , besides the black hole parameters r_H and λ , which could in principle be a sign for primary hair. However, having a black hole solution means one only has a coordinate (but not curvature) singularity at the horizon. From the dilaton field equation (4), then, also $\phi(r)$ and $\phi'(r)$ must be nonsingular at r_H . From (8) we see that, in order to avoid ϕ' becoming infinite at $r = r_H$, one must choose an adequate value for Σ , given by

$$\Sigma = -\frac{(d-2)^2}{4} \lambda r_H^{d-5}. \quad (11)$$

Equation (9) with Σ given by (11) is the solution for the dilaton in the background of a spherically symmetric black hole with $\mathcal{R}^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma}$ corrections in d dimensions. This dilaton solution acts as secondary hair, since it does not introduce any new physical parameter besides the ones of the black hole. The parameter $\frac{2}{d-3}$ is an integer for $d = 4, 5$; only for these values of d the function $B \left(\left(\frac{r_H}{r} \right)^{d-3}; \frac{2}{d-3}, 0 \right)$ can be written in terms of elementary functions of calculus.¹ In particular, for $d = 4$ our solution matches perfectly the result of [7], as it should.

At the horizon, ϕ is indeed regular and given by ²

$$\phi(r_H) = -\frac{\lambda}{r_H^2} \frac{(d-2)^2}{8(d-1)} \left(d^2 - 2d + 2(d-1) \left(\psi^{(0)} \left(\frac{2}{d-3} \right) + \gamma \right) - 3 \right). \quad (12)$$

For $d = 4$, $\psi^{(0)}(2) = 1 - \gamma$; for $d = 5$, $\psi^{(0)}(1) = -\gamma$. Again, for higher values of d , $\phi(r_H)$ depends explicitly on γ and $\psi^{(0)} \left(\frac{2}{d-3} \right)$, but for $d = 4, 5$ this dependence can always be eliminated. The same is true in general for other expressions that we will meet later.

From (8) and (11), the derivative of the dilaton is given by

$$\phi'(r) = \lambda \frac{(d-3)(d-2)^2}{4} \frac{r_H^{d-3}}{r^d} \frac{1 - \left(\frac{r}{r_H} \right)^{d-1}}{1 - \left(\frac{r}{r_H} \right)^{d-3}},$$

a strictly positive function for $r > r_H$; we conclude that, outside the horizon, ϕ grows between $\phi(r_H)$ given by (12) and 0, its value at infinity.

The article [1] determines the equivalent dilaton solution in the string frame, where the field equations for the dilaton are different than in the Einstein frame we are considering. The final

¹It is interesting to notice that exactly the same argument can be used to show that, also for toroidally compactified string theory, finite-horizon-area black holes which are asymptotically flat only exist (in the supergravity approximation, without α' corrections) for $d = 4, 5$. In such approximation, one has $g_{rr} = \left(1 + \left(\frac{M}{r-r_H} \right)^{d-3} \right)^{\frac{2}{d-3}}$. Solutions obtained by string theory compactifications always give a positive integer as an exponent; therefore, in the same way $\frac{2}{d-3}$ must be an integer. See [6], chapter 11.

²The digamma function is given by $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\Gamma(z)$ being the usual Γ function. For positive n , one defines $\psi^{(n)}(z) = d^n \psi(z)/d z^n$. This definition can be extended for other values of n by fractional calculus analytic continuation. These are meromorphic functions of z with no branch cut discontinuities.

γ is Euler's constant, defined by $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right)$, with numerical value $\gamma \approx 0.577216$.

expression is relatively complicated; since we don't need it here, we refer to the appendix of [1]. The two solutions can be mapped by a transformation of the horizon radius and are equivalent up to a shift by a constant value which depends on d . (The horizon radii are different in the two frames; the relation between them can be seen explicitly in [2].) Because of such shift, the solution (9) we present here, besides being more concise and elegant, is the one which is normalized to vanish asymptotically, according to (10).

