ON THE WELL-POSEDNESS FOR SOME PERTURBATIONS OF THE KDV EQUATION WITH LOW REGULARITY DATA

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ABSTRACT. We study some well-posedness issues of the initial value problem (IVP) associated to the equation

$$u_t + u_{xxx} + \eta Lu + uu_x = 0, \quad x \in \mathbb{R},\, t \geq 0, \tag{$\star$}$$

where $\eta > 0$, $\widehat{Lu}(\xi) = -\Phi(\xi)\hat{u}(\xi)$ and $\Phi \in \mathbb{R}$ is bounded above

Using the theory developed by Bourgain and Kenig Ponce and Vega, we prove that the IVP associated to ($\star$) is locally well-posed for given data in Sobolev spaces $H^s(\mathbb{R})$ with regularity below L^2. Examples of the model ($\star$) are the Ostrovsky-Stepanyams-Tsimring equation for $\Phi(\xi) = |\xi| - |\xi|^3$, the derivative Korteweg-de Vries-Kuramoto-Sivashinsky equation for $\Phi(\xi) = \xi^2 - \xi^4$ and the Korteweg-de Vries-Burguers equation for $\Phi(\xi) = -\xi^2$.

1. Introduction

In this paper we consider the initial value problem (IVP)

$$\begin{cases} u_t + u_{xxx} + \eta Lu + uu_x = 0, & x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $\eta > 0$ is a constant and the linear operator L is defined via the Fourier transform by $\widehat{Lu}(\xi) = -\Phi(\xi)\hat{u}(\xi)$.

The Fourier symbol

$$\Phi(\xi) = \sum_{j=0}^{n}\sum_{i=0}^{2m} c_{i,j}\xi^i|\xi|^j, \quad c_{i,j} \in \mathbb{R},\ c_{2m,n} = -1. \tag{1.2}$$

is a real valued function which is bounded above, i.e., there is a constant C such that $\Phi(\xi) < C$. Without loss of generality, we can suppose that $\Phi(\xi) < 1$. For this, let us

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perform the following scale change

$$v(x,t) = \frac{1}{\lambda^2} u\Big(\frac{x}{\lambda}, \frac{t}{\lambda^3}\Big).$$

Then v satisfies the equation

$$\lambda^3 v_t + \lambda^3 v_{xxx} + \eta T v + \lambda^3 v v_x = 0, \tag{1.3}$$

where

$$\widehat{Tv}(\xi) = \Phi(\lambda\xi)\hat{v}(\xi).$$

If we take $\lambda^3 = C$, where C is as earlier, then the Fourier symbol of the new operator T in (1.3) is bounded above by 1. Finally, inverting the scale change, we obtain well-posedness result for the original IVP (1.1) from that of (1.3). So, throughout this work we consider the IVP (1.1) with $\Phi(\xi)$ in (1.2) satisfying $\Phi(\xi) < 1$.

Our interest here is to obtain well-posedness results to the IVP (1.1) for given data u_0 in Sobolev spaces $H^s(\mathbb{R})$ with regularity below L^2. The L^2-based Sobolev space $H^s(\mathbb{R})$ is defined by

$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi,$$

and $\hat{f}(\xi)$ is the usual Fourier transform given by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

However, from here onwards, we will neglect the factor 2π in the definition of the Fourier transform because it does not alter our analysis.

Also, we consider the homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$ defined via the norm

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

Before stating the main results of this work, we give some examples that belong to the class considered in (1.1).

The first example of this type of equations is the generalized Ostrovsky-Stepanyams-Tsimring (OST) equation.

$$\begin{cases} u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + u^k u_x = 0, \quad x \in \mathbb{R},\, t \geq 0,\, k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x), \end{cases} \tag{1.4}$$

where $\mathcal{H}$ denotes the Hilbert transform.

$$\mathcal{H}g(x) = \text{ P.V.}\frac{1}{\pi}\int \frac{g(x-\xi)}{\xi}d\xi,$$

$u = u(x,t)$ is a real-valued function and $\eta > 0$ is a constant.

Equation (1.1) with $k = 1$ was derived by Ostrovsky et al. in [17] to describe the radiational instability of long waves in a stratified shear flow. Recently, Carvajal and Scialom in [6] considered the IVP (1.4) and proved the local well-posedness results for given data in H^s, $s \geq 0$ when $k = 1, 2, 3$. They also obtained the global well-posedness result for data in L^2 when $k = 1$. The earlier well-posedness results for the IVP (1.4) with $k = 1$ can be found in [1], where for given data in $H^s(\mathbb{R})$, local result when $s > 1/2$ and global result when $s \geq 1$ have been obtained.

Another model that fits in the class (1.1) is the derivative Korteweg-de Vries-Kuramoto Sivashinsky equation

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + uu_x = 0, \quad x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.5}$$

where $u = u(x,t)$ is a real-valued function and $\eta > 0$ is a constant.

This equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (see [8, 18]). The equation (1.5) is a particular case of Benney-Lin equation [2, 18], i.e.

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + \beta u_{xxxxx} + uu_x = 0, \quad x \in \mathbb{R},\, t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.6}$$

when $\beta = 0$.

The IVP associated to (1.5) was studied by Biagioni, Bona, Iorio and Scialom in [3]. They also determined the limiting behavior of solutions as the dissipation tends to zero. Biagioni and Linares proved global well-posedness for the IVP (1.6) for initial data in L^2 in [4].

Another example is the Korteweg-de Vries-Burguers equation

$$\begin{cases} u_t + u_{xxx} - \eta u_{xx} + uu_x = 0, \quad x \in \mathbb{R},\, t \geq 0,\, \eta > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.7}$$

Recently, Molinet and Ribaud considered the IVP (1.7) in [15] and proved that it is locally well-posed for given data in H^s, $s > -1$. The equation (1.7) is also known as the parabolic regularization of the KdV equation with $\eta > 0$. Some years ago, when the interest was to obtain local results for given data in larger Sobolev spaces, this regularization was used to obtain well-posedness results for $\eta > 0$ and then pass the limit $\eta \downarrow 0$. However, this limit is a delicate matter.

Now, we state the main results of this work. The first result deals with the local well-posedness for given data in the Sobolev spaces of negative index.

Theorem 1.1. *The IVP (1.1) with $\eta > 0$ and $\Phi(\xi)$ given by (1.2) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$, $s > -3/4$.*

To prove this theorem we follow the theory developed by Bourgain [5] and Kenig, Ponce and Vega [11]. The main ingredients in the proof are estimates in the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$ (see IVP (1.12) below). The proof we presented here does not use the Bourgain type space associated to the linear part of the IVP (1.1); instead it uses the usual Bourgain space associated to the KdV equation. To carry out this scheme, the Proposition 2.2 plays a fundamental role which permits us to use a bilinear estimate for $\partial_x(u^2)$ (see [11]), that is a central part of our arguments.

The result of the Theorem 1.1 improves the known local well-posedness results for the IVP (1.4) and (1.5) described above. Note that, the value $s > -3/4$, in the case of the Korteweg-de Vries (KdV) equation, is sharp in the sense that for $s < -3/4$, the IVP associated to the KDV equation is ill-posed. We should mention that, the lack of conserved quantities in the spaces with regularity below L^2, prevents us to get global solution using the usual technique.

The second result is concerned with the particular case of the IVP (1.1) for given data in the homogeneous Sobolev space when the Fourier symbol is of the form $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$.

