ISOMORPHISM PROBLEMS FOR TRANSFORMATION SEMIGROUPS

SUZANA MENDES-GONÇALVES
Centro de Matemática,
Campus de Gualtar, Universidade do Minho,
4710-057 Braga, Portugal
E-mail: smendes@math.uminho.pt

Many people have studied the problem of describing all isomorphisms between transformation semigroups defined on sets and between linear transformation semigroups defined on vector spaces. In this paper, we summarise some of that work, as well as recent work showing that Baer-Levi semigroups defined on sets are never isomorphic to their linear analogue, despite all appearances to the contrary.

1. Introduction

Let X be an arbitrary set and let $P(X)$ denote the set of all *partial transformations* of X: that is, all transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $\operatorname{ran}\alpha$, are subsets of X. As usual, the *composition* $\alpha\circ\beta$ of $\alpha,\beta\in P(X)$ is the transformation with domain $Y=(\operatorname{ran}\alpha\cap\operatorname{dom}\beta)\alpha^{-1}$ such that, for all $x\in Y$, $x(\alpha\circ\beta)=(x\alpha)\beta$, and we write $\alpha\circ\beta$ more simply as $\alpha\beta$. It is well-known that $(P(X),\circ)$ is a semigroup. Let $T(X)$ denote the subsemigroup of $P(X)$ consisting of all $\alpha\in P(X)$ with domain X, and let $I(X)$ denote the *symmetric inverse semigroup* on X: that is, the set of all injective elements of $P(X)$.

If $\alpha\in T(X)$, we define the *rank* of α to be $r(\alpha)=|\operatorname{ran}\alpha|$ and we define another two cardinal numbers, called the *defect* and the *collapse* of α, respectively, as follows.

$$\begin{aligned} D(\alpha)&=X\setminus X\alpha, & d(\alpha)&=|D(\alpha)|,\\ C(\alpha)&=\bigcup\{y\alpha^{-1}:|y\alpha^{-1}|\geq 2\}, & c(\alpha)&=|C(\alpha)|.\end{aligned}$$

If X is an infinite set and $|X|=p\geq q\geq\aleph_0$, we write

$$BL(p,q)=\{\alpha\in T(X):c(\alpha)=0,\ d(\alpha)=q\}.$$

That is, $BL(p,q)$ consists of all one-to-one transformations α of X into itself such that $X \setminus X\alpha$ has cardinal q. Following Clifford and Preston, we call this the *Baer-Levi semigroup of type* (p,q) *on the set* X. It is well known that $BL(p,q)$ is a right simple, right cancellative semigroup without idempotents; and, according to Clifford and Preston [4] Vol. 2, p. 82, Teissier showed in 1953 that any semigroup with these properties can be embedded in some Baer-Levi semigroup.

Now let V be a vector space over a field F and let $P(V)$ denote the set of all *partial linear transformations* of V: that is, all linear transformations $\alpha : A \to B$ where A, B are subspaces of V. It is not difficult to see that $(P(V), \circ)$ is a semigroup, with composition of transformations defined as before. Let $T(V)$ denote the linear analogue of $T(X)$, and $I(V)$ that of $I(X)$. Note that we use the 'V' in place of 'X' to denote the fact that we are considering *linear* transformations.

In this paper we briefly survey some results on isomorphisms between transformation semigroups and between linear transformation semigroups. In section 2, we give a short account of work on the first problem, focusing on semigroups that are 'normal' in the usual group-theoretic sense (that is, which are invariant with respect to conjugation by permutations of the underlying set) and on semigroups that contain 'sufficient' constant maps. It is well-known that some important transformation semigroups have these properties. Likewise, in section 3, we illustrate the ongoing interest in the isomorphism problems between linear transformation semigroups, focusing on semigroups which are closed under conjugation by elements of the general linear group defined on the underlying vector space, as well as on semigroups which contain a 'big enough' number of linear transformations with rank 1. This motivates the definition of a linear version of the Baer-Levi semigroup: in section 4, we present some results which suggest that the two types of Baer-Levi semigroup are similar and finally, we show that the two versions – one defined on sets, the other on vector spaces – are never isomorphic. All the work in section 4 has appeared in [16].