3 A black hole solution with \mathcal{R}^2 corrections for toroidal compactifications

It would be interesting to obtain the α' -corrected black hole solution coupled to the α' -corrected dilaton. In ref. [5] only approximations (at asymptotic infinity and close to the horizon) are obtained. But in this article the authors consider a primary-hair kind of dilaton, like (10), but with Σ as an independent parameter. The dilaton solution of [5] seems to be nonvanishing already at order $\lambda = 0$, but the true physical solution (the only one which is nonsingular at the horizon) is the one we have taken, with Σ given by (11). Because Σ depends on λ , this solution vanishes at order $\lambda = 0$. Having a nonvanishing dilaton only at order λ means that, when solving the field equations, one can discard several terms depending on ϕ in the perturbative expansion, which were not discarded in [5].

The article [1] presents the λ -corrected metric, for the system we are considering, in the Einstein frame, in a system of coordinates such that the horizon radius r_H is fixed and has no α' corrections. The result (the Callan–Myers–Perry solution) is of the form (1), with $f(r) = g(r) \equiv g_{CMP}(r)$, where

$$g_{CMP}(r) = \left(1 - \left(\frac{r_H}{r}\right)^{d-3}\right) \left[1 - \lambda \frac{(d-3)(d-4)}{2} \frac{r_H^{d-5} r^{d-1} - r_H^{d-1}}{r^{d-1} r^{d-3} - r_H^{d-3}}\right]. \quad (13)$$

This article [1] only considers (bosonic and heterotic) string theory black hole solutions on arbitrary spacetime dimensions in the presence of a dilaton. No other string effects are considered. Recently, the article [2] has considered black holes in any dimension formed by a fundamental string compactified on an internal circle with any momentum n and winding w , both at leading order and with leading α' corrections, by adding an additional coordinate to the solution of [1], boosting along this direction, reducing again to d dimensions, T -dualizing (to change string momentum into winding) and then boosting one other time to add momentum charge. Einstein–Maxwell–dilaton black holes with \mathcal{R}^2 corrections in any dimension have been considered in [8]. But no solution considers the effects of string compactification from 10 or 26 to d dimensions.

String theories live in d_s dimensions, with $d_s = 26$ for bosonic and $d_s = 10$ for heterotic strings. When one talks about a black hole in string theory in d dimensions, the original d_s -dimensional spacetime must have been compactified on some $(d_s - d)$ -dimensional manifold, with internal coordinates y^m and internal metric $g_{mn}(y)$. When passing from the string to the Einstein frame, one needs a transformation under which

$$g_{\mu\nu} \rightarrow \exp\left(\frac{4}{d-2}\Phi\right) g_{\mu\nu}, \quad \mathcal{R}_{\mu\nu}{}^{\rho\sigma} \rightarrow \tilde{\mathcal{R}}_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - \delta_{[\mu}^{[\rho} \nabla_{\nu]} \nabla^{\sigma]} \Phi. \quad (14)$$

If one takes this as a conformal transformation of the entire d_s -dimensional metric (rather than just on the d -dimensional black hole part, as it was done in [1] to obtain (13)), it involves the total dilaton field Φ , including the Kaluza–Klein part depending on the internal coordinates y^m (rather than just the d -dimensional part ϕ as we have been considering). This way the size of the compact space becomes spatially varying, being governed by a function h . The total metric is then of the form

$$ds^2 = -f(r) dt^2 + g^{-1}(r) dr^2 + r^2 d\Omega_{d-2}^2 + h g_{mn}(y) dy^m dy^n. \quad (15)$$

Taking the field equations for the whole spacetime, the compact space and the black hole are no longer decoupled, due to the term $g_{\mu\nu} \mathcal{R}_{\rho\sigma\lambda\tau} \mathcal{R}^{\rho\sigma\lambda\tau}$ in (5). In order to avoid this problem, we take the internal space to be a flat torus, with vanishing internal curvature to leading order. If this is the case, the function h can be shown to depend only on the d -dimensional part of the dilaton ϕ , i.e. $h = h(\phi)$.