Theorem 1.2. *The IVP (1.1) with $\eta > 0$ and $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$, is locally well-posed for any data $u_0 \in \dot{H}^s(\mathbb{R})$, $s > -1/2$.*

Although this theorem does not improve the result obtained in Theorem 1.1, it is interesting on its own because the proof we present here uses different tools, that are simpler than the ones used in the proof of Theorem 1.1. The main ingredients in the proof are the refined local smoothing effect (see (3.4) in Corollary 3.2 below), and a Strichartz type estimate (see Proposition 4.1 below). Using these estimates we are able to apply fixed point argument to obtain a local well-posedness result in the homogeneous Sobolev spaces of negative order without the use of Bourgain type spaces.

Now we introduce function spaces that will be used to prove the Theorem 1.1. We consider the following IVP associated to the Linear KdV equation

$$\begin{cases} u_t + u_{xxx} = 0, \quad x,\, t \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{1.8}$$

The solution to the IVP (1.8) is given by $u(x,t) = U(t)u_0(x)$, where the unitary group $U(t)$ is defined as

$$\widehat{U(t)u_0}(\xi) = e^{it\xi^3}\widehat{u_0}(\xi). \tag{1.9}$$

For $s, b \in \mathbb{R}$, we define the space $X_{s,b}$ as the completion of the Schwartz space $S(\mathbb{R}^2)$ with respect to the norm

$$\begin{aligned} \|u\|_{X_{s,b}} \equiv \|U(-t)u\|_{H_{s,b}} &:= \|\langle\tau\rangle^b\langle\xi\rangle^s\widehat{U(-t)u}(\xi,\tau)\|_{L^2_\tau L^2_\xi} \\ &= \|\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\widehat{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi}, \end{aligned} \tag{1.10}$$

where $\widehat{u}(\xi,\tau)$ is the Fourier transform of u in both space and time variables. The space $X_{s,b}$ is the usual Bourgain space for the KdV equation (see [5]).

Note that, the IVP (1.1) is defined only for $t \geq 0$. To use Bourgain's type space, we should be able to write the IVP (1.1) for all $t \in \mathbb{R}$. For this, we define

$$\eta(t) \equiv \eta\,\mathrm{sgn}(t) = \begin{cases} \eta & \text{if } t \geq 0, \\ -\eta & \text{if } t < 0 \end{cases} \tag{1.11}$$

and write the IVP (1.1) in the following form

$$\begin{cases} u_t + u_{xxx} + \eta(t)Lu + uu_x = 0, \quad x,\, t \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{1.12}$$

Now we consider the IVP associated to the linear part of (1.12)

$$\begin{cases} u_t + u_{xxx} + \eta(t)Lu = 0, \quad x,\, t \in \mathbb{R}, \\ u(0) = u_0. \end{cases} \tag{1.13}$$

The solution to (1.13) is given by $u(x,t) = V(t)u_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)u_0}(\xi) = e^{it\xi^3+\eta|t|\Phi(\xi)}\widehat{u_0}(\xi). \tag{1.14}$$

Observe that, defining $\widetilde{U}(t)$ by $\widehat{\widetilde{U}(t)u_0}(\xi) = e^{\eta|t|\Phi(\xi)}\widehat{u_0}(\xi)$, the semigroup $V(t)$ can be written as $V(t) = U(t)\widetilde{U}(t)$ where $U(t)$ is the unitary group associated to the KdV equation (see (1.9)).

This paper is organized as follows: In Section 2, we prove Theorem 1.1. In Section 3, we present a refined local smoothing effect when $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, $k \in \mathbb{Z}^+$, in (1.2). In Section 4, we to obtain some Stricharz type estimates. In Section 5, we prove Theorem 1.2.

2. Local Well-posedness in H^s for $s > -3/4$

This section is devoted to supply the proof of the Theorem 1.1. We start by proving some preliminary results.

2.1. **Preliminary estimates.**

Proposition 2.1. *Let $s > -\frac{3}{4}$. There exist $b' \in (-\frac{1}{2}, 0)$ and $\epsilon_s > 0$ such that for any $b \in (\frac{1}{2}, b'+1]$ with $1 - b + b' \leq \epsilon_s$, and $u, v \in X_{s,b}$*

$$\|(uv)_x\|_{X_{s,b'}} \leq c\,\|u\|_{X_{s,b}}\,\|v\|_{X_{s,b}}.$$

Proof. See [11]. □

We consider a cut-off function $\psi \in C^\infty(\mathbb{R})$, such that $0 \leq \psi(t) \leq 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2. \end{cases} \tag{2.1}$$

Let us define $\psi_T(t) \equiv \psi(\frac{t}{T})$ and $\tilde{\psi}_T(t) = \operatorname{sgn}(t)\psi_T(t)$.

The following Proposition plays a central role in the proof of our first main result, the Theorem 1.1. This Proposition allows us to work in the usual $X_{s,b}$ space associated to the KdV equation instead of the Bourgain space associated to the IVP (1.12).

Proposition 2.2. *Let $-1/2 < b' \leq 0$, $T \in [0,1]$. Then for all $\epsilon \geq 3(b - 1/2)$ we have*

$$\|\psi(t)V(t)u_0\|_{X_{s-\epsilon,b}} \leq c\|u_0\|_s. \tag{2.2}$$

If $1/2 < b \leq b'/3 + 2/3$, $s \in \mathbb{R}$ then

$$\|\psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'\|_{X_{s,b}} \leq c\,T^{1+b'/2-3b/2}\|uu_x\|_{X_{s,b'}}, \tag{2.3}$$

where c is a constant.

Before providing proof of this Proposition, we record the following results.

Lemma 2.1. *Let* $1/2 < b < 1$ *and* $a < 1$ *then*

$$\|\psi_T(t)\|_{H_t^b} \leq c(T^{1/2} + T^{1/2-b}), \tag{2.4}$$

$$\|\psi_T(t)\, e^{a|t|}\|_{H_t^b} \leq \begin{cases} c\Big(\frac{T}{|a|}\Big)^{1/2}\Big\langle \frac{1}{T}\Big\rangle^b \langle a\rangle^b & \text{if } a < 0, \\ T^{1/2}\Big\langle \frac{\langle T\rangle}{T^b}\Big\rangle & \text{if } 0 \leq a < 1. \end{cases} \tag{2.5}$$

$$|\mathcal{F}(|t|\, \psi_T(t)\, e^{a|t|})(\tau)| \leq \frac{c\,T^2}{1 + (\tau^2 + a^2)T^2}, \tag{2.6}$$

$$|\mathcal{F}(|t|\, \tilde{\psi}_T(t)\, e^{a|t|})(\tau)| \leq \frac{c\,T^2}{1 + (\tau^2 + a^2)T^2}, \tag{2.7}$$

where c *is a constant independent of* T *and* a*.*

Proof. Using the definition of H^b space we have

$$\|\psi_T(t)\|_{H_t^b} \leq c\,\|\psi_T\|_{L^2} + c\,\|D_t^b \psi_T\|_{L^2} = cT^{1/2}\|\psi\|_{L^2} + cT^{1/2-b}\|D_t^b\psi\|_{L^2},$$

where we used that $\widehat{h(t/T)}(\tau) = T\hat{h}(T\,\tau)$.
If $a < 0$ and $b > 1/2$ we get

$$\|\psi_T(t)\, e^{a|t|}\|_{H_t^b} \leq c\,\|\psi_T(t)\|_{H_t^b}\|e^{-|at|}\|_{H_t^b} \leq c\,(T^{1/2} + T^{1/2-b})(|a|^{-1/2} + |a|^{b-1/2}),$$

and if $0 \leq a < 1$

$$\|\psi_T(t)\, e^{a|t|}\|_{H_t^b} \leq c\,T^{1/2}\|\psi\|_{L^2} + c\|D_t^b(\psi_T(t)\, e^{a|t|})\|_{L^2} \leq c\,(T^{1/2} + T^{1/2-b}\|D_t^b h\|_{L^2}),$$

where $h(t) = \psi(t)e^{aT|t|}$, integrating by parts twice we get

$$|\hat{h}(\tau)| \leq \frac{c\langle T\rangle}{\langle \tau\rangle^2}.$$

This proves the inequality (2.5).

