GENERALIZED INVERSES OF A SUM IN RINGS

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Abstract
We study properties of the Drazin index of regular elements in a ring with a unity 1. We give expressions for generalized inverses of $1 - ba$ in terms of generalized inverses of $1 - ab$. In our development we prove that the Drazin index of $1 - ba$ is equal to the Drazin index of $1 - ab$.

Keywords and phrases: Regular element, reflexive inverse, Drazin index, Drazin inverse, EP elements.

1. Introduction
Let $R$ be a ring with a unity 1. An element $a$ is said to be regular if there is an element $x$ such that $axa = a$. If it exists, then it is called an inner inverse of $a$ (von Neumann inverse). We will denote by $a\{1\} = \{x \in R \mid axa = a\}$ the set of all inner inverses of $a$ and we will write $a^-$ to designate a member of $a\{1\}$. A reflexive inverse $a^+$ of $a$ is an inner and outer inverse of $a$, that is, $a^+ \in a\{1\}$ and $a^+ aa^+ = a^+$.

An element $a$ is said to be Drazin invertible provided there is a common solution for the equations
\[ xax = x, \quad ax = xa, \quad a^k xa = a^k \] for some $k \geq 0$.

If a common solution exists, then it is unique and it will be denoted by $a^D$ (see [2]). The smallest integer $k$ for which the above equations hold is called the Drazin index of $a$, denoted by $\text{ind}(a)$.

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The Drazin index can be characterized in terms of right and left ideals generated by a power of $a$ as follows [7]: $\text{ind}(a) = k$ if and only if $k$ is the smallest non-negative integer for which $a^k R = a^{k+1} R$ and $R a^k = R a^{k+1}$, or equivalently, $a^k \in a^{k+1} R \cap R a^{k+1}$.

If $a$ is Drazin invertible with $\text{ind}(a) = 1$, then $a$ is regular. In the former case the Drazin inverse of $a$ is known as the group inverse of $a$, denoted by $a^\#$. It is well known that the smallest $k$ for which $(a^k)^\#$ exists equals $\text{ind}(a) = k$, and $a D = (a^k)^\# a^{k-1} = a^{k-1} (a^k)^\#$.

If there exists an element $a^\pi \in R$ such that $aa^\pi = a^\pi a$, $aa^\pi$ is nilpotent, and $a + a^\pi$ is nonsingular, then it is called a spectral idempotent of $a$; such element is unique (if it exists). We know that $a$ is Drazin invertible if and only the spectral idempotent of $a$ exists. In this case we have $a^D = (a + a^\pi)^{-1} (1 - a^\pi)$ and $a^\pi = 1 - aa^D$. Characterizations of ring elements with related spectral idempotents are given in [4], [5].

Let $R$ be a ring with an involution $x \mapsto x^*$ such that $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^* x^*$, for all $x, y \in R$. We say that $a$ is Moore-Penrose invertible if the equations

$bab = b, \quad aba = a, \quad (ab)^* = ab, \quad (ba)^* = ba$

have a common solution; such solution is unique if it exists (see [2], [6]), and it will be denoted by $a^\dagger$.

We say that an element $a$ is EP if $a$ is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$. An element $a$ is generalized EP if there exists $k \in \mathbb{N}$ such that $a^k$ is EP.

Barnes [1] proved that the ascents (descents) of $I - RS$ and $I - SR$ are equal for bounded operators on Banach spaces $R \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, X)$. Consequently, the Drazin indices of $I - RS$ and $I - SR$ are equal. In this paper we deal with the Drazin index of $1 - ab$ and $1 - ba$ in rings, and therefore neither functional calculi and operator theory can be used. Moreover, we provide a formula for the reflexive inverse, the group inverse and the Drazin inverse of $1 - ba$ in terms of the corresponding generalized inverse of $1 - ab$.

In our development, we extend the following characterization of the Drazin index given by Puystjens and Hartwig [10]: Given a regular element $a \in R$, then $\text{ind}(a) \leq 1 \iff \text{ind}(a + 1 - aa^-) = 0$, for one and hence all choices of $a^- \in a(1)$.

2. Auxiliary results

In this section we give some auxiliary lemmas. We start with an elementary known result.
**Lemma 2.1.** Let $a, b \in \mathcal{R}$. Then $1 - ab$ is invertible if and only if $1 - ba$ is invertible.

**Lemma 2.2.** Let $a$ be a regular element. Then, given a natural $n$,

$$(a + 1 - aa^-)^n = (a^2a^- + 1 - aa^-)^n + \sum_{i=1}^{n} a^i(1 - aa^-). \quad (2.1)$$

**Proof.** The proof is by induction on $n$. Denote $z = a + 1 - aa^-$ and $x = a^2a^- + 1 - aa^-$. It is clear that $z = x + a(1 - aa^-)$. Assuming (2.1) to hold for $k$, we will prove it for $k + 1$.