The solution (15) to (5) is then [1]

$$h(\phi) = \left(1 - \frac{2}{d_s - 2}\phi\right)^2, \quad (16)$$

$$f(r) = g(r) + 4 \left(1 - \left(\frac{r_H}{r}\right)^{d-3}\right) \frac{d_s - d}{(d_s - 2)^2} (\phi - r\phi'), \quad (17)$$

with $g(r)$ being equal to $g_{CMP}(r)$ given by (13). Explicitly, using the dilaton solution (9) for ϕ ,

$$\begin{aligned} f(r) = & \left(1 - \left(\frac{r_H}{r}\right)^{d-3}\right) \left(1 - \frac{(d-3)(d-4)}{2} \frac{\lambda}{r_H^2} \left(\frac{r_H}{r}\right)^{d-3} \frac{1 - \left(\frac{r_H}{r}\right)^{d-1}}{1 - \left(\frac{r_H}{r}\right)^{d-3}}\right. \\ & + (d-2)^2 \frac{d_s - d}{(d_s - 2)^2} \frac{\lambda}{r_H^2} \left[\ln \left(1 - \left(\frac{r_H}{r}\right)^{d-3}\right) + B \left(\left(\frac{r_H}{r}\right)^{d-3}; \frac{2}{d-3}, 0\right) \right. \\ & \left. \left. - \frac{d-3}{2} \left(\frac{r_H}{r}\right)^2 - \frac{d-3}{d-1} \left(\frac{r_H}{r}\right)^{d-1} - (d-3) \left(\frac{r_H}{r}\right)^{d-3} \frac{1 - \left(\frac{r_H}{r}\right)^{d-1}}{1 - \left(\frac{r_H}{r}\right)^{d-3}} \right] \right). \quad (18) \end{aligned}$$

As one sees from (17), when $d = d_s$ one has $f(r) = g(r)$. This is to be expected: in this case there is no compactification, and (15) reduces to (1), with $f(r) = g(r) = g_{CMP}(r)$ given by (13).

4 Thermodynamical properties

In this section, we compute several thermodynamical quantities for the black hole solution we have just found. In each case we compare the result to the corresponding one of the solution (13) obtained in [1], since the parameters are the same. This way we can evaluate the physical effects introduced by the toroidal compactification.

The free energy of the black hole solution (17) is obtained from the euclideanized Einstein-frame action (3), to which one adds a surface term consisting of an integral (on the boundary) of the trace of the second fundamental form, subtracted by the same trace for the boundary embedded on flat space, to render the total surface contribution finite. This surface term also includes contributions for the higher-derivative terms, but these contributions do not affect this calculation. Also, there exists a choice of fields such that all the terms in the euclidean action directly involving the dilaton, the Kaluza-Klein scalar and the $(d_S - d)$ -torus metric are of order λ^2 and can, therefore, be neglected. This means in particular that the result for the free energy for our solution is the same that for the solution (13), whose calculation is described in [1]; the result is $(\Omega_{d-2} = 2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2}))$ being the area of the unit $(d-2)$ -sphere) [2]

$$F = \left(1 - \frac{d(d-3)}{2} \frac{\lambda}{r_H^2}\right) \frac{\Omega_{d-2}}{16\pi G} r_H^{d-3}. \quad (19)$$

The entropy of this black hole solution can be obtained by Wald's formula [9]

$$S = -2\pi \int_H \frac{\partial \mathcal{L}}{\partial R^{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} \sqrt{h} d\Omega_{d-2},$$

since from (3) we are dealing with a lagrangian \mathcal{L} with higher derivatives. H is the black hole horizon, with area $A_H = r_H^{d-2} \Omega_{d-2}$ and metric h_{ij} induced by the spacetime metric $g_{\mu\nu}$. For the metric (15), the nonzero components of the binormal $\varepsilon^{\mu\nu}$ to H are $\varepsilon^{tr} = -\varepsilon^{rt} = -\sqrt{\frac{g}{f}}$. From (3) one also needs

$$8\pi G \frac{\partial \mathcal{L}}{\partial R^{\mu\nu\rho\sigma}} = \frac{1}{4} (g_{\mu\rho} g_{\sigma\nu} - g_{\mu\sigma} g_{\rho\nu}) + e^{\frac{4}{d-2}\phi} \frac{\lambda}{2} R_{\mu\nu\rho\sigma}.$$

This way, taking only nonzero components, one gets from (15)

$$8\pi G \frac{\partial \mathcal{L}}{\partial R^{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} = 4 \times 8\pi G \frac{\partial \mathcal{L}}{\partial R^{trtr}} \varepsilon^{tr} \varepsilon^{tr} = \left(-\frac{f}{g} + e^{\frac{4}{d-2}\phi} \lambda f''\right) \frac{g}{f}.$$