In order to prove the inequality (2.6) we have that

$$\mathcal{F}(|t|\, \psi_T(t)\, e^{a|t|})(\tau) = T^2\hat{p}(T\tau) \tag{2.8}$$

where $p(t) = |t|\psi(t)e^{aT|t|}$.

Integrating by parts we obtain

$$|\hat{p}(\tau)| \leq \frac{c}{|\tau - iaT|^k}, \quad k = 0, 1, 2.$$

Therefore,

$$|\hat{p}(\tau)| \leq \frac{c}{1 + \tau^2 + a^2T^2}. \tag{2.9}$$

Combining (2.8) and (2.9) yields the desired inequality (2.6). The proof of (2.7) is similar. □

Remark. *It's not possible to obtain similar inequalities as (2.4) and (2.5) for* $\tilde{\psi}_T(t)$ *because of the discontinuity.*

In the following estimates, without loss of generality, we can suppose $\eta = 1$

Lemma 2.2. *Let* $-1/2 < b' \le 0$, $1/2 < b \le b'/3 + 2/3$, $T \in (0,1]$, $a < 1$. *Then*

$$\|\psi_T(t)\int_0^t e^{|t-t'|a} f(t')dt'\|_{H_t^b} \le c\, T^{1+b'/2-3b/2}\|f\|_{H^{b'}}, \tag{2.10}$$

where c is a constant independent of a, f *and* T.

Proof. It is sufficient to prove the Lemma 2.2 when $|a| \le 1$. In fact, let us suppose that Lemma 2.2 has been established in the case $|a| \le 1$. Then when $a < -1$, we use the change of variable $t'a \equiv t'$, to obtain

$$\psi_T(t)I_a(t) := \psi_T(t)\int_0^t e^{|t-t'|a} f(t')dt' = \frac{1}{a}\psi_{aT}(at)\int_0^{at} e^{|at-t'|} f_a(t')dt' = \frac{1}{a}J(at). \tag{2.11}$$

where $f_a(t') = f(t'/a)$ and $J(t) = \psi_{aT}(t)\int_0^t e^{|t-t'|} f_a(t')dt'$.

Note that for $a < -1$,

$$\|J(t)\|_{H^b} \le c\,|aT|^{1+b'/2-3b/2}\|f_a\|_{H^{b'}} \le c|a|^{3/2-b'/2-3b/2}T^{1+b'/2-3b/2}\|f\|_{H^{b'}}. \tag{2.12}$$

Since $b' + b > 0$ and $|a| > 1$, from (2.11) and (2.12) we obtain

$$\|\psi_T(t)I_a(t)\|_{H^b} = \frac{1}{|a|}\|J(at)\|_{H^b} \le c\,\frac{\langle a\rangle^b}{|a|^{3/2}}\|J(t)\|_{H^b} \le \frac{c}{|a|^{(b'+b)/2}}T^{1+b'/2-3b/2}\|f\|_{H^{b'}}.$$

Hence, we arrived at (2.10) in this case too.

From here onwards, we consider $|a| \leq 1$. Now let $b > 1/2$, then we have

$$\begin{aligned} I_a(t) :=& \int_0^t e^{|t-t'|a} f(t')dt' = \int_0^t e^{(t-t')\text{sgn}(t)a} f(t')dt' \\ =& e^{a|t|} \int_0^t e^{-\text{sgn}(t)at'} \int_{\mathbb{R}} e^{it'\tau} \hat{f}(\tau) d\tau dt' \\ =& e^{a|t|} \int_{\mathbb{R}} \hat{f}(\tau) \int_0^t e^{(i\tau - \text{sgn}(t)a)t'} dt' d\tau \\ =& e^{a|t|} \int_{\mathbb{R}} \hat{f}(\tau) \frac{e^{(i\tau-\text{sgn}(t)a)t} - 1}{i\tau - \text{sgn}(t)a} d\tau \\ =& \int_{\mathbb{R}} \hat{f}(\tau) \frac{e^{i\tau t} - e^{a|t|}}{i\tau - \text{sgn}(t)a} d\tau. \end{aligned}$$

We have

$$\frac{1}{\text{sgn}(t)a - i\tau} = \text{sgn}(t) \frac{a}{a^2+\tau^2} + i \frac{\tau}{a^2+\tau^2}.$$

If we define

$$p_a(t) = \frac{a}{a^2+t^2}, \quad q_a(t) = \frac{t}{a^2+t^2}$$

and replace τ by t' we get,

$$\begin{aligned} I_a(t) =& \text{sgn}(t) \int_{\mathbb{R}} p_a(t') \big[e^{a|t|} - e^{it't}\big] \widehat{f}(t') dt' + ic \int_{\mathbb{R}} q_a(t') \big[e^{a|t|} - e^{it't}\big] \widehat{f}(t') dt' \\ :=& I_{a,1}(t) + I_{a,2}(t). \end{aligned} \tag{2.13}$$

Estimate for $I_{a,1}$: We estimate it in two different cases.

Case 1: $|t'| > 1/T$. Let $\widehat{f}(t') \equiv \widehat{f}(t') \chi_{\{|t'|>1/T\}}$. From the definition of $I_{a,1}$ we have

$$\begin{aligned} \psi_T(t) I_{a,1}(t) =& a\,\text{sgn}(t) \psi_T(t) \int_{\mathbb{R}} \frac{\widehat{f}(t')}{a^2+t'^2} \big[e^{a|t|} - e^{itt'}\big] dt' \\ =& ah\Big(\frac{t}{T}\Big), \end{aligned} \tag{2.14}$$

where $h(t) = \text{sgn}(t)\psi(t) \int_{\mathbb{R}} \{\widehat{f}(t')/(a^2+t'^2)\} \big[e^{aT|t|} - e^{iTtt'}\big] dt'$.