2. Isomorphisms between transformation semigroups

In this section, we consider the problem of describing all isomorphisms between certain semigroups of transformations defined on *unstructured* sets. That is, we omit all mention of work on the corresponding problem for transformations which preserve some structure on the sets (for example, an order, graph, ultrafilter or topology). We begin with some background

concerning automorphisms of transformation semigroups, as a preamble to our discussion of isomorphisms.

We denote the *symmetric group* on X by $G(X)$, and we let $\mathrm{Alt}(X)$ denote the *alternating group* on X: that is, the set of all $\alpha \in G(X)$ whose *shift* $S(\alpha) = \{x \in X : x\alpha \neq x\}$ is finite and for which $\alpha|S(\alpha)$ is an even permutation (see [21] p. 301). Given a subsemigroup S of $P(X)$, we denote the automorphism group of S by $\mathrm{Aut}(S)$. We say that $\phi \in \mathrm{Aut}(S)$ is *inner* if there exists some $g \in G(X)$ such that $\alpha\phi = g^{-1}\alpha g$ for every $\alpha \in S$.

In [22] Sullivan observed that many people had described the automorphism group of various transformation semigroups defined on a set X. For example, Lyapin in 1955, Magill in 1967, and Schreier in 1936, each described the automorphisms of the full transformation semigroup $T(X)$, and Mal'cev in 1952 determined the ideals of $T(X)$ and their automorphism groups. One year later, Liber characterised the automorphism group of each ideal of $I(X)$ and, several years later, Šutov studied the automorphisms of the subsemigroup of $T(X)$ consisting of all (total) transformations which shift at most a finite number of elements. Also, Magill in 1967 described all automorphisms of $P(X)$, and Šutov in 1961 determined all ideals of $P(X)$ and their automorphism groups. In all these cases, the automorphisms are inner and each semigroup has $G(X)$ as its automorphism group.

To unify these results, in [22] Sullivan introduced two simple concepts: a semigroup S of partial transformations on X is said to *cover* X if for every $x \in X$ there exists an idempotent map in S with range $\{x\}$; and S is $G(X)$-*normal* if $g^{-1}\alpha g \in S$ for every $\alpha \in S$ and every $g \in G(X)$. Using these ideas, he proved the following in [22] Theorems 1 and 2 (this generalises the result in [9] which was unknown to him at the time).

Theorem 2.1. *If S is a subsemigroup of $P(X)$ covering X then every automorphism of S is inner. Moreover, if S is also $G(X)$-normal, then its automorphism group is isomorphic to $G(X)$.*

Subsequently, in [28], Symons characterised all $G(X)$-normal subsemigroups of $T(X)$ for finite X and, for this case, he observed a partial converse of Theorem 2.1. That is, for comparison later with its linear analogue, we quote his remark in [28] p. 390 as a formal result.

Theorem 2.2. *Let X be a finite set and let S be a subsemigroup of $T(X)$ which is not contained in $G(X)$. Then the following are equivalent.*
(a) S is $G(X)$-normal;
(b) every automorphism of S is induced by a permutation of X.

It may be possible for some transformation semigroups without constant maps (and which are not permutation groups) to have all its automorphisms inner; and in fact, in [5], Fitzpatrick and Symons proved that if X is infinite and S is a subsemigroup of $T(X)$ containing $G(X)$, then every automorphism of S is inner. Later, this result was extended in [24] to include subsemigroups of $I(X)$ which contain $\mathrm{Alt}(X)$.

As noted by Sullivan in [26] p. 213, the Baer-Levi semigroups $BL(p,q)$ are $G(X)$-normal and they contain no constants or permutations, hence "they became a test case for Sullivan's Conjecture in the early 1980s" that every automorphism of a $G(X)$-normal transformation semigroup is inner. In fact, Levi, Schein, Sullivan and Wood together showed in [12] that the automorphisms of a Baer-Levi semigroup are inner; and a few years later, Levi proved Sullivan's Conjecture in [10] and [11].