At order $\lambda = 0$, $\phi = 0$, $f = g$ and $f'' = -\frac{1}{r_H^2}(d-3)(d-2)$. Therefore

$$S = \frac{1}{4G} \int_H \left(1 + \frac{\lambda}{r_H^2}(d-3)(d-2) \right) \sqrt{h} d\Omega_{d-2} = \frac{A_H}{4G} \left(1 + (d-3)(d-2) \frac{\lambda}{r_H^2} \right). \quad (20)$$

Because the λ -correction to the entropy depends only on the $\lambda = 0$ part of the metric, it is no surprise that this same result was obtained (by a different process, though, and for a metric (13) with a different α' correction) in [1].

In order to compute the black hole temperature, one first Wick-rotates the metric (15) to Euclidean time $t = i\tau$. The resulting manifold has no conical singularities as long as τ is a periodic variable, with a period β related to the black hole temperature as $T = \frac{1}{\beta}$. The precise smoothness condition is $2\pi = \lim_{r \rightarrow r_H} \frac{\beta}{g^{-\frac{1}{2}}(r)} \frac{df^{\frac{1}{2}}(r)}{dr}$, from which one gets $T = \lim_{r \rightarrow r_H} \frac{\sqrt{g} d\sqrt{f}}{2\pi dr}$. In our particular case,

$$\begin{aligned} T &= \frac{d-3}{4r_H\pi} \left[1 + \frac{\lambda}{r_H^2} \delta T(d, d_s) \right], \\ \delta T(d, d_s) &= \frac{1}{4(d-1)(d_s-2)^2} [3d^5 - (3d_s+18)d^4 + (-2d_s^2+26d_s+27)d^3 \\ &+ (12d_s^2-83d_s+28)d^2 - 2(9d_s^2-46d_s+38)d + 4(2d_s^2-7d_s+8) \\ &+ 2(d-2)^2(d-1)(d-d_s) \left(\psi^{(0)} \left(\frac{2}{d-3} \right) + \gamma \right)]. \end{aligned} \quad (21)$$

We have checked that δT and, therefore, the correction term to the temperature, are always negative: α' corrections decrease the temperature for every relevant values of d and d_s . If one takes the approximate expression (21) as exact, one may even get $T < 0$ to first order in λ for some values of d , d_s and λ/r_H^2 . From our evaluation of δT , we concluded that the approximate expression for T given by (21) is positive as long as $\alpha' < 0.148148r_H^2$ (for $d_s = 10$) or $\alpha' < 0.00727273r_H^2$ (for $d_s = 26$). But (21) is only a first-order perturbative approximation; a complete analysis would require a full knowledge of T to all orders. Nonetheless, the leading string correction being negative suggests that the temperature may reach a maximum, for each particular given value of d , d_s , approximately for $r_H = \sqrt{-3\delta T(d, d_s)\lambda}$ (again taking (21) as an exact expression, a good approximation if the higher-order α' corrections are much smaller than the first-order one we are considering, something that should be true at least for large black holes). For all possible values of d and d_s , we evaluated $T = \frac{d-3}{6r_H\pi}$ (which is what one obtains after replacing $r_H = \sqrt{-3\delta T(d, d_s)\lambda}$ in (21)). For $d_s = 10$ we obtained a maximum $T_{\max} = \frac{0.082}{\sqrt{\alpha'}}$ for $d = 10$, while for $d_s = 26$ we obtained a maximum $T_{\max} = \frac{0.071}{\sqrt{\alpha'}}$ for $d = 4$. Like the ones corresponding to the solution (13) determined in [1], these values are smaller than the critical Hagedorn temperatures, obtained from the free string spectrum, and given by $T_{\text{crit}} = \frac{0.16}{\sqrt{\alpha'}}$ (for the heterotic string, with $d_s = 10$) and $T_{\text{crit}} = \frac{0.08}{\sqrt{\alpha'}}$ (for the bosonic string, with $d_s = 26$).