We have,

$$\widehat{h(t)}(\tau) = \int_{\mathbb{R}} \frac{\widehat{f}(t')}{a^2+t'^2} K(a,T,\tau,t') dt', \tag{2.15}$$

where

$$K(a,T,\tau,t') = \int_{\mathbb{R}} \text{sgn}(t)\psi(t) \big[e^{aT|t|} - e^{iTtt'}\big] e^{-it\tau} dt.$$

Integrating by parts we get

$$|K(a,T,\tau,t')| \leq c\frac{\langle t'\rangle}{\langle \tau\rangle}, \quad \text{and} \quad |K(a,T,\tau,t')| \leq c\frac{\langle t'\rangle}{\langle \tau\rangle^2} + c\frac{\langle t'\rangle^2}{\langle \tau\rangle^2} \leq c\frac{\langle t'\rangle^2}{\langle \tau\rangle^2}.$$

Hence

$$|K(a,T,\tau,t')| \leq c\frac{\langle t'\rangle^{2b}}{\langle \tau\rangle^{2b}}.$$

Therefore, from (2.15) we obtain

$$|\widehat{h(t)}|(\tau) \leq \frac{c}{\langle\tau\rangle^{2b}}\int_{|t'|>1/T} \frac{|\widehat{f}(t')|}{a^2+t'^2}\langle t'\rangle^{2b}dt' \leq c\frac{T^{3/2+b'-2b}}{\langle\tau\rangle^{2b}}\|f\|_{H^{b'}}.$$

Now, using (2.14) we have

$$\|\psi_T(t)I_a(t)\|_{H^b} = |a|\, \|h\Big(\frac{t}{T}\Big)\|_{H^b} \leq |a|T^{1/2-b}\|h(t)\|_{H^b} \leq cT^{3/2+b'-2b}T^{1/2-b}\|f\|_{H^{b'}}.$$

Hence

$$\|\psi_T(t)I_a(t)\|_{H^b} \leq cT^{2+b'-3b}\|f\|_{H^{b'}} \leq cT^{1+b'/2-3b/2}\|f\|_{H^{b'}}.$$

Case 2: $|t'| \leq 1/T$. Let $\widehat{f}(t') \equiv \widehat{f}(t')\chi_{\{|t'|\leq 1/T\}}$ and as earlier $\tilde{\psi}_T(t) = \operatorname{sgn}(t)\psi_T(t)$. We have

$$\begin{aligned}
\mathcal{F}(\psi_T(t)I_{a,1}(t))(\tau) =& \int_{\mathbb{R}} e^{-it\tau}\tilde{\psi}_T(t)\int_{\mathbb{R}} p_a(t')\big[e^{a|t|}-e^{it't}\big]\hat{f}(t')dt'dt\\
=& \int_{\mathbb{R}} p_a(t')\hat{f}(t')\int_{\mathbb{R}} \tilde{\psi}_T(t)e^{-it\tau}\big[e^{a|t|}-e^{it't}\big]dtdt'\\
=& \int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\, e^{a|t|}\big)(\tau) - \mathcal{F}\big(\tilde{\psi}_T(t)\big)(\tau-t')\}dt'\\
=& \int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\, e^{a|t|}\big)(\tau) - \mathcal{F}\big(\tilde{\psi}_T(t)\, e^{a|t|}\big)(\tau-t')\}dt'\\
&+\int_{\mathbb{R}} p_a(t')\widehat{f}(t')\{\mathcal{F}\big(\tilde{\psi}_T(t)\, e^{a|t|}\big)(\tau-t') - \mathcal{F}\big(\tilde{\psi}_T(t)\big)(\tau-t')\}dt'\\
:=& I_{a,11}(\tau) + I_{a,12}(\tau).
\end{aligned}$$

Since $|p_a(t')| \leq 1/|t'|$, we can estimate the term $I_{a,11}(\tau)$ as in [9]. Therefore we will estimate only the term $I_{a,12}(\tau)$.

Let us define $h(t',\tau) := \mathcal{F}\big(\tilde{\psi}_T(t)[e^{a|t|}-1]\big)(\tau-t')$, then we have

$$h(t',\tau) = \int_0^a \mathcal{F}\big(|t|\,\tilde{\psi}_T(t)\, e^{s|t|}\big)(\tau-t')ds. \tag{2.16}$$

From (2.7) we have that

$$|\mathcal{F}\big(|t|\,\tilde{\psi}_T(t)\,e^{s|t|}\big)(\tau-t')|\le \frac{c\,T^2}{(1+(|\tau-t'|+|s|)T)^2}, \tag{2.17}$$

where c is independent of s, τ, t' and T.

Observe that $0\le s\le a$ if $a\ge 0$ and $a\le s\le 0$ if $a\le 0$. Thus we obtain

$$\begin{aligned}|h(t',\tau)|\le& c\,T^2\int_0^{|a|}\frac{1}{(1+(|\tau-t'|+|s|)T)^2}ds\\ =&c\,T^2\frac{|a|}{(1+|\tau-t'|T)(1+|\tau-t'|T+|a|T)}.\end{aligned}$$

As $|Tt'|\le 1$ we have

$$\frac{1}{1+|\tau-t'|T}\le\frac{2}{1+|\tau|T}.$$

Hence

$$|h(t',\tau)|\le c\,T^2\frac{|a|}{(1+|\tau|T)^2}.$$

Using the Hölder's inequality we get,

$$\begin{aligned}|I_{a,12}(\tau)|&=|\int_{\mathbb{R}}p_a(t')\hat{f}(t')h(t',\tau)dt'|\\ &\le\|f\|_{H^{b'}}\Big(\int\limits_{|t'|\le 1/T}\frac{\langle t'\rangle^{-2b'}|h(t',\tau)|^2}{(a^2+t'^2)^2}dt'\Big)^{1/2}\\ &\le\frac{c\,T^2}{(1+|\tau|T)^2}\,\|f\|_{H^{b'}}\Big(\int\limits_{|t'|\le 1/T}(1+|t'|^{-2b'})dt'\Big)^{1/2}\\ &\le\frac{c\,T^{3/2+b'}}{(1+|\tau|T)^2}\,\|f\|_{H^{b'}}.\end{aligned}\tag{2.18}$$

Finally, we arrive at

$$\begin{aligned}\Big(\int\limits_{\mathbb{R}}(1+|\tau|)^{2b}|I_{a,12}(\tau)|^2\,d\tau\Big)^{1/2}\le& c\,T^{3/2+b'}\,\|f\|_{H^{b'}}\Big(\int\limits_{\mathbb{R}}\frac{1+|\tau|^{2b}}{(1+|\tau|T)^4}d\tau\Big)^{1/2}\\ \le& c\,T^{3/2+b'}\,\|f\|_{H^{b'}}\Big(\frac{1}{T^{1/2}}+\frac{1}{T^{b+1/2}}\Big)\\ \le& c\,T^{1-b+b'}\,\|f\|_{H^{b'}}.\end{aligned}$$

Therefore, in this case we have

$$\|\psi_T I_{a,1}\|_{H^b}\le c\,T^{1-b+b'}\,\|f\|_{H^{b'}}.$$

Estimate for $I_{a,2}$: The estimate for $I_{a,2}$ is similar to $I_{a,1}$ exchanging p_a by q_a and $\tilde{\psi}_T(t)$ by $\psi_T(t)$. So, we omit the calculation details. □

In the following remark we present improvement of the estimate obtained in the Lemma 2.2 in some particular cases. Although, this improvement does not help to improve our main result, it will be of interest on its own.