We note that $zx = x^2 + a(1 - aa^-)$ and $za = a^2$. Now, by the induction step

$$z^{k+1} = z\left(x^k + \sum_{i=1}^{k} a^i(1 - aa^-)\right)$$

$$= x^{k+1} + a(1 - aa^-) + \sum_{i=1}^{k} a^{i+1}(1 - aa^-)$$

$$= x^{k+1} + \sum_{i=1}^{k+1} a^i(1 - aa^-).$$

□

**Lemma 2.3.** Let $a, b \in \mathcal{R}$. Then, given a natural $n$,

$$(1 - ba)^n = 1 - bra \quad \text{and} \quad (1 - ab)^n = 1 - rab,$$

where $r = \sum_{j=0}^{n-1}(1 - ab)^j$.

**Proof.** It can be easily proved by induction on $n$. □

In [5] the authors give the following characterization of EP elements in a ring.

**Lemma 2.4.** Let $\mathcal{R}$ be a ring with an involution $x \rightarrow x^*$. For $a \in \mathcal{R}$ the following conditions are equivalent:

(i) $a$ is EP.

(ii) $a$ is Drazin and Moore-Penrose invertible and $a^D = a^\dagger$.

(iii) $a$ is group invertible and $a^\pi = (a^*)^\pi$. 


3. Main results

The following theorem is an answer to a question raised by Patricio and Veloso in [8] about the equivalence between \(\text{ind}(a^2a^- + 1 - aa^-) = k\) and \(\text{ind}(a + 1 - aa^-) = k\), and provides a new characterization of the Drazin index.

**Theorem 3.1.** Let \(a\) be a regular non-invertible element. The following conditions are equivalent:

(i) \(\text{ind}(a) = k + 1\).

(ii) \(\text{ind}(a^2a^- + 1 - aa^-) = k\), for one and hence all choices of \(a^- \in a[1]\).

(iii) \(\text{ind}(a + 1 - aa^-) = k\), for one and hence all choices of \(a^- \in a[1]\).

**Proof.** The equivalence (i)\(\iff\) (ii) is proved in [8, Theorem 2.1]. We proceed to show that (ii)\(\implies\) (iii). Denote \(x = a^2a^- + 1 - aa^-\) and \(z = a + 1 - aa^-\). Assume \(\text{ind}(x) = k\), or equivalently, \(\text{ind}(a) = k + 1\). Then \(x^k = x^{k+1}R\) and \(a^{k+1} = a^{k+2}w\) for some \(w \in R\). By (2.1),

\[
\begin{align*}
  z^kR & = \left(1 + \sum_{i=1}^{k} a^i (1 - aa^-)\right) x^k R \\
  & = \left(1 + \sum_{i=1}^{k} a^i (1 - aa^-)\right) x^{k+1} R \\
  & = \left(z^{k+1} - \sum_{i=1}^{k+1} a^i (1 - aa^-) + \sum_{i=1}^{k} a^i (1 - aa^-)\right) R \\
  & = \left(z^{k+1} - a^{k+1}(1 - aa^-)\right) R = (z^{k+1} - a^{k+2}w(1 - aa^-)) R \\
  & = z^{k+1}(1 - aw(1 - aa^-)) R \subseteq z^{k+1}R.
\end{align*}
\]

This gives \(z^kR = z^{k+1}R\). On the other hand, since \(\text{ind}(x) = k\) we also have \(x^k = ux^{k+1}\) for some \(u \in R\). By (2.1),

\[
\begin{align*}
  Rz^k & = R \left(x^k + \sum_{i=1}^{k} a^i (1 - aa^-)\right) \\
  & = R \left(ux^{k+1} + \sum_{i=1}^{k} a^i (1 - aa^-)\right) \\
  & = R \left(u - u \sum_{i=1}^{k} a^i (1 - aa^-) + \sum_{i=1}^{k} a^i (1 - aa^-)\right) z^{k+1} \subseteq Rz^{k+1}.
\end{align*}
\]

From this we conclude that \(Rz^k = Rz^{k+1}\). Consequently, \(\text{ind}(z) \leq k\).
By symmetrical arguments, we can show that ind($z$) = $k$ implies that ind($x$) $\leq k$. Further, suppose ind($z$) < $k$, having ind($x$) = $k$, then we would get that ind($x$) $\leq k - 1$, and we would arrive to a contradiction. Therefore ind($z$) = $k$. □

We can state the symmetrical of Theorem 3.1.