It is interesting to compare the value $\delta T(d, d_s)$ we obtained with the corresponding one for the noncompactified solution (13) from [1]. For this solution, the temperature is given by

$$T_{CMP} = \frac{d-3}{4r_H\pi} \left[1 - \frac{\lambda}{r_H^2} \frac{(d-1)(d-4)}{2} \right]. \quad (22)$$

We have checked that $\delta T(d, d_s) < -\frac{(d-1)(d-4)}{2}$, i.e. the decrease in T due to α' corrections is larger for (17) than for (13), for every relevant values of d and d_s . The only exception is precisely when $d = d_s$, when $\delta T(d, d_s) \equiv -\frac{(d-1)(d-4)}{2}$, for the reasons we have already mentioned.

The specific heat is given by $C = T \frac{\partial S}{\partial T} = T \frac{dS}{dr_H}$. In our case, one is left with

$$\begin{aligned} C &= -(d-2) \frac{A_H}{4G} \left[1 + \frac{\lambda}{r_H^2} \delta C(d, d_s) \right], \\ \delta C(d, d_s) &= -\frac{d-2}{2(d-1)(d_s-2)^2} [62d_s - 16d_s^2 - 64 + 3d^4 - 3(d_s+4)d^3 - (2d_s(2d_s-14)+5)d^2 \\ &+ (d_s(20d_s-91)+82)d + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)} \left(\frac{2}{d-3} \right) + \gamma \right)]. \end{aligned} \quad (23)$$

We checked that $\delta C(d, d_s)$ is always positive for every relevant value of d and d_s , which means α' -corrected black holes keep being thermodynamically unstable.

For the noncompactified solution (13), the specific heat is given by

$$C_{CMP} = -(d-2) \frac{A_H}{4G} \left[1 + 2(d-4)(d-2) \frac{\lambda}{r_H^2} \right]. \quad (24)$$

It is also interesting to compare the value $\delta C(d, d_s)$ we obtained with the corresponding one for the noncompactified solution (13). We checked that $\delta C(d, d_s) > 2(d-4)(d-2)$, for every relevant value of d and d_s except when $d = d_s$. This means the α' correction is bigger, i.e. C becomes more negative with (17) than with (13).

The black hole inertial mass matches the result of solution (13) of [1]:

$$M_I = M_{CMP} = \frac{(d-2)\Omega_{d-2}}{16\pi G} \lim_{r \rightarrow \infty} r^{d-3} (1 - g(r)) = \left(1 + \frac{(d-3)(d-4)}{2} \frac{\lambda}{r_H^2} \right) \frac{(d-2)\Omega_{d-2}}{16\pi G} r_H^{d-3}. \quad (25)$$

Since $g(r) \neq f(r)$, one expects the black hole inertial and gravitational masses to be different. This situation is usual when one is dealing with compactifications and originates from the integration of Kaluza-Klein modes in the full d_s -dimensional action, resulting in a d -dimensional action with nondiagonal kinetic terms. Indeed, from (10) and (17), one gets

$$M_G = \frac{(d-2)\Omega_{d-2}}{16\pi G} \lim_{r \rightarrow \infty} r^{d-3} (1 - f(r)) = M_I + \frac{d_s - d}{(d_s - 2)^2} \frac{(d-2)^4 \Omega_{d-2}}{16\pi G} \frac{\lambda}{r_H^2} r_H^{d-3}. \quad (26)$$

The actual physical mass is obtained by the relation $M = ST + F$. From (19), (20), (21),

$$\begin{aligned} M &= \left[1 + \frac{\lambda}{r_H^2} \delta M(d, d_s) \right] \frac{(d-2)\Omega_{d-2}}{16\pi G} r_H^{d-3}, \\ \delta M(d, d_s) &= \left[3d^4 - 3(d_s + 4)d^3 + (2d_s(d_s + 2) + 19)d^2 + (d_s(-10d_s + 29) - 38)d + 2d_s(4d_s - 17) \right. \\ &\quad \left. + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)} \left(\frac{2}{d-3} \right) + \gamma \right) + 32 \right] \frac{(d-3)}{4(d-1)(d_s-2)^2}. \end{aligned} \quad (27)$$

