# The (2,2,0) Group Inverse Problem*

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**Abstract**

We characterize the existence of the group inverse of a two by two matrix with zero (2,2) entry, over a ring by means of the existence of the inverse of a suitable function of the other three entries. Some special cases are derived.

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## 1 Introduction

In this paper we shall examine the existence and representation of the group inverse of the block matrix $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$, in which the (2,2) block is zero. We aim for results in terms of "words" in the three blocks $a$, $b$, $c$ and their g-inverses, such as inner inverses or Drazin inverses (D-inverses for short). We shall use the results of [3] to create suitable unit matrices.

We shall need the concept of regularity, which guarantees solutions to $aa^-a = a$ and $aa^+a = a$, $a^+ = a^+aa^+$. If in addition $aa^+ = a^+a$ then $a^+$ is known as the group inverse of $a$ and is traditionally denoted by $a^\#$. We will use $rk(\cdot)$, $R(\cdot)$ and $RS(\cdot)$ to denote rank, range and row space, respectively, and write $\approx$ for similarity.

## 2 The group inverse of a (2,2,0) matrix

Consider the matrices $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$ and $M^2 = \begin{bmatrix} a^2 + cb & ac \\ ba & bc \end{bmatrix}$, where we assume that $b$, $c$ and the cornerstone $w = (1 - cc^+)a(1 - b^+b)$ are regular.

Over a (skew) field it is known that $M^\#$ exists if and only if $M$ and $M^2$ have equal rank, and so we could apply the block rank formula of [2]. This, however, only seems to give a tractable result when $c = 1$.

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**Proposition 2.1.** *Let* $M = \begin{bmatrix} A & I_n \\ B & 0 \end{bmatrix}$ *be over a skew field* $\mathbb{F}$*. Then the following are equivalent:*

1. $M^{\#}$ *exists*
2. $rk[(I_n - BB^-)A(I_n - B^-B)] + rk(B) = n$
3. $R(I_n - BB^-) = R[(I_n - BB^-)A] = R[(I_n - BB^-)A(I_n - B^-B)]$
4. $RS(I_n - B^-B) = RS[A(I_n - B^-B)] = RS[(I_n - BB^-)A(I_n - B^-B)]$

*Proof.* Let $N = \begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}$. Then $MN = \begin{bmatrix} 0 & I_n \\ B & 0 \end{bmatrix}$ and $M^2N = \begin{bmatrix} B & A \\ 0 & B \end{bmatrix}$. We now recall the rank formulæ

$$rk[P, Q] = rk(P) + rk[(I_n - PP^-)Q]$$

and

$$rk \begin{bmatrix} P \\ Q \end{bmatrix} = rk(Q) + rk[P(I_n - Q^-Q)].$$

These show that

$$rk \begin{bmatrix} B & A \\ 0 & B \end{bmatrix} = rk(B) + rk[B, A(I_n - B^-B)] = 2rk(B) + rk[(I_n - BB^-)A(I_n - B^-B)].$$

It is now clear that $rk(M^2) = rk(M)$ exactly when condition (2) holds.

The remaining results follow from standard range-rank conditions. From $R(B^-B) \oplus R(I_n - B^-B) = R(I_n) = R(BB^-) \oplus R(I_n - BB^-)$, and since $rk(BB^-) = rk(B) = rk(B^-B)$, we see that $rk(I_n - B^-B) = n - rk(B) = rk(I_n - BB^-)$. Condition (2) means that $rk[(I_n - BB^-)A(I_n - B^-B)] = rk(I_n - BB^-) = rk(I_n - B^-B)$, which is equivalent to $R(I_n - BB^-) = R[(I_n - BB^-)A(I_n - B^-B)]$ and to $RS(I_n - B^-B) = RS[(I_n - BB^-)A(I_n - B^-B)]$. □

We shall not use rank to investigate the general (2,2,0) case but instead apply the "unit"-conditions as given in [3], for the existence of the group inverse of the triplet $M = PAQ$, where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Q = I$ and $A = \begin{bmatrix} b & 0 \\ a & c \end{bmatrix}$. We shall make use of the fact that $A$ is regular as a matrix precisely when $w$ is regular as an element [4].

We shall repeatedly use the fact that in any ring with 1,

**Lemma 2.1.**

$$(1 + ab)x = 1 \text{ if and only if } (1 + ba)(1 - bxa) = 1. \tag{1}$$

Hence, $1 + ba$ is a unit if and only if $1 + ab$ is a unit, with $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$.

Using [3, Corollary 1], $(PA)^{\#}$ exists if and only if $U = APAA^- + 1 - AA^-$ is a unit, which is equivalent to $V = A^-APA + I - A^-A$ being a unit. In this case,

$$M^{\#} = (PA)^{\#} = PU^{-1}AV^{-1} = P(U^{-2})A = (PA)V^{-2} = MV^{-2}.$$

Now, $U$ can be written as $U = I + (AP - I)AA^- = I + YX$, which by Lemma (2.1) tells us that

$$U^{-1} = (I + YX)^{-1} = I - Y(I + XY)^{-1}X.$$

But

$$I + XY = I + (AA^-)(AP - I) = I - AA^- + AP = G.$$

As such,

$$U^{-1} = I + (I - AP)G^{-1}AA^-$$

and we shall have to compute $I - AP$, $AA^-$ and $G^{-1}$.