Remarks. *1) The proof in the case $|t'| \leq 1/T$ is valid for all $a < 1$.*
2) We know that

$$\widehat{\mathcal{H}g}(\eta) = -i\, sgn(\eta)\, \widehat{g}(\eta) \quad and \quad \widehat{q_a}(t) = -i\, sgn(t)\, e^{a|t|}.$$

Thus

$$\begin{aligned} I_{a,2}(t) =& - \tilde{\psi}_T(t)\widehat{q_a}(t) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\tilde{\psi}_T(t)\mathcal{F}^{-1}\mathcal{H}(q_a\widehat{f}\,)(t) \\ =& \tilde{\psi}_T(t)\mathcal{F}^{-1}(q_a)(t) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\tilde{\psi}_T(t)\mathcal{F}^{-1}\mathcal{H}(q_a\widehat{f}\,)(t), \end{aligned}$$

where $\tilde{\psi}_T(t) = sgn(t)\psi_T(t)$. Consequently,

$$\begin{aligned} \widehat{I_{a,2}}(\tau) =& \widehat{\tilde{\psi}_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\widehat{\tilde{\psi}_T} \star \mathcal{H}(q_a\widehat{f}\,)(\tau) \\ =& \widehat{\tilde{\psi}_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \sqrt{2\pi}\mathcal{H}(\widehat{\tilde{\psi}_T}) \star (q_a\widehat{f}\,)(\tau). \end{aligned}$$

Similarly,

$$\widehat{I_{a,1}}(\tau) = \frac{1}{i}\widehat{\psi_T} \star q_a(\tau) \int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt' + \frac{\sqrt{2\pi}}{i}\mathcal{H}(\widehat{\psi_T}) \star (q_a\widehat{f}\,)(\tau).$$

3) If $1/2 < b < b'/3 + 2/3$, then $1 - b + b' > 3/4 + b'/2 - b > 1 + b'/2 - 3b/2 > 0$.
4) If $|a| > 1$, then $|q_a(t')| \leq c/\langle t' \rangle$, hence

$$\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq c \int_{\mathbb{R}} \frac{|\widehat{f}(t')|\,\langle t' \rangle^{1-b}}{\langle t' \rangle\,\langle t' \rangle^{1-b}}dt' \leq c\,\|f\|_{H^{b-1}} \left\|\frac{1}{\langle t' \rangle^{b}}\right\|_{L^2},$$

and therefore we obtain a more refined estimate than (2.19).

5) In the case $|t'| > 1/T$ we can to obtain a better estimate for $I_{a,2}$ because ψ_T is regular (using the inequalities (2.4) and (2.5)). In fact, let $\widehat{f}(t') \equiv \widehat{f}(t')\chi_{\{|t'|>1/T\}}$. We have that

$$\|\psi_T(t)I_{a,2}(t)\|_{H_t^b} \leq \|\psi_T(t)e^{|t|a}\|_{H_t^b}\left|\int_{\mathbb{R}} q_a(t')\widehat{f}(t')dt'\right| + \|\psi_T(t)\|_{H_t^b}\|\mathcal{F}^{-1}(q_a\widehat{f}\,)(t)\|_{H_t^b}.$$

Since $|t'| > 1/T$ implies $|t'| \simeq \langle t' \rangle$, using the Cauchy-Schwartz inequality we get

$$\begin{aligned}\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq \int\limits_{|t'|>1/T} \frac{|\widehat{f}(t')|}{|t'|}dt' \lesssim \int\limits_{|t'|>1/T} \frac{|\widehat{f}(t')|\,|\,t'|^{-b'}}{|\,t'|\,|\,t'|^{-b'}}dt' \\ \leq \|f\|_{H^{b'}} \Big(\int\limits_{|t'|>1/T} \frac{dt'}{|t'|^{2(1+b')}} \Big)^{1/2} \lesssim T^{1/2+b'}\, \|f\|_{H^{b'}}.\end{aligned} \tag{2.19}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq & c\,\|f\|_{H^{b'}} \Big(\int\limits_{|t'|>1/T} \frac{dt'}{|t'|^{2b'}(a^2+t'^2)} \Big)^{1/2} \\ \leq & c\,\frac{1}{|a|^{1/2+b'}} \Big(\int\limits_{\mathbb{R}} \frac{dt'}{|t'|^{2b'}(1+t'^2)} \Big)^{1/2} \|f\|_{H^{b'}} \\ \leq & c\,\frac{1}{|a|^{1/2+b'}}\,\|f\|_{H^{b'}}.\end{aligned} \tag{2.20}$$

Hence from (2.19) and (2.20) we get

$$\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq\; c\Big(\frac{T}{|a|}\Big)^{1/4+b'/2} \|f\|_{H^{b'}}, \tag{2.21}$$

and

$$\begin{aligned}\|\mathcal{F}^{-1}(q_a\widehat{f}\,)(t)\|^2_{H^b_t} \lesssim \int\limits_{|t'|>1/T} \frac{|\widehat{f}(t')|^2}{\langle t'\rangle^{2(1-b)}}dt' \lesssim \int\limits_{|t'|>1/T} \frac{|\widehat{f}(t')|^2}{\langle t'\rangle^{-2b'}|t'|^{2(1-b+b')}}dt' \\ \leq T^{2(1-b+b')}\|f\|^2_{H^{b'}}.\end{aligned}$$

So from inequality (2.4) we have that

$$\|\psi_T(t)\|_{H^b_t}\|\mathcal{F}^{-1}(q_a\widehat{f}\,)(t)\|_{H^b_t} \leq c\,T^{1/2-b}\,T^{(1-b+b')}\|f\|_{H^{b'}} \leq c\,T^{3/2-2b+b'}\|f\|_{H^{b'}}.$$

On the other hand, if $a < -1$ from inequalities (2.5) and (2.21) we get

$$\begin{aligned}\|\psi_T(t)\,e^{a|t|}\|_{H^b_t}\int_{\mathbb{R}} |q_a(t')\widehat{f}(t')|dt' \leq & c\Big(\frac{T}{|a|}\Big)^{1/2}\Big(1+\frac{1}{T^b}\Big)(1+|a|^b)\Big(\frac{T}{|a|}\Big)^{1/4+b'/2}\|f\|_{H^{b'}} \\ \leq & c\Big(\frac{T}{|a|}\Big)^{3/4+b'/2}\Big(1+\frac{1}{T^b}\Big)(1+|a|^b)\|f\|_{H^{b'}} \\ \leq & c\,T^{3/4+b'/2-b}\Big(1+\frac{1}{|a|^{3/4+b'/2-b}}\Big)\|f\|_{H^{b'}} \\ \leq & c\,T^{3/4+b'/2-b}\|f\|_{H^{b'}}.\end{aligned}$$

Now if $|a|<1$ using the inequality (2.5) we obtain:

$$
\begin{aligned}
\|\psi_T(t)\, e^{a|t|}\|_{H_t^b}\int_{\mathbb{R}}|q_a(t')\widehat{f}(t')|dt' \leq & c\,T^{1/2+b'}\|f\|_{H^{b'}}T^{1/2}\Big(1+\frac{1}{|T|^b}\Big)\\
\leq & c\,T^{1/2+b'}\,T^{1/2-b}\|f\|_{H^{b'}}\\
\leq & c\,T^{1-b+b'}\|f\|_{H^{b'}}\\
\leq & c\,T^{3/4+b'/2-b}\|f\|_{H^{b'}}.
\end{aligned}
$$

Therefore in this case

$$
\|\psi_T(t)I_a(t)\|_{H_t^b}\leq c\,T^{3/4+b'/2-b}\,\|f\|_{H^{b'}},
$$

where c is a constant independent of a, f and T.

Now we prove Proposition 2.2 which plays a crucial role in the proof of the first main result of this work.