The sign of $\delta M(d, d_s)$ depends on its parameters. For $d = 4$ and $d = 5, d_s = 10$ it is negative, i.e. M decreases with α' corrections; for $d = 5, d_s = 26$ and $d > 5$ it is positive, which means α' corrections increase M . It is important to verify that, taking (27) as an exact expression, one does not get a negative mass to first order in λ , i.e. when $\delta M(d, d_s)$ is negative. We verified that for such cases, exactly like we did with the temperature. The limits are much less restrictive this time: M given by (27) is positive as long as $\alpha' < 8.82759 r_H^2$ (for $d_s = 10$) or $\alpha' < 10.8339 r_H^2$ (for $d_s = 26$). Again, a complete analysis would require a full knowledge of M to all orders.

We also compared the value $\delta M(d, d_s)$ we obtained with the corresponding one for the noncompactified solution (13) from [1] given by (25). We checked that $\delta M(d, d_s) < \frac{(d-3)(d-4)}{2}$, i.e. the increase in M due to α' corrections is smaller for (17) than for (13), for every relevant value of d and d_s , except when $d = d_s$.

One can invert (27) to get the horizon radius as a function of the mass, obtaining

$$\begin{aligned} r_H &= \frac{8^{\frac{1}{d-3}}}{\sqrt{\pi}} \left(\frac{GM\Gamma\left(\frac{d-1}{2}\right)}{d-2} \right)^{\frac{1}{d-3}} \left[1 - \frac{4^{\frac{d}{3-d}} \pi}{(d-1)(d_s-2)^2} \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{2}{d-3}} \lambda \right. \\ &\quad \left. \left[3d^4 - 3(d_s + 4)d^3 + (2d_s(d_s + 2) + 19)d^2 + (d_s(-10d_s + 29) - 38)d \right. \right. \\ &\quad \left. \left. + 2d_s(4d_s - 17) + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)} \left(\frac{2}{d-3} \right) + \gamma \right) + 32 \right] \right]. \end{aligned} \quad (28)$$

This expression must be interpreted with care. In (27) we obtained the leading perturbative correction to M as a function of r_H , but only if we knew the full expression $M(r_H)$, including all the string corrections, could we eventually invert it, and obtain an expression for r_H as a function of the full physical string-corrected mass (and not just the classical mass, given by setting $\lambda = 0$). Equation (28)

represents the leading term in a series, but it does not represent by itself a string correction to r_H . This is because (28) hides the fact that M itself has string corrections. If one considers those string corrections on M , they should be such that they would eventually be cancelled, at every order in λ , when taken all together. Indeed, by assumption, r_H receives no α' corrections [1] - it is the only free parameter of the solution and, together with λ, d, d_s determines all the physical quantities.

It is useful to express the thermodynamical quantities we have been computing in terms of the physical mass, by replacing (28) in (19), (20), (21) and (23). The temperature is expressed as

$$T = \frac{2^{\frac{3-2d}{d-3}}(d-3)}{\sqrt{\pi}} \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{1}{d-3}} \left[1 + \frac{2^{-\frac{2d}{d-3}}(d-2)}{(d_s-2)^2} \pi \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{2}{d-3}} \lambda [8d_s^2 - 31d_s + 32 + 3d^3 - (3d_s+6)d^2 - (2d_s^2 - 14d_s + 9)d + 2(d-1)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) \right] \right], \quad (29)$$

while the free energy, entropy and specific heat are given by

$$F = \frac{M}{d-2} \left[1 - \frac{4^{\frac{d}{3-d}}(d-3)(d-2)\pi}{(d-1)(d_s-2)^2} \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{2}{d-3}} \lambda \left[2(d-1)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) + 3d^3 - (3d_s+6)d^2 + (4d_s^2 - 10d_s + 15)d + d_s(-4d_s + 17) - 16 \right] \right]; \quad (30)$$

$$S = 2^{\frac{2d-3}{d-3}} \frac{M}{d-2} \sqrt{\pi} \left(\frac{GM\Gamma\left(\frac{d-1}{2}\right)}{d-2} \right)^{\frac{1}{d-3}} \left[1 - \frac{4^{\frac{d}{3-d}}(d-2)^2\pi\lambda}{(d-1)(d_s-2)^2} \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{2}{d-3}} \times [3d^3 - (3d_s+6)d^2 - (2d_s(d_s-7)+9)d - 7d_s + 2d_s^2 + 8 + 2(d-1)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) \right] \right]; \quad (31)$$