**Corollary 3.2.** Let $a$ be a regular non-invertible element. The following conditions are equivalent:

(i) ind($a$) = $k + 1$.
(ii) ind($a^2 + 1 - a^{-}a$) = $k$, for one and hence all choices of $a^{-} \in a[1]$.
(iii) ind($a + 1 - a^{-}a$) = $k$, for one and hence all choices of $a^{-} \in a[1]$.

The following corollary is an extension of the analogous result for the Drazin index of a complex partitioned matrix over $\mathbb{C}$ [3, Theorem 7.7.5].

**Corollary 3.3.** Let $R$ be any ring with unity. If $M = \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} \in R_{n \times n}$, where $A \in R_{r \times r}$ is invertible, then ind($M$) = ind($A + BCA^{-1}$) + 1.

**Proof.** We have $M^{-} = \begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{pmatrix}$ is an inner inverse of $M$ and

$$M + I - MM^{-} = \begin{pmatrix} A + BCA^{-1} & 0 \\ C - CA^{-1}(I - BCA^{-1}) & I \end{pmatrix}.$$ 

Using the following known result for block triangular matrices,

$$\max\{\text{ind}(I), \text{ind}(A + BCA^{-1})\} \leq \text{ind}(M + I - MM^{-}) \leq \text{ind}(A + BCA^{-1}) + \text{ind}(I),$$

we conclude that ind($M + I - MM^{-}$) = ind($A + BCA^{-1}$). Now, that ind($M$) = ind($A + BCA^{-1}$) + 1 follows from Theorem 3.1. □

It is well known that $1 - ba$ is regular if and only if $1 - ab$ is regular. Moreover, if $(1 - ab)^{-}$ is an inner inverse of $1 - ab$ then $(1 - ba)^{-} = 1 + b(1 - ab)^{-}a$ is an inner inverse of $1 - ba$. In the sequel, we will extend the same reasoning to other generalized inverses, namely reflexive, group and Drazin inverse.
THEOREM 3.4. Let \( a, b \in \mathbb{R} \). If \((1 - ab)^+\) is a reflexive inverse of \( 1 - ab \), then a reflexive inverse of \( 1 - ba \) is given by

\[
(1 - ba)^+ = 1 + b ((1 - ab)^+ - pq) a,
\]

where \( p = 1 - (1 - ab)^+ (1 - ab) \) and \( q = 1 - (1 - ab)(1 - ab)^+ \).

PROOF. Let \( x = 1 + b ((1 - ab)^+ - pq) a \). Then

\[
(1 - ba)x = 1 - bqa.
\]

Further,

\[
(1 - ba)x(1 - ba) = 1 - ba - bqa(1 - ba)a = 1 - ba
\]

and

\[
x(1 - ba)x = x - xbqa
= x - bqa - b ((1 - ab)^+ - pq) abqa
= x,
\]

where we have simplified writing \( ab = 1 - (1 - ab) \) and using relations \((1 - ab)(1 - ab)^+ (1 - ab) = (1 - ab)\) and \((1 - ab)^+ (1 - ab)(1 - ab)^+ = (1 - ab)^+\). \(\square\)

THEOREM 3.5. Let \( a, b \in \mathbb{R} \). If \( 1 - ab \) is group invertible, then \( 1 - ba \) is group invertible and

\[
(1 - ba)^\# = 1 + b \left((1 - ab)^\# - (1 - ab)^\# a\right)
\]

where \((1 - ab)^\# = 1 - (1 - ab)^\# (1 - ab)\).

PROOF. Let \( x = 1 + b \left((1 - ab)^\# - (1 - ab)^\# a\right) a \). First, we note that \((1 - ab)^\#\) is a reflexive inverse that commutes with \( 1 - ab \). In view of the preceding theorem we have that \( x \) is reflexive inverse of \( 1 - ba \). Next, we will prove that \( x \) commutes with \( 1 - ba \). We have

\[
x(1 - ba) = 1 - ba + b(1 - ab)^\# (1 - ab)a = 1 - b(1 - ab)^\# a
\]

and, similarly, \((1 - ba)x = 1 - b(1 - ab)^\# a\) which gives \( x(1 - ba) = (1 - ba)x \). Therefore \( x \) verifies the three equations involved in the definition of group inverse. \(\square\)
THEOREM 3.6. Let \( a, b \in \mathcal{R} \). If \( 1 - ab \) is Drazin invertible with \( \text{ind}(1 - ab) = k \), then \( 1 - ba \) is Drazin invertible with \( \text{ind}(1 - ba) = k \) and

\[
(1 - ba)^D = 1 + b \left( (1 - ab)^D - (1 - ab)^\pi r \right) a,
\]

where \( r = \sum_{j=0}^{k-1} (1 - ab)^j \).