Proof of Proposition 2.2. In order to prove (2.2), using (2.5) with $T=1$ it is not difficult to see that:

$$
\|\psi(t)\, e^{\eta\Phi(\xi)|t|}\|_{H_t^b}^2\leq c(\eta)\langle\Phi(\xi)\rangle^{2b-1}\leq c(\eta)\langle\xi\rangle^{3(2b-1)},
$$

where $c(\eta)$ is a constant. Therefore,

$$
\|\psi_T(t)V(t)u_0\|_{X_{s-\epsilon,b}}\leq c(\eta)\Big(\int_{\mathbb{R}}\langle\xi\rangle^{2(s-\epsilon)}\langle\xi\rangle^{3(2b-1)}|\widehat{u_0}(\xi)|^2d\xi\Big)^{1/2}\leq c(\eta)\|u_0\|_{H^s}.
$$

Now, we move to prove (2.3). From definition (1.10) of the $X_{s,b}$ norm, we have

$$
\begin{aligned}
&\|\psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'\|_{X_{s,b}}\\
&\qquad=\|U(-t)\psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt'\|_{H_{s,b}}\\
&\qquad=\|\langle\tau\rangle^b\langle\xi\rangle^s\mathcal{F}_{\xi\tau}\big[\psi_T(t)\int_0^t U(-t')\tilde{U}(t-t')(uu_x)(t')dt'\big]\|_{L^2_\tau L^2_\xi}\\
&\qquad=\|\langle\xi\rangle^s\|\psi_T(t)\int_0^t e^{-it'\xi^3}e^{|t-t'|\Phi(\xi)}\widehat{uu_x}(t',\xi)dt'\|_{H_t^b}\|_{L^2_\xi}.
\end{aligned}
\tag{2.22}
$$

If we fix the variable ξ and suppose $f_\xi(t')=e^{-it'\xi^3}\widehat{uu_x}(t',\xi)$ the estimate (2.3) follows from (2.22) using (2.10). □

2.2. **Proof of the Theorem 1.1.**

Proof. As discussed in the introduction, we will use Bourgain's space associated to the KdV group to prove well-posedness for the IVP (1.1), therefore we need to consider the IVP (1.12) that is defined for all t. Now consider the IVP (1.12) in its equivalent integral form

$$u(t) = V(t)u_0 - \int_0^t V(t-t')(uu_x)(t')dt', \tag{2.23}$$

where $V(t)$ is the semigroup associated with the linear part given by (1.14).

Note that, if for all $t \in \mathbb{R}$, $u(t)$ satisfies

$$u(t) = \psi(t)V(t)u_0 - \psi_T(t)\int_0^t V(t-t')(uu_x)(t')dt',$$

then $u(t)$ satisfies (2.23) in $[-T,T]$. We define an application

$$\Psi(u)(t) = \psi(t)\,V(t)u_0 - \psi_T(t)\,\int_0^t V(t-t')(uu_x)(t')dt'.$$

Let $s > -3/4$, and $u_0 \in H^s$. Let b and b' be two numbers given by Proposition 2.1, such that $\theta \equiv \min\{1 + b'/2 - 3b/2,\, 3/4 + s/3 - b\} > 0$. We will prove that Ψ is a contraction in the following space

$$X^M_{s-\epsilon,b} = \{u \in X_{s-\epsilon,b};\ \|u\|_{X_{s-\epsilon,b}} \le M\},$$

where $\epsilon \in [3(b-1/2), s+3/4)$. First we will prove that $\Psi : X^M_{s-\epsilon,b} \to X^M_{s-\epsilon,b}$. Let $u \in X^M_{s-\epsilon,b}$. By using Propositions 2.1, 2.2 and the definition of $X^M_{s-\epsilon,b}$ we get for all ϵ such that $3(b-1/2) \le \epsilon < s+3/4$

$$\begin{aligned}\|\Psi(u)\|_{X_{s-\epsilon,b}} &\le c\|u_0\|_s + c\,T^\theta\|(uu_x)\|_{X_{s-\epsilon,b'}}\\ &\le \frac{M}{4} + cT^\theta M^2 \le \frac{M}{2},\end{aligned}$$

where we took $M = 4c\|u_0\|_{H^s}$ and $cT^\theta M = 1/4$. Therefore, $\|\Psi(u)\|_{X_{s-\epsilon,b}} \le M$. A similar argument proves that Ψ is a contraction. Hence Ψ has a fixed point u which is a solution of the IVP (1.1) such that $u \in C([-T,T], H^{s-\epsilon})$, with

$$\|u\|_{X_{s-\epsilon,b}} \le M = 4c\|u_0\|_{H^s}.$$

As b can be chosen arbitrarily near to 1/2, ϵ can be chosen arbitrarily near to 0. Hence by Fatou's lemma we have

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^s} \le c\|u_0\|_{H^s}. \tag{2.24}$$

Thus by (2.24) and Lebesgue's dominated convergence theorem we get

$$u \in C([-T,T],H^s).$$

The rest of the proof follows in an analogous way to [11], so we omit the details. □

3. A refined local smoothing effect

In this section we prove the following local smoothing effect for the semigroup $V_k(t)$ defined by (1.14) with $\Phi(\xi)=|\xi|^k-|\xi|^{k+2}$. Similar results can also be obtained for more general Φ as in (1.2). Our proof follows the ideas of [6].

Theorem 3.1. *Let $T>0$, $u_0\in L^q$, $0\le s<(k+3-p)/p+1/p_1$ and $p\ge 2$, $p_1\ge 2$, then*

$$\|D_x^sV_k(t)u_0\|_{L^p_TL^{p_1}_x}\le \frac{c(\eta)}{(k+3-p(s+1)+p/p_1)^{1/p}}\,(T^{1/p}e^{2\eta T}+T^{\epsilon})\,\|u_0\|_{L^q}, \tag{3.1}$$

where $\epsilon=\epsilon(p,k,s,p_1)=(k+3-p(s+1))/(k+2)+p/((k+1)p_1)$ and $1/p+1/q=1$.

Corollary 3.1. *Let $u_0\in L^q$, $T>0$, $2\le p<k+3$, and $0\le s<(k+3-p)/p$, then*

$$\|D_x^sV_k(t)u_0\|_{L^p_TL^\infty_x}\le \frac{c(\eta)}{(k+3-p(s+1))^{1/p}}\,(T^{1/p}e^{2\eta T}+T^{\epsilon_0})\,\|u_0\|_{L^q}, \tag{3.2}$$

where $\epsilon_0=\epsilon(p,k,s)=(k+3-p(s+1))/(k+2)$ and $1/p+1/q=1$.

In particular, the case when $p=2$ is interesting, in fact we have

Corollary 3.2. 1) *If $u_0\in L^2$, $p_1\ge 2$, $0\le s<1+(k-1)/2+1/p_1$, $0<T<1$ and $\gamma=\min\{1/2,\epsilon(2,k,s,p_1)\}$ then*

$$\|D_x^sV_k(t)u_0\|_{L^2_TL^{p_1}_x}\le \frac{c(\eta)\,T^\gamma}{(1+(k-1)/2+1/p_1-s)^{1/2}}\|u_0\|_{L^2}. \tag{3.3}$$

2) *If $u_0\in \dot H^s$, $-k/2<s\le 0$, $0<T<1$ and $\gamma=\min\{1/2,\epsilon(2,k,1-s,p_1)\}$ then*

$$\|D_xV_k(t)u_0\|_{L^2_TL^{p_1}_x}\le \frac{c(\eta)\,T^\gamma}{((k-1)/2+1/p_1+s)^{1/2}}\|D^su_0\|_{L^2}, \tag{3.4}$$

in the following cases:

i) when $-(k-1)/2\le s\le 0$ and $2\le p_1$.

ii) when $-k/2<s<-(k-1)/2$ and $2\le p_1\le (-s-(k-1)/2)^{-1}$.

In the proof of Theorem 3.1 we will use the following result

Proposition 3.1. *Let $p \geq 2$, and $1/p + 1/q = 1$, then*

$$\|\hat{u}\|_{L^p} \leq c\|u\|_{L^q}, \tag{3.5}$$

Proof. See Corollary 1.43 in [13]. □

Now, we give a proof of Theorem 3.1.