We note in passing that when $p = 1$, $a^\#$ exists if and only if $u = a^2a^- + 1 - aa^- = 1 + a(aa^- - a^-)$ is a unit if and only if $v = a^-a^2 + 1 - a^-a = 1 + (a^-a - a^-)a$ is a unit, which, on account of the above lemma, occurs precisely when $g = a + (1 - aa^-)$ or $h = a + (1 - a^-a)$ is a unit.

In this case $a^\# = u^{-1}av^{-1} = u^{-2}a = av^{-2}$ where $u^{-1} = 1 - ah^{-1}(aa^- - a^-)$ and $v^{-1} = 1 - (a^-a - a^-)g^{-1}a$.

Since $w$ is regular, we know from [4] that there exists an inner inverse $A^-$, such that $AA^-$ is lower triangular. Indeed, $A^- = \begin{bmatrix} 1 & 0 \\ -c^+a & 1 \end{bmatrix} \begin{bmatrix} b^+ & (1 - b^+b)w^-(1 - cc^+) \\ 0 & c^+ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(1 - cc^+)ab^+ & 1 \end{bmatrix}$ will do, where again $w = (1 - cc^+)a(1 - b^+b)$.

Next we compute

$$AP = \begin{bmatrix} 0 & b \\ c & a \end{bmatrix} \quad \text{and} \quad AA^- = \begin{bmatrix} bb^+ & 0 \\ (1 - ww^-)(1 - cc^+)ab^+ & cc^+ + ww^-(1 - cc^+) \end{bmatrix} \tag{2}$$

giving

$$G = \left[\begin{array}{c|c} 1 - bb^+ & b \\ \hline c - (1 - ww^-)(1 - cc^+)ab^+ & a + (1 - ww^-)(1 - cc^+) \end{array}\right].$$

We then form

$$G \begin{bmatrix} 1 & 0 \\ b^+ & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ \alpha & \delta \end{bmatrix} = G',$$

where

$$\alpha = c + ab^+ - (1 - ww^-)(1 - cc^+)[ab^+ - b^+]$$

and

$$\delta = a + (1 - ww^-)(1 - cc^+).$$

As such, $G$ will be a unit if and only if the (1,1) Schur complement of $G'$,

$$z = \delta - \alpha b = a(1 - b^+b) - cb + (1 - ww^-)(1 - cc^+)[1 + ab^+b - b^+b]$$

is a unit.

We may state the following theorem:

**Theorem 2.1.** *Assume that $b$, $c$ and $w = (1 - cc^+)a(1 - b^+b)$ are regular. Then $M = \begin{bmatrix} a & c \\ b & 0 \end{bmatrix}$ has a group inverse if and only if*

$$a(1 - b^+b) - cb + (1 - ww^-)(1 - cc^+)[1 + ab^+b - b^+b]$$

*is a unit.*

Two special cases are of interest (cf. [1]):

**Corollary 2.1.**

1. $\begin{bmatrix} a & a \\ b & 0 \end{bmatrix}^{\#}$ *exists if and only if* $z = (a + 1 - aa^{+})(1 - b^{+}b) - ab$ *is a unit.*

2. *If in addition* $a$ *has a group inverse and* $e = aa^{\#}$ *then the following are equivalent:*

   (a) $\begin{bmatrix} a & a \\ b & 0 \end{bmatrix}^{\#}$ *exists.*

   (b) $x = 1 - b^{+}b - b^{+}beb$ *is a unit.*

   (c) $y = 1 - b^{+}b - eb$ *is a unit.*

   (d) $bebR = bR$ *and* $Rbeb = Rb$.

   *In this case,* $be$ *and* $eb$ *have group inverses and are similar.*

*Proof.* 1. Set $c = a$ and $w = 0$.

2. Multiply through by $a^{\#} + 1 - aa^{\#}$ – the inverse of $a + 1 - aa^{\#}$– and then use Lemma (2.1). The equivalence of (b) and (c) follows from Lemma (2.1).

   If $x$ is a unit, then $xb^{+}b = b^{+}beb$ implies $b^{+}b \in Rbeb$, which in turn implies $Rb = Rbeb$. Also, from $bx = beb$ we obtain $b \in bebR$, which implies $bR = bebR$.

   Conversely, from $bebR = bR$ we see that $(eb)^2R = ebR$ and $beR = bR$, while $Rbeb = Rb$ implies that $R(be)^2 = Rbe$ and $Reb = Rb$.

   We then observe that $beR = bR = bebR = be(beb)R \subseteq bebeR \subseteq beR$ and thus $be$ has a group inverse. Likewise $(eb)^{\#}$ exists.

   These observations imply $(-eb)^{\#}$ exists and $Rb = R(-e)b$, which, by [3, Corollary 1], is equivalent to the invertibility of $x = 1 - b^{+}b - b^{+}beb$.