Most of the above work was motivated by earlier results on infinite permutation groups. For example, in [21] Theorem 11.4.6, Scott shows that if $|X| > 3$ and $|X| \neq 6$, then every automorphism of a subgroup of $G(X)$ which contains $\mathrm{Alt}(X)$ is inner. Also, in [21] sections 10.8 and 11.3, he describes all normal subgroups of $G(X)$ for any set X with $|X| > 4$. In particular, such subgroups always contain $\mathrm{Alt}(X)$. Consequently, every automorphism of a normal subgroup N of $G(X)$ is inner if $|X| > 4$ and $|X| \neq 6$; and moreover, in this case, $\mathrm{Aut}(N)$ is isomorphic to $G(X)$ (see [21] Theorem 11.4.8).

Often, these results on automorphisms of transformation semigroups can be converted to results concerning isomorphisms. For example, it is well-known that $T(X)$ is isomorphic to $T(Y)$ if and only if $|X| = |Y|$ (this follows from [4] Vol. 1, Exercise 1.1.7). In fact, each isomorphism $\phi : T(X) \to T(Y)$ is *induced* by a bijection $g : X \to Y$ in the sense that $\alpha\phi = g^{-1}\alpha g$ for every $\alpha \in T(X)$. And it is not difficult to see that this result can be extended to $P(X)$ and $I(X)$, and their ideals. More generally, Mendes-Gonçalves has proven the following extension of Theorem 2.1 above.

Theorem 2.3. *Let S be a subsemigroup of $P(X)$ covering X and S' a subsemigroup of $P(Y)$ covering Y. If $\phi : S \to S'$ is an isomorphism from S onto S', then ϕ is induced by a bijection $g : X \to Y$.*

In [12], the authors proved that every automorphism of a Baer-Levi semigroup is inner. Our next Theorem extends their result to isomorphisms between Baer-Levi semigroups: its proof can be found in [16] Theorem 3.2. For clarity, we write $BL(X,p,q)$ instead of $BL(p,q)$.

Theorem 2.4. *Let X and Y be infinite sets with $|X| = p$ and $|Y| = m$ and let q and n be infinite cardinals such that $q \leq p$ and $n \leq m$. Then, the semigroups $BL(X, p, q)$ and $BL(Y, m, n)$ are isomorphic if and only if $p = m$ and $q = n$. Moreover, for each isomorphism $\theta : BL(X, p, q) \to BL(Y, m, n)$, there is a bijection $h : X \to Y$ such that $\alpha\theta = h^{-1}\alpha h$ for every $\alpha \in BL(X, p, q)$.*

In passing, we recall that in 1967 Magill considered a major generalisation of $T(X)$ and in [15] Theorem 3.1 he described the isomorphisms between such semigroups. Also, Schein in [19] described the homomorphisms between certain semigroups of endomorphisms of various algebraic systems; and, as he observed in [19] p. 31, in general "every homomorphism (excluding some trivial ones) is an isomorphism induced by an isomorphism or an anti-isomorphism" between the underlying sets. For a similar result concerning infinite permutation groups, see [21] Theorem 11.3.7.

3. Isomorphisms between linear transformation semigroups

There are significant results in the theory of transformation semigroups that have corresponding results in the context of linear algebra. For example, in [18] Reynolds and Sullivan showed that Howie's characterisation in 1966 of the elements of $E(X)$, the semigroup generated by the non-identity idempotents of $T(X)$, has an analogue for the linear case.

In [6], Fountain and Lewin found a way to unify these areas: they introduced the concept of a *strong independence algebra* $\mathcal{A}$, of which sets and vector spaces are prime examples; and in [6] and [7] they described the semigroup generated by the non-identity idempotents in the semigroup of endomorphisms of $\mathcal{A}$. Likewise, in [14], Lima extended the work by Howie and Marques-Smith in 1984 on the semigroup generated by all nilpotents of $P(X)$ of index 2 to strong independence algebras, and thus also to vector spaces.