Proof of Theorem 3.1. We can suppose $u_0 \in S(\mathbb{R})$. We consider a cut-off function $\varphi \in C(\mathbb{R} \setminus \{0\})$, $0 \leq \varphi \leq 1$ defined by

$$\varphi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t < 0 \text{ or } t \geq 2. \end{cases} \tag{3.6}$$

Let us define $\varphi_T(t) \equiv \varphi(\frac{t}{T})$, then

$$\|D_x^s V_k(t) u_0(x)\|_{L_T^p L_x^{p_1}} \leq \|\varphi_T(t) D_x^s V_k(t) u_0(x)\|_{L_t^p L_x^{p_1}}.$$

Let $1/p_1 + 1/q_1 = 1$, using duality it is enough to prove for u_0 in L^q and g in $L_t^q L_x^{q_1}$

$$J \equiv \Big| \int_{\mathbb{R}^2} \varphi_T(t) D_x^s V_k(t) u_0(x) g(x,t) dx\, dt \Big| \leq c\|u_0\|_{L^q} \|g\|_{L_t^q L_x^{q_1}}.$$

Using (1.14) we have

$$D_x^s V_k(t) u_0(x) = i \int_{\mathbb{R}} |\xi|^s\, e^{it\xi^3 + \eta t\Phi(\xi) + ix\xi} \hat{u}_0(\xi) d\xi.$$

Therefore, by Fubini's theorem, Proposition 3.1 and Hölder's inequality we get

$$J \leq c\|u_0\|_{L^q} \|Lg\|_{L^q},$$

where $Lg(\xi)$ is defined by

$$Lg(\xi) \equiv |\xi|^s \Big| \int_{\mathbb{R}^2} \varphi_T(t) g(x,t) e^{it\xi^3 + \eta t\Phi(\xi) + ix\xi} dx\, dt \Big|$$

and

$$|Lg(\xi)| \leq |\xi|^s \int_{\mathbb{R}} \varphi_T(t) e^{\eta t\Phi(\xi)} |\mathcal{F}^{-1}(g(\cdot,t))(\xi)| dt. \tag{3.7}$$

We have

$$\|Lg(\xi)\|_{L^q(\mathbb{R})} \leq \|Lg(\xi)\|_{L^q(|\xi| \leq 2)} + \|Lg(\xi)\|_{L^q(|\xi| > 2)} \equiv J_1 + J_2. \tag{3.8}$$

In J_1 by (3.7), Minkowski and Hölder's inequalities and Proposition (3.1) we get

$$\begin{aligned} J_1 \leq & c \int_{\mathbb{R}} \varphi_T(t) \|e^{\eta t\Phi(\xi)}\|_{L^{r_1}(|\xi| \leq 2)} \|\mathcal{F}^{-1}(g(\cdot,t))(\xi)\|_{L^{p_1}(|\xi| \leq 2)} dt \\ \leq & c\, e^{2\eta T} \|\varphi_T\|_{L^p}\, \|g\|_{L_t^q L_x^{q_1}} \leq c\, e^{2\eta T} T^{1/p}\, \|g\|_{L_t^q L_x^{q_1}}, \end{aligned} \tag{3.9}$$

where c is a constant, $1/q = 1/r_1 + 1/p_1$ and $1/p_1 + 1/q_1 = 1$.
Similarly in J_2 we have

$$J_2 \leq \int_{\mathbb{R}} \varphi_T(t) \| \, \xi^s \, e^{\eta t \Phi(\xi)} \|_{L^{r_1}(|\xi|>2)} \|g(\cdot, t)\|_{L_x^{q_1}} dt,$$

and for $t > 0$ we have

$$\| \, |\xi|^s e^{-\eta t |\xi|^{k+2}/2} \|_{L^{r_1}(|\xi|>2)} \leq \frac{c(\eta)}{t^{s/(k+2)+1/((k+2)r_1)}}.$$

Therefore, for $0 \leq s < (k+3-p)/p + 1/p_1$ we get

$$J_2 \leq c(\eta) \Big\| \frac{\varphi_T(t)}{t^{s/(k+2)+1/((k+2)r_1)}} \Big\|_{L^p} \|g\|_{L_t^q L_x^{q_1}} \leq c(\eta) \, T^{\epsilon} \, \|g\|_{L_t^q L_x^{q_1}}, \tag{3.10}$$

where $\epsilon = (k+3-p(s+1))/(k+2) + p/((k+1)p_1)$.

From (3.8), (3.9) and (3.10) we obtain

$$\|Lg\|_{L^q} \leq \frac{c(\eta)}{(k+3-p(s+1)+p/p_1)^{1/p}} \left(T^{1/p} e^{2\eta T} + T^{\epsilon}\right) \|g\|_{L_t^q L_x^{q_1}}.$$

□

4. Some Strichartz type estimates

Proposition 4.1. *Let* $2 \leq p$, $k \geq 1$, $c_{p,k} = \frac{p-2}{2p(k+2)}$, $0 < T < 1$, $s \leq 0$ *and*

$$\frac{1}{r} + \frac{s}{(k+2)} - c_{p,k} > 0$$

then

$$\|V_k(t)u_0\|_{L_T^r L_x^p} \leq c(\eta, sqr_0) T^{\left\{\frac{1}{r} + \frac{s}{(k+2)} - c_{p,k}\right\}} \|u_0\|_{H^s},$$

where $1/q + 1/p = 1$, $r_0 = 2/(2-q)$.

Proof. Let $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$, by (3.5) we have

$$\begin{aligned} \|V_k(t)u_0\|_{L_T^r L_x^p} &\leq c \, \|\widehat{V_k(t)u_0}\|_{L_T^r L_\xi^q} \leq c \, \|e^{\eta t \Phi(\xi)} \widehat{u_0}\|_{L_T^r L_\xi^q(|\xi| \leq 2)} \\ &\quad + c \, \|e^{\eta t \Phi(\xi)} \widehat{u_0}\|_{L_T^r L_\xi^q(|\xi|>2)} \equiv J_1 + J_2. \end{aligned}$$

In J_1, using Hölder's inequality we have

$$J_1 \leq c 3^{-s} e^{\eta T} T^{1/r}.$$

To estimate J_2, by Hölder's inequality we obtain

$$\begin{aligned}\int_{|\xi|>2} e^{q\eta t\Phi(\xi)}|\widehat{u_0}(\xi)|^q d\xi &\le \int_{|\xi|>2} e^{-q\eta t|\xi|^{k+2}/2}(1+|\xi|)^{-sq}(1+|\xi|)^{sq}|\widehat{u_0}(\xi)|^q d\xi \\ &\le \|e^{-q\eta t|\xi|^{(k+2)}/2}(1+|\xi|)^{-sq}\|_{L^{r_0}(|\xi|>2)}\|u_0\|_{H^s}^q \\ \le\ c(\eta)\Big[\frac{1}{t^{1/((k+2)r_0)}} &+ \frac{1}{t^{1/((k+2)r_0)-sq/(k+2)}}\Big(\int_{\mathbb{R}} e^{-|y|^{(k+2)}}|y|^{-sqr_0}dy\Big)^{1/r_0}\Big]\|u_0\|_{H^s}^q,\end{aligned}$$

where $r_0 = 2/(2-q)$.