Part (d) can be completed with aid of the following result:

**Lemma 2.2.** *Let* $e^2 = e$ *and* $beR = bR$ *and* $Reb = Rb$. *Then* $be \approx eb$.

*Proof.* If $bek = b = \ell eb$, then $be(uv) = (uv)eb$, where $u = 1 + (1 - e)\ell e$ and $v = 1 + ek(1 - e)$. The latter are clearly units. □

Our second special case concerns the (flipped) companion matrix [3].

**Corollary 2.2.** $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}^{\#}$ *exists if and only if* $[b - a(1 - b^{+}b)]$ *is a unit.*

*Proof.* Set $c = 1$ and $w = 0$ so that $z$ reduces to $z = -[b - a(1 - b^{+}b)]$. □

To find the actual expressions for the group inverse of $M = PA$, we still have to compute $G^{-1}$. Now

$$G = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b^{+} & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ \alpha & \delta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b^{+} & 1 \end{bmatrix}$$

with $z = \delta - \alpha b$, and thus

$$G^{-1} = \begin{bmatrix} 1 & 0 \\ b^+ & 1 \end{bmatrix} \begin{bmatrix} 1 + bz^{-1}\alpha & -bz^{-1} \\ -z^{-1}\alpha & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 + bz^{-1}\alpha & -bz^{-1} \\ b^+ - (1 - bb^+)z^{-1}\alpha & (1 - bb^+)z^{-1} \end{bmatrix}.$$

We can now either compute $U^{-1}$ and then $M^{\#} = P(U^{-1})^2 A$, or we can first simplify this expression and use $G^{-1}$.

For the former case we need

$$\begin{aligned}
(I - AP)G^{-1} &= \begin{bmatrix} 1 & -b \\ -c & 1 - a \end{bmatrix} \begin{bmatrix} 1 + bz^{-1}\alpha & -bz^{-1} \\ b^+ - (1 - bb^+)z^{-1}\alpha & (1 - bb^+)z^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 1 - bb^+ + (2b - b^2 b^+)z^{-1}\alpha & -b(2 - bb^+)z^{-1} \\ -c - [cb + (1 - a)(1 - bb^+)]z^{-1}\alpha + (1 - a)b^+ & [cb + (1 - a)(1 - bb^+)]z^{-1} \end{bmatrix} \\
&= \begin{bmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{bmatrix},
\end{aligned}$$

followed by

$$U^{-1} = I + (I - AP)G^{-1}AA^- = \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix},$$

where

$$\begin{aligned}
u_1 &= 1 + (2b - b^2 b^+)z^{-1}\alpha bb^+ + \gamma_3(1 - ww^-)(1 - cc^+)ab^+, \\
u_2 &= \gamma_2 bb^+ + \gamma_4(1 - ww^-)(1 - cc^+)ab^+, \\
u_3 &= \gamma_3[cc^+ + ww^-(1 - cc^+)], \\
u_4 &= 1 + \gamma_4[cc^+ + w^-(1 - cc^+)],
\end{aligned}$$

while for the latter case we compute

$$U^{-2}A = A + 2(I - AP)G^{-1}A + (I - AP)G^{-1}(AA^- - AP)G^{-1}A.$$

Recalling that $A = PM$ and $P^{-1} = P$ we arrive at

$$M^{\#} = M + 2(I - M)H^{-1}M + (I - M)H^{-1}(MM^- - M)H^{-1}, \tag{3}$$

where $H^{-1} = PG^{-1}P^{-1} = \begin{bmatrix} (1 - bb^+)z^{-1} & b^+ - (1 - bb^+)z^{-1}\alpha \\ -bz^{-1} & 1 + bz^{-1}\alpha \end{bmatrix}$.

## Remarks and questions

We close with some remarks and questions.

1. For the case where $c = a$, we may postmultiply $Rz = R$ by $b^-b$ and premultiplying $zR = R$ by $1 - aa^-$ and $aa^-$ in succession. This yields the necessary conditions

   (i) $Rb = Rab$,

   (ii) $(1 - aa^-)R = (1 - aa^-)(1 - bb^-)R$, and

(iii) $aR = a[b - (1 - b^- b)]R$.

The second condition is equivalent to $aR + (1 - b^- b)R = R$ or $R(1 - aa^-) \cap Rb = (0)$ and hence we must also have

(iv) $baR = bR$.

2. If $c = a$ and in addition $a^\#$ exists then we can solve the range and row-space equations $M^2X = M$ and $YM^2 = M$ directly to give $M^\# = YMX = YM(FKF^{-1})$, where

$$Y = \begin{bmatrix} (1 - rr^\#)a^\# & r^\# aa^\# \\ br^\# a^\# & s^\# \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } K = \begin{bmatrix} aa^\# ss^\# b & aa^\# s^\# \\ a^\# s^\# b & a^\#( -ss^\#) \end{bmatrix},$$

in which $r = eb$ and $s = be$.

3. It would be of interest to find the conditions on $a$, $b$ , $c$ and $d$ in the general case, for $M$ to have a group inverse. That is to say, when does $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ have a group inverse?

4. When does $(n + e)$ have a group inverse, where $n$ is nilpotent and $e$ is idempotent?

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