To give a brief account of some results on isomorphisms between semigroups of linear transformations, we introduce some concepts. Let V and W be vector spaces over fields F and K, respectively. A *semilinear transformation* from V to W is a bijection $g : V \to W$ for which there is an isomorphism $\omega : F \to K$ such that, for every $u, v \in V$ and $k \in F$,

$$(u + v)g = ug + vg, \qquad (kv)g = (k\omega)(vg).$$

Let S and S' be subsemigroups of $P(V)$ and $P(W)$, respectively. As for the set case, we say that an isomorphism θ from S onto S' is *induced* by

a semilinear transformation $g : V \to W$ if $\alpha\theta = g^{-1}\alpha g$ for every $\alpha \in S$. According to Baer [3] chapter 6, p. 201, several people studied the isomorphisms and automorphisms of certain groups of linear transformations. To state a major result in this regard, we let $G(V)$ denote the general linear group of V. Then, in [3] chapter 6, p. 229, Structure Theorem, it is shown that if neither F nor K has characteristic 2, and if V and W are vector spaces with dimension at least 3, then $G(V)$ and $G(W)$ are isomorphic if and only if $\dim V = \dim W$ and one of the following holds.

(a) F and K are isomorphic fields.
(b) $\dim V$ is finite and F and K are anti-isomorphic fields.

Moreover, a complete description of the isomorphisms from $G(V)$ onto $G(W)$ is given in [3] chapter 6, p. 231, Isomorphism Theorem.

Given a vector space V over a field F, we let $K(V)$ denote the subsemigroup of $T(V)$ which consists of zero and all linear transformations of V into itself with rank one. In [8] section 2, Gluskin studied the structure of $K(V)$ and some of its subsemigroups: in particular, he considered certain right ideals of $K(V)$ which we denote by $K^*(V)$ and of which $K(V)$ is a particular case (see [8] 2.7, pp. 113-114). In [8] Theorem 3.5, he proved the following.

Theorem 3.1. *Let V and W be vector spaces over the fields F and K, respectively, with $\dim V \geq 2$. If S and S' are subsemigroups of $T(V)$ and $T(W)$ containing $K^*(V)$ and $K^*(W)$, respectively, as two-sided ideals, then every isomorphism ϕ of S onto S' is induced by a semilinear transformation of V onto W. Moreover, $K^*(V)\phi = K^*(W)$.*

Subsequently, a weaker version of this result was proved in [20]. In fact, [20] Theorem 6.2 is an analogue of Theorem 2.3 for vector spaces. Recall our remark before Theorem 2.3 and note that, from Theorem 3.1, it follows that $T(V)$ and $T(W)$ are isomorphic if and only if there is a semilinear transformation from V onto W. Likewise, using Theorem 3.1, in [2] Araújo and Silva proved a linear version of Theorem 2.2 above. By analogy with the set case, we say that a subsemigroup S of $P(V)$ is $G(V)$-*normal* if $g^{-1}\alpha g \in S$ for every $\alpha \in S$ and every $g \in G(V)$.

Theorem 3.2. *Let V be a finite dimensional vector space over a field F and let S be a subsemigroup of $T(V) \setminus G(V)$. Then the following are equivalent.*
(a) S is $G(V)$-normal;
(b) S is an ideal of $T(V)$;
(c) every automorphism of S is induced by a semilinear transformation.

These results for semigroups of transformations and for semigroups of linear transformations are clearly connected. But, as observed by Sullivan in [25] p. 291, there are some 'puzzling' differences. For example, if X is an infinite set then certain total transformations of X can be written as a product of four idempotents in $T(X)$, and '4' is best possible. On the other hand, Reynolds and Sullivan showed in [18] that, if V is infinite-dimensional, then similarly-defined total linear transformations of V can be written as a product of three idempotents, and '3' is best possible. Consequently, in these cases, $E(X)$ and its linear analogue $E(V)$ can never be isomorphic.