PROOF. Assume \( \text{ind}(1 - ab) = k \geq 2 \). Then \( (1 - ab)^k \) is group invertible and Theorem 3.1 leads to \( \text{ind}(1 - (1 - (1 - ab)^k)(1 - ab)^\pi((1 - ab)^k)^\pi)) = 0 \). By Lemma 2.1 we have that

\[
1 - (1 - ab)^k = rab \quad \text{and} \quad 1 - (1 - ba)^k = bra,
\]

(3.1)

where \( r = \sum_{j=0}^{k-1} (1 - ab)^j \). According to the above relations, \( 1 - rab(1 - ab)^\pi((1 - ab)^k)^\pi \) is invertible and by Lemma 2.1 we have that \( 1 - b(1 - ab)(1 - ab)^D ra \) is invertible. Further,

\[
(1 - b(1 - ab)(1 - ab)^D ra)(1 - ba)^k = (1 - ba)^k - b(1 - ab)(1 - ab)^D ra(1 - ba)^k
\]

\[
= (1 - ba)^k - b(1 - ab)\pi ra
\]

\[
= (1 - bra)(1 - ba)^k = (1 - ba)^{2k}.
\]

From this it follows that \( (1 - ba)^k = (1 - b(1 - ab)(1 - ab)^D ra)^{−1}(1 - ba)^{2k} \in \mathcal{R}(1 - ba)^k+1 \). On the other hand,

\[
(1 - ba)^k(1 - b(1 - ab)(1 - ab)^D ra) = (1 - ba)^k - (1 - ba)^k b(1 - ab)(1 - ab)^D ra
\]

\[
= (1 - ba)^k - b(1 - ab)\pi ra = (1 - ba)^{2k}
\]

and hence \( (1 - ba)^k = (1 - ba)^{2k}(1 - b(1 - ab)(1 - ab)^D ra)^{−1} \in (1 - ba)^k+1 \mathcal{R} \).

Therefore \( (1 - ba)^k \in \mathcal{R}(1 - ba)^k+1 \cap (1 - ba)^k+1 \mathcal{R} \), which implies \( \text{ind}(1 - ba) \leq k \).

Further, analysis similar to that of the last part of the proof of Theorem 3.1 shows that \( \text{ind}(1 - ab) = k \). Now, \( (1 - ba)^D = ((1 - ba)^k)^\pi(1 - ba)^k−1 \). In view of (3.1) and applying Theorem 3, it follows

\[
((1 - ba)^k)^\pi = (1 - bra)^\pi = 1 + b \left( (1 - rab)^\pi - (1 - rab)^\pi \right) ra
\]

\[
= 1 + b \left( ((1 - ab)^k)^\pi - ((1 - ab)^k)^\pi \right) ra
\]

\[
= 1 + b \left( ((1 - ab)^D)^k - (1 - ab)^\pi \right) ra.
\]
Hence,
\[
(1 - ba)^D = \left(1 + b\left((1 - ab)^D\right)^k - (1 - ab)^\pi\right)ra (1 - ba)^{k-1}
\]
\[
= (1 - ba)^{k-1} + b\left((1 - ab)^D\right)^k - (1 - ab)^\pi (1 - ab)^{k-1}ra
\]
\[
= 1 - br'a + b\left((1 - ab)^D r - (1 - ab)^\pi (1 - ab)^{k-1}\right) a
\]
\[
= 1 + b\left((1 - ab)^D - (1 - ab)^\pi r' - (1 - ab)^\pi (1 - ab)^{k-1}\right) a
\]
\[
= 1 + b\left((1 - ab)^D - (1 - ab)^\pi r\right) a,
\]
where \(r' = \sum_{j=0}^{k-2}(1 - ab)^j\), completing the proof. \(\square\)

Let \(\mathcal{R}_{n\times n}\) the ring of \(n \times n\) matrices over \(\mathcal{R}\). Any matrix \(A \in \mathcal{R}_{r\times n}\) \((B \in \mathcal{R}_{n\times r})\) with \(r < n\) may be enlarged to square \(n \times n\) matrix \(A'\) \((B')\) by adding zeros. Then we can compute a generalized inverse of \(I - BA = I - B'A'\) using preceding results in the ring \(\mathcal{R}_{n\times n}\). Finally, we can rewrite the corresponding expression for the generalized inverse of \(I - B'A'\) in terms of \(A\) and \(B\), getting that formulas similar to that in the preceding theorems hold for rectangular matrices \(A\) and \(B\).