Therefore

$$J_2 \le c(\eta, sqr_0)\left\|\frac{1}{t^{1/((k+2)r_0q)-s/(k+2)}}\right\|_{L_T^r}\|u_0\|_{H^s} \le c(\eta, sqr_0)\,T^{\frac{1}{r}+\frac{s}{(k+2)}-c_{p,k}}\|u_0\|_{H^s},$$

where $c_{p,k} = \frac{p-2}{2p(k+2)}$. ☐

Corollary 4.1. *Let $0<T<1$, $s\le 0$, $1<r<2(k+2)/(1-2s)$, then*

$$\|V_k(t)u_0\|_{L_T^rL_x^\infty} \le c(\eta, s)T^{1/r-(1-2s)/(2(k+2))}\|u_0\|_{H^s}.$$

Proposition 4.2. *Let $u_0\in L^2$, then*

$$\|V_k(t)u_0\|_{L_T^\infty L_x^2} \le e^{\eta T}\|u_0\|_{L^2}.$$

Proof. Using Plancherel identity

$$\|V_k(t)u_0\|_{L_x^2} = \|e^{\eta t(|\xi|^k-|\xi|^{k+2})}\widehat{u_0}(\xi)\|_{L_\xi^2} \le e^{\eta T}\|u_0\|_{L^2},$$

where we used the estimate $e^{\eta t(|\xi|^k-|\xi|^{k+2})}\le e^{\eta T}$. ☐

In the following section we give an application of the above results.

5. Proof of Theorem 1.2.

This section is devoted to give proof of the local well-posedness result for given data in homogeneous Sobolev space with regularity below L^2. We consider the IVP

$$\begin{cases}\partial_t u + \partial_x^3 u + \eta L_k(u) + u\partial_x u = 0, & x\in\mathbb{R},\ t\ge 0,\\ u(x,0) = u_0(x),\end{cases} \tag{5.1}$$

which is a special case of the IVP (1.1) with Fourier symbol $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$.

To prove Theorem 1.2, we need the following proposition

Proposition 5.1. *Let $0\le -s<1/2$. If $u\in L^{2/(1-2s)}$, then*

$$\|D_x^s(u)\|_{L^2} \le c\|u\|_{L^{2/(1-2s)}}. \tag{5.2}$$

If $u \in L^1 \cap L^2$, then

$$\|D_x^s(u)\|_{L^2} \le c(\|u\|_{L^1} + \|u\|_{L^2}). \tag{5.3}$$

Proof. The inequality (5.2) follows using the Hardy, Sobolev, Littlewood inequality and

$$\widehat{\Big(\frac{1}{|\xi|^{1+s}}\Big)}(\eta) = \frac{1}{|\eta|^{-s}}.$$

The inequality (5.3) follows from

$$\|D_x^s(u)\|_{L^2}^2 = \int_{|\eta|<1} \frac{|\widehat{u}(\eta)|^2}{|\eta|^{-2s}} d\eta + \int_{|\eta|\ge 1} \frac{|\widehat{u}(\eta)|^2}{|\eta|^{-2s}} d\eta.$$

□

Now we are in position to supply proof of our second main result.

Proof of Theorem 1.2. Now let $u_0 \in \dot{H}^s$, with $0 \le -s < 1/2$. For $0 < T < 1$, define a ball

$$Z_{a,T} = \{w \in C([0,T], \dot{H}^s); \quad \|\!|w|\!\|_T \le a\},$$

where

$$\|\!|w|\!\|_T = \|w\|_{\dot{H}^s} + \|w_x\|_{L_T^2 L_x^{p_1}} + T^{\left\{\frac{q_1-2}{2q_1(k+2)} - \frac{1}{2} - \frac{s}{k+2}\right\}} \|w\|_{L_T^2 L_x^{q_1}},$$

$-1/s \le p_1 < \infty$, $q_1 \ge 2$, $1/p_1 + 1/q_1 = (1-2s)/2$ and p_1 is chosen as in inequality (3.4) of Corollary 3.2.

Using Corollary 3.2 and Proposition 4.1 we get

$$\|\!|V_k(t)u_0|\!\|_T \le c\|D_x^s(u_0)\|_{L^2}. \tag{5.4}$$

Also, using the inequality (5.2) we obtain

$$\begin{aligned}
\int_0^T \|D_x^s(vv_x)\|_{L_x^2} &\le c \int_0^T \|vv_x\|_{L_x^{2/(1-2s)}} \\
&\le c\|v_x\|_{L_T^2 L_x^{p_1}} \|v\|_{L_T^2 L_x^{q_1}} \\
&\le cT^{\left\{-\frac{q_1-2}{2q_1(k+2)} + \frac{1}{2} + \frac{s}{k+2}\right\}} a^2.
\end{aligned} \tag{5.5}$$

Now, define an application

$$\Psi(v)(t) = V_k(t)u_0 - \int_0^t V_k(t-\tau) vv_x(\tau) d\tau,$$

where $V_k(t)$ is the evolution operator defined in (1.14) with $\Phi(\xi) = |\xi|^k - |\xi|^{k+2}$.

With the help of the inequalities (5.4) and (5.5), it can be shown that the application Ψ maps $Z_{a,T}$ into $Z_{a,T}$ and is a contraction considering $a = 2c\|D_x^s(u_0)\|_{L^2}$, and $cT^{\left\{-\frac{q_1-2}{2q_1(k+2)}+\frac{1}{2}+\frac{s}{k+2}\right\}}a < 1/2$.

The rest of the proof follows from an standard argument. □

Remark. *If we use the inequality (5.3) we can also take the following space in the proof of the Theorem 1.2*

$$Z_{a,T} = \{w \in C([0,T], \dot{H}^s); \quad \|\!|w|\!\|_T \leq a\},$$

where

$$\begin{aligned}\|\!|w|\!\|_T =&\|w\|_{\dot{H}^s} + \|w_x\|_{L_T^2 L_x^{p_1}} + \|w_x\|_{L_T^2 L_x^2} + T^{\left\{\frac{q_1-2}{2q_1(k+2)}-\frac{1}{2}-\frac{s}{k+2}\right\}}\|w\|_{L_T^2 L_x^{q_1}}\\ &+ T^{\left\{-\frac{1}{2}-\frac{s}{k+2}\right\}}\|w\|_{L_T^2 L_x^2},\end{aligned}$$

$-1/s < p_1 < \infty$, $q_1 \geq 2$, $1/p_1 + 1/q_1 = 1/2$ *and* p_1 *is chosen as in Corollary 3.2 inequality (3.4). By Corollary 3.2 and Proposition 4.1 we get*

$$\|\!|V_k(t)u_0|\!\|_T \leq c\|D_x^s(u_0)\|_{L^2}.$$

And we have using the inequality (5.3)

$$\begin{aligned}\int_0^T \|D_x^s(vv_x)\|_{L_x^2} \leq& c\int_0^T (\|vv_x\|_{L_x^1} + \|vv_x\|_{L_x^2})\\ \leq& c\|v_x\|_{L_T^2 L_x^2}\|v\|_{L_T^2 L_x^2} + c\|v_x\|_{L_T^2 L_x^{p_1}}\|v\|_{L_T^2 L_x^{q_1}}\\ \leq& cT^{\left\{\frac{1}{2}+\frac{s}{k+2}\right\}}a^2 + cT^{\left\{-\frac{q_1-2}{2q_1(k+2)}+\frac{1}{2}+\frac{s}{k+2}\right\}}a^2\\ \leq& cT^{\left\{-\frac{q_1-2}{2q_1(k+2)}+\frac{1}{2}+\frac{s}{k+2}\right\}}a^2.\end{aligned}$$

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