As stated before, in [5] Fitzpatrick and Symons showed that, if X is infinite, then all automorphisms of a subsemigroup S of $T(X)$ which contains $G(X)$ are inner (note that if $G(X) \subseteq S$ then S is $G(X)$-normal). In fact, they first proved a weaker result: namely, that every automorphism of $G(X)$ can be extended to at most one automorphism of S (for contrast, we note that an automorphism of a normal subgroup of a group G is not necessarily 'extendible' to an automorphism of G: for example, see [21] exercise 9.2.28).

A simple analogue of their result for vector spaces would be as follows: if V is an infinite-dimensional vector space over a field F and S is a subsemigroup of $T(V)$ which contains $G(V)$, then every automorphism of $G(V)$ can be extended to at most one automorphism of S. However, as observed by Araújo in [1] pp. 57-58 this result does not hold: in [1] Lemmas 21 and 39, Araújo gives two simple examples of subsemigroups A_1 and A_2 of $T(V)$ which contain $G(V)$, but where the identity automorphism of $G(V)$ can be extended in infinitely many ways to automorphisms of A_1 and A_2, respectively. In order to produce a result close to the linear version of Fitzpatrick and Symons' result, Araújo in [1] Theorem 12 assumes that S is a subsemigroup of $P(V)$ containing $G(V) \cup E$, where E is the subset of $P(V)$ consisting of all identity transformations on one-dimensional subspaces of V, and he shows that every automorphism of $G(V)$ can be extended to at most one automorphism of S. In fact, [1] Theorem 12 is more general: the result is proved for strong independence algebras with at most one constant and rank at least 3.

4. Isomorphisms between Baer-Levi semigroups

In the remainder of this paper, we examine a semigroup related to the Baer-Levi semigroup $BL(p, q)$, which we define as follows. Let V be a

vector space over a field F and suppose $\dim V = m \geq \aleph_0$. If $\alpha \in T(V)$, we write $\ker \alpha$ for the *kernel* of α, and put

$$n(\alpha) = \dim \ker \alpha,\ r(\alpha) = \dim \operatorname{ran} \alpha,\ d(\alpha) = \operatorname{codim} \operatorname{ran} \alpha.$$

As usual, these are called the *nullity, rank* and *defect* of α, respectively. For each cardinal n such that $\aleph_0 \leq n \leq m$, we write

$$GS(m,n) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) = n\}$$

and call this the *linear Baer-Levi semigroup* on V. It can be shown that this is indeed a semigroup with the same properties as $BL(p,q)$: that is, $GS(m,n)$ is a right cancellative, right simple semigroup without idempotents. This fact extends work by Lima [14] Proposition 4.1 on $GS(m,m)$. More importantly, however, these two types of Baer-Levi semigroup – one defined on sets, the other on vector spaces – are never isomorphic. Hence, it is natural to look for properties of $GS(m,n)$ which mimic those of $BL(p,q)$. Next, we consider two of these: namely, the left ideals of $GS(m,n)$ and some of its maximal subsemigroups (for details, see [16] sections 4 and 5).

First we transfer results of Šutov [27] and Sullivan [23] on the left ideals of $BL(p,q)$ to the linear Baer-Levi semigroup on V. By analogy with their work, the most natural way to do this is to show that the left ideals of $GS(m,n)$ are the subsets L of $GS(m,n)$ which satisfy the condition:

$$\big[\alpha \in L,\ \beta \in GS(m,n),\ \operatorname{ran}\beta \subseteq \operatorname{ran}\alpha, \\ \dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n\big] \quad \text{implies} \quad \beta \in L.$$

Although this result is valid, to obtain more information about the left ideals of $GS(m,n)$ we proceed as follows.