**Example 3.7.** We consider the following matrices with entries in the univariate polynomial ring in \(x\) over \(\mathbb{Z}_8\), the ring of integers modulo 8:

\[
A = \begin{pmatrix} x & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 7x \\ 2 \\ x^2 + 3 \end{pmatrix}.
\]

Then

\[
I - BA = \begin{pmatrix} x^2 + 1 & 2x & x \\ 6x & 5 & 6 \\ 7x^2 + 5x & 6x^2 + 2 & 7x^2 + 6 \end{pmatrix} \quad \text{and} \quad 1 - AB = 2.
\]

The zero degree polynomial equal to 2 is nilpotent of index 3 and, so, \(\text{ind}(1 - AB) = 3\) and \((1 - AB)^D = 0\). Applying Theorem 3 we get

\[
(I - BA)^D = I + \begin{pmatrix} 7x \\ 2 \\ x^2 + 3 \end{pmatrix} (0 - 1(1 + 2 + 2^2)) \begin{pmatrix} x & 2 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 7x^2 + 1 & 6x & 7x \\ 2x & 5 & 2 \\ x^3 + 3x & 2x^2 + 6 & x^2 + 4 \end{pmatrix}.
\]
We know that in general $1 - ab$ is EP may not imply that $1 - ba$ is EP. In the following result we give a necessary and sufficient condition for such implication to hold.

**Corollary 3.8.** Let $R$ be a ring with an involution $x \rightarrow x^*$. If $1 - ab$ is EP, then $1 - ba$ is EP if and only if $a^*(1 - ab)^\natural b^* = b(1 - ab)^\natural a$. In this case, 

$$(1 - ba)^\dagger = 1 + b\left((1 - ab)^\dagger - (1 - (1 - ab)(1 - ab)^\dagger)\right)a.$$ 

**Proof.** Since $1 - ab$ is EP, by Lemma 2.4 we have that $1 - ab$ is group invertible and Moore-Penrose invertible and $(1 - ab)^\# = (1 - ab)^\dagger$. Now, from Theorem 3 it follows that $1 - ba$ is also group invertible and $(1 - ba)^\# = 1 + b((1 - ab)^\# - (1 - ab)^\natural a)$, and consequently, $(1 - ba)^\natural = b(1 - ab)^\natural a$. Thus, by Lemma 2.4, $1 - ba$ is EP if and only if $((1 - ba)^*)^\natural = (1 - ba)^\natural$, that is, 

$$(b(1 - ab)^\natural a)^* = b(1 - ab)^\natural a.$$ 

Hence, using that $((1 - ab)^*)^\natural = (1 - ab)^\natural$, the result follows. $\square$

**Corollary 3.9.** Let $R$ be a ring with an involution $x \rightarrow x^*$. If $1 - ab$ is generalized EP, then $1 - ba$ is generalized EP if and only if $(ra)^*(1 - ab)^\natural b^* = b(1 - ab)^\natural ra$, where $r = \sum_{j=0}^{k-1}(1 - ab)^j$ and $k = \text{ind}(1 - ab)$.

**Proof.** Since $1 - ab$ is generalized EP then there exists the smallest integer $k \in \mathbb{N}$ such that $(1 - ab)^k$ is EP. From Lemma 2.4 we can deduce that $\text{ind}(1 - ab) = k$. Now, by Lemma 2.3 we have $(1 - ab)^k = 1 - rab$, where $r$ is defined as in the statement of this corollary. By preceding corollary, $(1 - ba)^k = 1 - bra$ is EP if and only if $(b(1 - ab)^\natural ra)^* = b(1 - ab)^\natural ra$, completing the proof. $\square$

In this example we show that the existence of the Moore-Penrose of $1 - ab$ does not imply the existence of the Moore-Penrose of $1 - ba$.

**Example 3.10.** Consider the following matrices over the field $\mathbb{C}$ of complex numbers, with the involution defined by $A^* = A^T$:

$$A = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$ 

Then

$$I - AB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad I - BA = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},$$ 

where $I$ is the identity matrix.
and, further,

\[(I - AB)*(I - AB) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (I - BA)*(I - BA) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\]

Since rank \((I - AB) = 1\) and rank \((I - AB)^*(I - AB) = rank (I - AB)(I - AB)^* = 1\) we conclude, applying [9, Theorem 1], that \(I - AB\) is Moore-Penrose invertible. On the other hand, since rank \((I - BA) = 1\) and rank \((I - BA)^*(I - BA) = 0\) we conclude that \(I - BA\) is not Moore-Penrose invertible.

**References**


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