$$C = -2^{\frac{2d-3}{d-3}} M \sqrt{\pi} \left(\frac{GM\Gamma\left(\frac{d-1}{2}\right)}{d-2} \right)^{\frac{1}{d-3}} \left[1 - \frac{3(d-2)4^{\frac{d}{3-d}}\pi}{(d-1)(d_s-2)^2} \left(\frac{d-2}{GM\Gamma\left(\frac{d-1}{2}\right)} \right)^{\frac{2}{d-3}} \lambda [3d^4 - 3(d_s+4)d^3 + (-2d_s(d_s-10)+3)d^2 + (d_s(10d_s-51)+42)d + 30d_s - 8d_s^2 + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) - 32] \right]. \quad (32)$$

These variables can also be expressed in terms of the temperature: by inverting (21) instead of (27), one obtains an expression analogous to (28) which, when replaced in (19), (20), and (23), gives

$$F = \frac{2^{3-2d}\pi^{\frac{3-d}{2}}}{G\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{d-3}{T} \right)^{d-3} \left[1 + \frac{4(d-2)^2\pi^2\lambda T^2}{(d-3)(d-1)(d_s-2)^2} \left[2(d-1)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) + (3d^3 - (3d_s+6)d^2 - (2d_s(d_s-7)+9)d - 7d_s + 2d_s^2 + 8) \right] \right]; \quad (33)$$

$$S = \frac{2^{3-2d}\pi^{\frac{3-d}{2}}}{G\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{d-3}{T} \right)^{d-2} \left[1 + \frac{4(d-2)^2\pi^2\lambda T^2}{(d-3)^2(d-1)(d_s-2)^2} [3d^4 - 3(d_s+4)d^3 + (-2d_s(d_s-10)+3)d^2 + (d_s(12d_s-59)+50)d - 2(d_s(5d_s-19)+20) + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) \right] \right]; \quad (34)$$

$$C = -\frac{2^{3-2d}(d-2)\pi^{\frac{3-d}{2}}}{G\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{d-3}{T} \right)^{d-2} \left[1 + \frac{4(d-2)(d-4)\pi^2\lambda T^2}{(d-3)^2(d-1)(d_s-2)^2} [3d^4 - 3(d_s+4)d^3 + (-2d_s(d_s-10)+3)d^2 + (d_s(12d_s-59)+50)d - 2(d_s(5d_s-19)+20) + 2(d-1)(d-2)(d-d_s) \left(\psi^{(0)}\left(\frac{2}{d-3}\right) + \gamma \right) \right] \right]. \quad (35)$$

For all these expressions (29)–(35), the same warning we made for (28) applies: they are exact just for $\lambda = 0$ (involving just the classical quantities, without any α' corrections). Beyond the classical limit, these expressions just give an indication of the first-order (in terms of mass or temperature) terms of unknown functions, whose full expressions could only be determined if we knew all these quantities to all orders in α' . The true first-order corrections in α' are those given in equations (19)–(27), in terms of λ/r_H^2 , whose signs and magnitudes we analyzed.

5 Conclusions

In this work, we derived the spherically symmetric solution to a dilaton in the presence of a black hole in string theory with curvature-squared corrections in d spacetime dimensions. We then obtained a spherically symmetric black hole solution with dilatonic charge and curvature-squared corrections from compactified string theory in d dimensions, and we computed its free energy, entropy, temperature, specific heat and mass. We compared the magnitude of the α' corrections to these quantities to the ones corresponding to the noncompactified solution (13) from [1], in order to estimate the effects of string compactification. Free energy is decreased and entropy is increased with α' corrections; the magnitude of the corrections is the same as in the solution (13). Also like in (13), the temperature decreases and the specific heat becomes more negative, but in our case the effects of the α' corrections are strengthened. The α' corrections to the mass, on the contrary, are weakened in comparison to (13) (whose value for the mass is always increased), and for a few values of d they even mean a decrease in M .

In a future work we plan to study some other features of this black hole, like its stability and scattering of gravitational waves.

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