If Y is a non-empty subset of $GS(m,n)$, we let $L_Y^+ = Y \cup L_Y$, where

$$L_Y = \{\beta \in GS(m,n) : \operatorname{ran}\beta \subseteq \operatorname{ran}\alpha, \\ \dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n \ \text{ for some } \ \alpha \in Y\}.$$

To show L_Y is non-empty, we need some notation: namely, we write $\{e_i\}$ for a linearly independent subset of V, and take this to mean that the subscript i belongs to some (unmentioned) index set I. Let $\alpha \in Y$, suppose $\{e_i\}$ is a basis for V with $|I| = m$, and write $e_i\alpha = a_i$ for each $i \in I$. Since α is one-to-one, $\{a_i\}$ is linearly independent and so it can be expanded to a basis $\{a_i\} \cup \{b_j\}$ for V. Note that $|J| = d(\alpha) = n \leq m$, therefore we can write $\{a_i\} = \{c_i\} \cup \{d_j\}$. Now let $e_i\beta = c_i$ for every i and extend this by linearity to the whole of V. Clearly, β is in $GS(m,n)$ since it

is one-to-one and $d(\beta) = \dim\langle d_j, b_j\rangle = n$. We have $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = \dim\langle d_j\rangle = n$. Hence $\beta \in L_Y$ and so L_Y is non-empty.

Theorem 4.1. *If Y is a non-empty subset of $GS(m,n)$, then L_Y^+ is a left ideal of $GS(m,n)$. Conversely, if I is a left ideal of $GS(m,n)$, then $I = L_I^+$.*

We can say a lot about the poset under $\subseteq$ of the (proper) left ideals in $GS(m,n)$. For example, it does not form a chain, and it has no minimal or maximal elements. In addition, it is not difficult to show that the principal left ideal generated by $\alpha \in GS(m,n)$ is $L_{\{\alpha\}}^+$.

In [13] Theorem 1, Levi and Wood described some maximal subsemigroups of $BL(p,q)$. To do this, they chose a non-empty subset A of X such that $|X \setminus A| \geq q$ and proved that the set

$$M_A = \{\alpha \in BL(p,q) : A \not\subseteq \operatorname{ran}\alpha \ \text{ or } \ (A\alpha \subseteq A \ \text{ or } \ |\operatorname{ran}\alpha \setminus A| < q)\}$$

is a maximal subsemigroup of $BL(p,q)$. By analogy with this, we let U be a non-zero subspace of V with $\operatorname{codim} U \geq n$ and define

$$M_U = \{\alpha \in GS(m,n) : U \not\subseteq \operatorname{ran}\alpha \ \text{ or } \ (U\alpha \subseteq U \ \text{ or } \ \dim(\operatorname{ran}\alpha/U) < n)\}.$$

Our next result determines some maximal subsemigroups of $GS(m,n)$.

Theorem 4.2. *For each non-zero subspace U of V with $\operatorname{codim} U \geq n$, M_U is a maximal subsemigroup of $GS(m,n)$.*

We note that although the proofs of [13] Theorem 1 and of [16] Theorem 5.1 are similar in outline, they are quite different in detail. In fact, despite the similarity between some results for $P(X)$ and others for $P(V)$, the ideas and techniques in these two areas are quite different.

The main result of [16] can be stated as follows.

Theorem 4.3. *The semigroups $BL(p,q)$ and $GS(m,n)$ are not isomorphic for any infinite cardinals p,q,m,n with $q \leq p$ and $n \leq m$.*

Section 2 illustrates the ongoing interest in automorphisms and isomorphisms for semigroups of transformations of sets; and the corresponding results in section 3 illustrate the development of the work on automorphisms and isomorphisms for semigroups of linear transformations of vector spaces. Clearly, the above result overlaps both.

In passing, we note that in [17] an entirely different method from that used in the proof of Theorem 4.3 is used to show that $I(X)$ and $I(V)$ are almost never isomorphic, and that any inverse semigroup can be embedded in some $I(V)$. Likewise, since the semigroups $BL(p,q)$ and $GS(m,n)$ are never isomorphic, it is worth observing the following result (see [16] Theorem 3.12).

Theorem 4.4. *Any right simple, right cancellative semigroup S without idempotents can be embedded in some $GS(m,m)$.*

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